Quizz 4

Math 332
Introduction to Partial Differential Equations

Spring 2018

University Of Wisconsin, Milwaukee

Nasser M. Abbasi

May 2018 Compiled on September 17, 2018 at 8:32pm

Contents

1 Problem 1

Problem Solve the PDE

$$u_t = u_{xx} + xt \qquad 0 \leq x \leq 1, t \geq 0 \tag{1}$$

With boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 0$$

And initial condition

$$u(x,0) = \sin(\pi x)$$

Solution

The corresponding homogeneous PDE $u_t = u_{xx}$ with the same homogeneous boundary conditions was solved before. It was found to have eigenfunctions

$$\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right)$$

With corresponding eigenvalues

$$\lambda_n = n^2\pi^2 \qquad n = 1, 2, 3, \cdots$$

Using eigenfunction expansion, it is now assumed that the solution to the given inhomogeneous PDE is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)$$

Substituting the above into the original PDE (1), and since term by term differentiation is justified (eigenfunctions are continuous) results in

$$\sum_{n=1}^{\infty} b_n'(t)\Phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\Phi_n''(x) + \sum_{n=1}^{\infty} \gamma_n(t)\Phi_n(x) \tag{1A}$$

Where $\sum_{n=1}^{\infty} \gamma_n(t)\Phi_n(x)$ is the expansion of the forcing function xt using same eigenfunctions

$$xt = \sum_{n=1}^{\infty} \gamma_n(t)\Phi_n(x) \tag{1B}$$

But $\Phi_n''(x) = -\lambda_n\Phi_n(x)$ since the eigenfunctions satisfy the eigenvalue ODE $X'' = -\lambda_n X$. Therefore (1A) simplifies to

$$\sum_{n=1}^{\infty} b_n'(t)\Phi_n(x) = \sum_{n=1}^{\infty} -\lambda_n b_n(t)\Phi_n(x) + \sum_{n=1}^{\infty} \gamma_n(t)\Phi_n(x)$$
$$b_n'(t) + \lambda_n b_n(t) = \gamma_n(t) \tag{2}$$

$\gamma_n(t)$ is now found by applying orthogonality to (1B), and using the weight $r(x) = 1$ gives

$$t\int_0^1 x\Phi_n(x)\,dx = \gamma_n(t)\int_0^1 \Phi_n^2(x)\,dx$$

Using $\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right) = \sin(n\pi x)$ and $\int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{2}$, the above simplifies to

$$t\int_0^1 x\sin(n\pi x)\,dx = \gamma_n(t)\frac{1}{2}$$

$$\gamma_n(t) = 2t\int_0^1 x\sin(n\pi x)\,dx \tag{3}$$

The integral on the right side above is found using $\int x\sin(ax)\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$, therefore

$$\begin{aligned}
\int_0^1 x\sin(n\pi x)\,dx &= \left(\frac{\sin n\pi x}{n^2\pi^2} - \frac{x\cos n\pi x}{n\pi}\right)_0^1\\
&= \left(\frac{\sin n\pi}{n^2\pi^2} - \frac{\cos n\pi}{n\pi}\right)\\
&= -\frac{\cos n\pi}{n\pi}\\
&= \frac{-(-1)^n}{n\pi}\\
&= \frac{(-1)^{n+1}}{n\pi}
\end{aligned}$$

Hence equation (3) now can be written as

$$\gamma_n(t) = \frac{2(-1)^{n+1}}{n\pi}t$$

Substituting the above in (2) gives the first order ODE to solve for $b_n(t)$

$$b_n'(t) + (n\pi)^2 b_n(t) = \frac{2(-1)^{n+1}}{n\pi}t$$

The integrating factor is $I = e^{n^2\pi^2 t}$. Hence the above becomes, after multiplying both sides by I

$$\frac{d}{dt}\left(e^{n^2\pi^2 t}b_n(t)\right) = \frac{2(-1)^{n+1}}{n\pi}te^{n^2\pi^2 t}$$

Integrating both sides gives

$$e^{n^2\pi^2 t}b_n(t) = \frac{2(-1)^{n+1}}{n\pi}\int_0^t se^{n^2\pi^2 s}ds + b_n(0) \tag{4}$$

Where $b_n(0)$ is the constant of integration. Dividing both sides by $e^{n^2\pi^2 t}$ gives

$$b_n(t) = \frac{2(-1)^{n+1}}{n\pi}\int_0^t se^{n^2\pi^2(s-t)}ds + b_n(0)e^{-n^2\pi^2 t}$$

But $\int_0^t se^{n^2\pi^2(s-t)}ds = \frac{n^2\pi^2 t - 1 + e^{-n^2\pi^2 t}}{n^4\pi^4}$ by integration by parts. The above now becomes

$$b_n(t) = 2(-1)^{n+1}\left(\frac{n^2\pi^2 t - 1 + e^{-n^2\pi^2 t}}{n^5\pi^5}\right) + b_n(0)e^{-n^2\pi^2 t}$$

Now that $b_n(t)$ is found, the final solution is

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)\\
&= \sum_{n=1}^{\infty}\left(2(-1)^{n+1}\left(\frac{n^2\pi^2 t - 1 + e^{-n^2\pi^2 t}}{n^5\pi^5}\right) + b_n(0)e^{-n^2\pi^2 t}\right)\sin(n\pi x)
\end{aligned} \tag{5}$$

$b_n(0)$ is determined from the given initial conditions $u(x,0) = \sin\pi x$. The above becomes at $t = 0$

$$\begin{aligned}
\sin\pi x &= \sum_{n=1}^{\infty}\left(2(-1)^{n+1}\left(\frac{-1+1}{n^5\pi^5}\right) + b_n(0)\right)\sin(n\pi x)\\
&= \sum_{n=1}^{\infty} b_n(0)\sin(n\pi x)
\end{aligned}$$

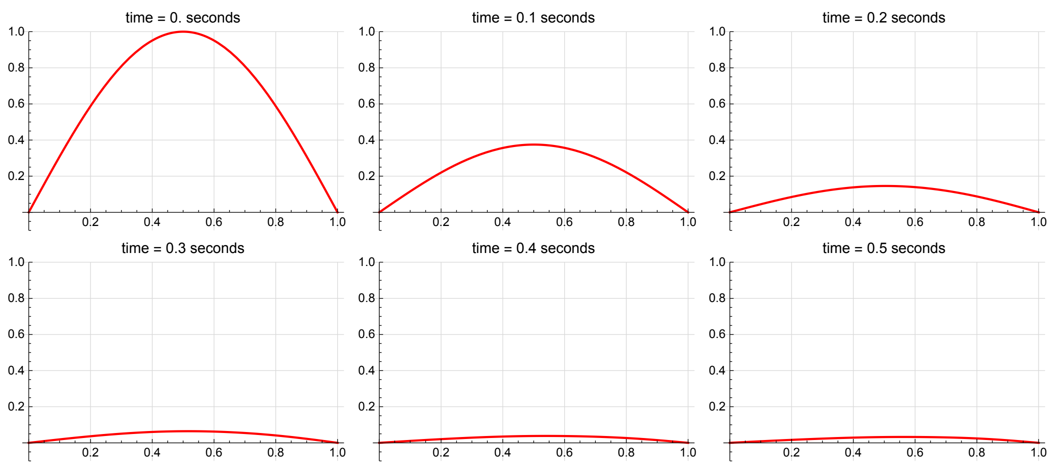
Therefore <u>when $n = 1$</u> (since LHS is $\sin\pi x$) the above gives

$$b_1(0) = 1$$

And $\underline{b_n(0) = 0 \text{ for all other } n}$. Equation (5) now simplifies to

$$u(x,t) = \overbrace{\left(2\left(\frac{\pi^2 t - 1 + e^{-\pi^2 t}}{\pi^5}\right) + e^{-\pi^2 t}\right)}^{n=1 \text{ term}} \sin(\pi x) + \frac{1}{\pi^5}\sum_{n=2}^{\infty} \frac{2}{n^5}(-1)^{n+1}\left(n^2\pi^2 t + e^{-n^2\pi^2 t} - 1\right)\sin(n\pi x)$$

To verify the above solution, it was plotted against numerical solution for different instances of time and also animated. It gave an exact match. A small number of terms was needed in the summation since convergence was fast and is of order $O\left(\frac{1}{n^3}\right)$. The following is a plot of the above solution for different instances of times using 5 terms.



2 Problem 2

Problem Show that

$$(\lambda - \mu)\int_0^1 x J_o\left(\sqrt{\lambda} x\right) J_o\left(\sqrt{\mu} x\right) dx = \sqrt{\mu} J_o'\left(\sqrt{\mu}\right) J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda} J_o'\left(\sqrt{\lambda}\right) J_o\left(\sqrt{\mu}\right)$$

Hint: Use the same method that proves orthogonality of eigenfunctions in 11.4

Solution

In the above, λ and μ are the eigenvalues, with the corresponding eigenfunctions

$$\Phi_\lambda(x) = J_o\left(\sqrt{\lambda} x\right) \tag{1}$$
$$\Phi_\mu(x) = J_o\left(\sqrt{\mu} x\right) \tag{2}$$

These come from the Sturm Liouville equation

$$-(xy')' = \lambda x y \tag{3}$$

Where

$$p(x) = x$$
$$q(x) = 0$$
$$r(x) = x$$

In operator form

$$L[\Phi_\lambda] = -\left(\Phi_\lambda'\right)' = \lambda x \Phi_\lambda \tag{4}$$

Similarly for any other eigenvalue such as μ. Multiplying both sides of (4) by $\Phi_\mu(x)$ and integrating gives

$$\int_0^1 L[\Phi_\lambda]\Phi_\mu dx = \int_0^1 \overbrace{-\left(\Phi_\lambda'\right)'}^{dv}\overbrace{\Phi_\mu}^{u} dx$$

Integrating by part the right side results in

$$\int_0^1 L[\Phi_\lambda]\Phi_\mu dx = \left[-\Phi_\lambda'\Phi_\mu\right]_0^1 - \int_0^1 -\Phi_\lambda'\Phi_\mu' dx$$

Integrating by parts again the second integral above, where now $dv = -\Phi'_\lambda, u = \Phi'_\mu$ gives

$$\begin{aligned}
\int_0^1 L[\Phi_\lambda]\Phi_\mu dx &= \left[-\Phi'_\lambda\Phi_\mu\right]_0^1 - \left(\left[-\Phi_\lambda\Phi'_\mu\right]_0^1 - \int_0^1 -\Phi_\lambda\Phi''_\mu dx\right) \\
&= \left[-\Phi'_\lambda\Phi_\mu\right]_0^1 - \left[-\Phi_\lambda\Phi'_\mu\right]_0^1 + \int_0^1 -\Phi_\lambda\Phi''_\mu dx \\
&= \left[-\Phi'_\lambda\Phi_\mu + \Phi_\lambda\Phi'_\mu\right]_0^1 + \int_0^1 \Phi_\lambda\left(-\Phi'_\mu\right)' dx
\end{aligned}$$

But $\left(-\Phi'_\mu\right)' = L\left[\Phi_\mu\right]$. Hence the above can be written as

$$\begin{aligned}
\int_0^1 L[\Phi_\lambda]\Phi_\mu dx &= \left[-\Phi'_\lambda\Phi_\mu + \Phi_\lambda\Phi'_\mu\right]_0^1 + \int_0^1 L\left[\Phi_\mu\right]\Phi_\lambda dx \\
\int_0^1 L[\Phi_\lambda]\Phi_\mu dx - \int_0^1 L\left[\Phi_\mu\right]\Phi_\lambda dx &= \left[-\Phi'_\lambda\Phi_\mu + \Phi_\lambda\Phi'_\mu\right]_0^1 \\
\int_0^1 \left(L[\Phi_\lambda]\Phi_\mu - L\left[\Phi_\mu\right]\Phi_\lambda\right) dx &= \left[-\Phi'_\lambda\Phi_\mu + \Phi_\lambda\Phi'_\mu\right]_0^1
\end{aligned}$$

But $L[\Phi_\lambda] = \lambda x\Phi_\lambda$ and $L\left[\Phi_\mu\right] = \mu x\Phi_\mu$, therefore the above can be written as

$$\begin{aligned}
\int_0^1 \left(\lambda x\Phi_\lambda\Phi_\mu - \mu x\Phi_\mu\Phi_\lambda\right) dx &= \left[-\Phi'_\lambda\Phi_\mu + \Phi_\lambda\Phi'_\mu\right]_0^1 \\
\int_0^1 (\lambda-\mu)\left(x\Phi_\lambda\Phi_\mu\right) dx &= \left[-\Phi'_\lambda\Phi_\mu + \Phi_\lambda\Phi'_\mu\right]_0^1 \\
(\lambda-\mu)\int_0^1 x\Phi_\lambda\Phi_\mu dx &= \left[-\Phi'_\lambda\Phi_\mu + \Phi_\lambda\Phi'_\mu\right]_0^1 \qquad (5)
\end{aligned}$$

Since $\Phi_\lambda(x) = J_o\left(\sqrt{\lambda}x\right), \Phi'_\lambda(x) = \sqrt{\lambda}J'_o\left(\sqrt{\lambda}x\right)$ and $\Phi_\mu(x) = J_o\left(\sqrt{\mu}x\right), \Phi'_\mu(x) = \sqrt{\mu}J'_o\left(\sqrt{\mu}x\right)$, then the above simplifies to

$$(\lambda-\mu)\int_0^1 xJ_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right)dx = \left[-\sqrt{\lambda}J'_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right) + J_o\left(\sqrt{\lambda}x\right)\sqrt{\mu}J'_o\left(\sqrt{\mu}x\right)\right]_0^1$$

What is left is to evaluate the boundary terms $\Delta = \left[-\sqrt{\lambda}J'_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right) + J_o\left(\sqrt{\lambda}x\right)\sqrt{\mu}J'_o\left(\sqrt{\mu}x\right)\right]_0^1$. This gives

$$\Delta = \left[-\sqrt{\lambda}J'_o\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right) + J_o\left(\sqrt{\lambda}\right)\sqrt{\mu}J'_o\left(\sqrt{\mu}\right)\right] - \left[-\sqrt{\lambda}J'_o(0)J_o(0) + J_o(0)\sqrt{\mu}J'_o(0)\right]$$

But $J'_o(0) = 0$ (since $J'_o(x) = -J_1(x)$ and $J_1(0) = 0$). Therefore the boundary terms reduces to

$$\Delta = J_o\left(\sqrt{\lambda}\right)\sqrt{\mu}J'_o\left(\sqrt{\mu}\right) - \sqrt{\lambda}J'_o\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)$$

Substituting this back in (5) gives the desired result

$$(\lambda-\mu)\int_0^1 xJ_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right)dx = \sqrt{\mu}J'_o\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda}J'_o\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)$$

3 Problem 3

Problem By letting $\mu \to \lambda$ in the formula of problem 2, derive a formula for $\int_0^1 xJ_0^2\left(\sqrt{\lambda}x\right)dx$. Then show that the normalized eigenfunctions of the eigenvalue problem in section 11.4 is

$$\hat{\Phi}_n(x) = \frac{\sqrt{2}J_0(j_n x)}{\left|J'_0(j_n)\right|}$$

where $0 < j_1 < j_2 < j_3 < \cdots$ denote the positive zeros of J_0

Solution

4 Part (a)

From problem 3, the formula obtained is

$$(\lambda-\mu)\int_0^1 xJ_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right)dx = \sqrt{\mu}J'_o\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda}J'_o\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)$$

Moving $(\lambda - \mu)$ to the right side gives

$$\int_0^1 xJ_o\left(\sqrt{\lambda}x\right)J_o\left(\sqrt{\mu}x\right)dx = \frac{\sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)}{(\lambda - \mu)}$$

Taking the limit $\lim \mu \to \lambda$ then the integral on the left becomes $\int_0^1 x\Phi_\lambda^2 dx$ resulting in

$$\int_0^1 xJ_o^2\left(\sqrt{\lambda}x\right)dx = \lim_{\mu\to\lambda}\frac{\sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)}{(\lambda - \mu)} \tag{1}$$

When $\mu \to \lambda$ the right side becomes indeterminate form $\frac{0}{0}$. Therefore L'hospital rule is used, which says that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Comparing the above to (1) shows that μ is now like x and λ is like a. Therefore $f'(x)$ is like

$$\begin{aligned}
f'(x) &\equiv \frac{d}{d\mu}\left(\sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)\right)\\
&\equiv \frac{d}{d\mu}\sqrt{\mu}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \frac{d}{d\mu}\sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\mu}\right)\\
&\equiv \frac{1}{2}\frac{1}{\sqrt{\mu}}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) + \sqrt{\mu}\frac{1}{2\sqrt{\mu}}J_o''\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \frac{1}{2\sqrt{\mu}}\sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o'\left(\sqrt{\mu}\right)
\end{aligned}$$

And $g'(x)$ is like $\frac{d}{d\mu}(\lambda - \mu) = -1$. Using the above result back in (1) gives

$$\begin{aligned}
\int_0^1 xJ_o^2\left(\sqrt{\lambda}\right)dx &\equiv \lim_{x\to a}\frac{f'(x)}{g'(x)}\\
&= \lim_{\mu\to\lambda}\left(-\frac{1}{2}\frac{1}{\sqrt{\mu}}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \sqrt{\mu}\frac{1}{2\sqrt{\mu}}J_o''\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) + \frac{1}{2\sqrt{\mu}}\sqrt{\lambda}J_o'\left(\sqrt{\lambda}\right)J_o'\left(\sqrt{\mu}\right)\right)\\
&= \lim_{\mu\to\lambda}\left(-\frac{1}{2}\frac{1}{\sqrt{\mu}}J_o'\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) - \frac{1}{2}J_o''\left(\sqrt{\mu}\right)J_o\left(\sqrt{\lambda}\right) + \frac{1}{2}J_o'\left(\sqrt{\lambda}\right)J_o'\left(\sqrt{\mu}\right)\right)
\end{aligned}$$

Now the limit is taken, since there is no indeterminate form. The above becomes

$$\begin{aligned}
\int_0^1 xJ_o^2\left(\sqrt{\lambda}x\right)dx &= -\frac{1}{2}\frac{1}{\sqrt{\lambda}}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right) - \frac{1}{2}J_o''\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right) + \frac{1}{2}J_o'\left(\sqrt{\lambda}\right)J_o'\left(\sqrt{\lambda}\right)\\
&= \frac{1}{2}\left(\left[J_o'\left(\sqrt{\lambda}\right)\right]^2 - \frac{1}{\sqrt{\lambda}}J_o'\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right) - J_o''\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right)\right)
\end{aligned} \tag{2}$$

To simplify the above, the following relations were obtained from dlmf.NIST.gov to simplify the above

$$\begin{aligned}
J_n'(x) &= J_{n-1}(x) - \frac{(n+1)}{x}J_n(x)\\
J_n'(x) &= -J_{n+1}(x) + \frac{n}{x}J_n(x)
\end{aligned}$$

Using these, then $J_o'\left(\sqrt{\lambda}\right) = -J_1\left(\sqrt{\lambda}\right)$ and $J_0''\left(\sqrt{\lambda}\right) = -J_0\left(\sqrt{\lambda}\right) + \frac{1}{\sqrt{\lambda}}J_1\left(\sqrt{\lambda}\right)$. Equation (2) now simplifies to

$$\begin{aligned}
\int_0^1 xJ_o^2\left(\sqrt{\lambda}x\right)dx &= \frac{1}{2}\left(\left[J_o'\left(\sqrt{\lambda}\right)\right]^2 - \frac{1}{\sqrt{\lambda}}\left(-J_1\left(\sqrt{\lambda}\right)\right)J_o\left(\sqrt{\lambda}\right) - \left(-J_0\left(\sqrt{\lambda}\right) + \frac{1}{\sqrt{\lambda}}J_1\left(\sqrt{\lambda}\right)\right)J_o\left(\sqrt{\lambda}\right)\right)\\
&= \frac{1}{2}\left(\left[J_o'\left(\sqrt{\lambda}\right)\right]^2 + \frac{1}{\sqrt{\lambda}}J_1\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right) + J_0\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right) - \frac{1}{\sqrt{\lambda}}J_1\left(\sqrt{\lambda}\right)J_o\left(\sqrt{\lambda}\right)\right)
\end{aligned}$$

The second term cancels with the last term above giving the final result

$$\int_0^1 xJ_o^2\left(\sqrt{\lambda}x\right)dx = \frac{1}{2}\left(\left[J_o'\left(\sqrt{\lambda}\right)\right]^2 + J_0^2\left(\sqrt{\lambda}\right)\right) \tag{3}$$

5 Part (b)

$\sqrt{\lambda_n}$ are the positive zeros of $J_0\left(\sqrt{\lambda_n}\right) = 0$. Below, $\sqrt{\lambda_n}$ is replaced by j_n where now j_n are the zeros of $J_0(j_n)$. One way to find the normalized eigenfunction $\hat{J}_0(j_n x)$ is by dividing $J_0(j_n x)$ by its norm. In other words,

$$\hat{J}_0(j_n x) = \frac{J_0(j_n x)}{\|J_0(j_n x)\|} \tag{1A}$$

But

$$\|J_0(j_n x)\| = \sqrt{\int_0^1 r(x) J_0^2(j_n x)\,dx}$$

Which is by the definition of the norm of a function with the corresponding weight $r(x)$. But from part(a) $\|J_0(j_n x)\| = \int_0^1 r(x) J_0^2(j_n x)\,dx$ was found to be $\frac{1}{2}\left(\left[J_o'(j_n)\right]^2 + J_0^2(j_n)\right)$. Therefore (1A) becomes

$$\begin{aligned}\hat{J}_0(j_n x) &= \frac{J_0(j_n x)}{\sqrt{\frac{1}{2}\left([J_o'(j_n)]^2 + J_0^2(j_n)\right)}}\\ &= \frac{\sqrt{2}J_0(j_n x)}{\sqrt{[J_o'(j_n)]^2 + J_0^2(j_n)}}\end{aligned}$$

But since j_n are the zeros of $J_0(j_n)$, then all the $J_0(j_n)$ terms above vanish giving

$$\begin{aligned}\hat{J}_0(j_n x) &= \frac{\sqrt{2}J_0(j_n x)}{\sqrt{[J_o'(j_n)]^2}}\\ &= \frac{\sqrt{2}J_0(j_n x)}{|J_o'(j_n)|}\end{aligned} \tag{1}$$

Another way to find the normalized eigenfunctions $\hat{J}_0(j_n x)$ is as was done in the text book, which is to first determine k_n as follows. Let $\hat{J}_0(j_n x) = k_n J_0(j_n x)$, then the following equation is solved for k_n

$$\int_0^1 r(x)\left[\hat{J}_0(j_n x)\right]^2 dx = 1 \tag{2}$$

But the weight $r(x) = x$, equation (2) becomes

$$k_n^2 \int_0^1 x J_0^2(j_n x)\,dx = 1$$

But from part(a), $\int_0^1 x J_0^2(j_n x)\,dx = \frac{1}{2}\left(\left[J_o'(j_n)\right]^2 + J_0^2(j_n)\right)$. Hence the above becomes

$$\begin{aligned}k_n^2 &= \frac{1}{\frac{1}{2}\left([J_o'(j_n)]^2 + J_0^2(j_n)\right)}\\ k_n &= \frac{\sqrt{2}}{\sqrt{[J_o'(j_n)]^2 + J_0^2(j_n)}}\end{aligned}$$

As above, since all $J_0(j_n) = 0$ then

$$k_n = \frac{\sqrt{2}}{\sqrt{[J_o'(j_n)]^2}}$$

And the normalized eigenfunction become

$$\begin{aligned}\hat{J}_0(j_n x) &= k_n J_0(j_n x)\\ &= \frac{\sqrt{2}J_0(j_n x)}{\sqrt{[J_o'(j_n)]^2}}\\ &= \frac{\sqrt{2}J_0(j_n x)}{|J_o'(j_n)|}\end{aligned}$$

Which is the same result as (1).

6 Problem 4

Problem Solve the inhomogeneous differential equation

$$-\left(\left(1-x^2\right)y'\right)' = y + x^3 \qquad -1 < x < 1$$

With boundary conditions $y(x), y'(x)$ bounded as $x \to -1^+$ and $x \to 1^-$.

Solution

This problem is solved using 11.3 method (Eigenfunction expansion). The ODE is written as

$$-\left(\left(1-x^2\right)y'\right)' = \mu y + x^3 \tag{1}$$

Where $\mu = 1$ in this case. The corresponding homogeneous eigenvalue ODE to solve is then

$$-\left(\left(1-x^2\right)y'\right)' = \lambda y \tag{2}$$

Comparing to Sturm-Liouville form $-(py')' + qy = r\lambda y$, then $p(x) = (1-x^2)$, $q = 0, r = 1$. Since $p(x)$ must be positive over all points in the domain, and since in this problem $p(-1) = 0$ and $p(1) = 0$, then both $x = -1, +1$ are singular points. They can be shown to be regular singular points.

Equation (2), where λ is now is an eigenvalue, is the Legendre equation

$$\left(1-x^2\right) y'' - 2xy' + \lambda y = 0$$

Comparing to the standard Legendre equation form in chapter 5

$$\left(1-x^2\right) y'' - 2xy' + n(n+1) y = 0 \tag{3}$$

There are two cases to consider. n is integer and n is not an integer.

Case n is not an integer. It is know that now the solution to (3) is

$$y(x) = c_1 \bar{P}_n(x) + c_2 \bar{Q}_n(x)$$

Where $\bar{P}_n(x)$ is called the Legendre function of order n and $\bar{Q}_n(x)$ is called the Legendre function of the second kind of order n. These solutions are valid for $|x| < 1$ since series expansion was about point $x = 0$. But both of these functions are unbounded at the end points ($\bar{Q}_n(x)$ blows up at $x = \pm 1$ and $\bar{P}_n(x)$ blows up at $x = -1$) leading to trivial solution.

This means n must be an integer. When n is an integer, then $\lambda_n = n(n+1)$. It is known (from chapter 5), that in this case the solution to (3) becomes a terminating power series (a polynomial), which is called the Legendre polynomial $P_n(x)$ .These polynomials are there bounded everywhere, including at the end points $x = \pm 1$, and therefore these solutions satisfy the boundary conditions. Hence the Legendre $P_n(x)$ are the eigenfunctions to (3). This table summaries the result found

n	eigenvalue	eigenfunctions
0	$\lambda_0 = 0$	$P_0(x) = 1$
1	$\lambda_1 = 2$	$P_1(x) = x$
2	$\lambda_2 = 6$	$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$
3	$\lambda_3 = 12$	$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$
$\vdots$	$\vdots$	$\vdots$
n	$\lambda_n = n(n+1)$	$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^2} \left(x^2 - 1\right)^n$

What the above says, is that the solution to

$$\left(1-x^2\right) P_n''(x) - 2xP_n'(x) + \lambda_n P_n(x) = 0$$

Is $P_n(x)$ with the corresponding eigenvalue $\lambda_n = n(n+1)$ as given by the above table. Now that the eigenfunctions of the corresponding homogeneous eigenvalue ODE are found, they are used to solve the given inhomogeneous ODE

$$-\left(\left(1-x^2\right) y'\right)' = \mu y + x^3 \tag{4}$$

Using eigenfunction expansion method. Since $\mu = 1$ and since there is no eigenvalue which is also 1, then a solution exists. Let the solution be

$$y(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

Substituting this solution into (4), and noting that $L[y] = -\left(\left(1-x^2\right) y'\right)' = \lambda_n y$ gives

$$\lambda_n \sum_{n=0}^{\infty} c_n P_n(x) = \mu \sum_{n=0}^{\infty} c_n P_n(x) + x^3$$

Expanding x^3 using the same eigenfunctions (this can be done, since x^3 is continuous function and the eigenfunctions are complete), then the above becomes

$$\begin{aligned}
\lambda_n \sum_{n=0}^{\infty} c_n P_n(x) &= \mu \sum_{n=0}^{\infty} c_n P_n(x) + \sum_{n=0}^{\infty} d_n P_n(x) \\
\lambda_n c_n &= \mu c_n + d_n \\
c_n &= \frac{d_n}{\lambda_n - \mu}
\end{aligned}$$

What is left is to determine d_n from

$$x^3 = \sum_{n=0}^{\infty} d_n P_n(x)$$

The above can be solved for d_n using orthogonality, or by direct expansion (otherwise called undetermined coefficients method). Since the force x^3 is already a polynomial in x and of a small order, then direct expansion is simpler. The above then becomes

$$x^3 = d_0 P_0(x) + d_1 P_1(x) + d_2 P_2(x) + d_3 P_3(x)$$

There is no need to expand for more than $n = 3$, since the LHS polynomial is of order 3. Substituting the known $P_n(x)$ expressions into the above equation gives

$$\begin{aligned} x^3 &= d_0 + d_1 x + d_2 \frac{1}{2}\left(3x^2 - 1\right) + d_3 \frac{1}{2}\left(5x^3 - 3x\right) \\ &= d_0 + d_1 x + d_2 \left(\frac{3}{2}x^2 - \frac{1}{2}\right) + d_3 \left(\frac{5}{2}x^3 - \frac{3}{2}x\right) \end{aligned}$$

Collecting terms of equal powers in x results in

$$x^3 = x^0 \left(d_0 - \frac{1}{2}d_2\right) + x\left(d_1 - \frac{3}{2}d_3\right) + x^2\left(\frac{3}{2}d_2\right) + x^3\left(\frac{5}{2}d_3\right)$$

Or

$$\begin{aligned} d_0 - \frac{1}{2}d_2 &= 0 \\ d_1 - \frac{3}{2}d_3 &= 0 \\ \frac{3}{2}d_2 &= 0 \\ \frac{5}{2}d_3 &= 1 \end{aligned}$$

From third equation, $d_2 = 0$. From first equation $d_0 = 0$, and substituting last equation in the second equation give $d_1 = \frac{3}{2}$. Therefore

$$\begin{aligned} d_1 &= \frac{3}{5} \\ d_3 &= \frac{2}{5} \end{aligned}$$

And all other d_n are zero. Now the c_n are found using $c_n = \frac{d_n}{\lambda_n - \mu}$. For $n = 1$

$$c_1 = \frac{d_1}{\lambda_1 - \mu} = \frac{\frac{3}{5}}{2 - 1} = \frac{3}{5}$$

And for $n = 3$

$$c_3 = \frac{d_3}{\lambda_3 - \mu} = \frac{\frac{2}{5}}{12 - 1} = \frac{2}{55}$$

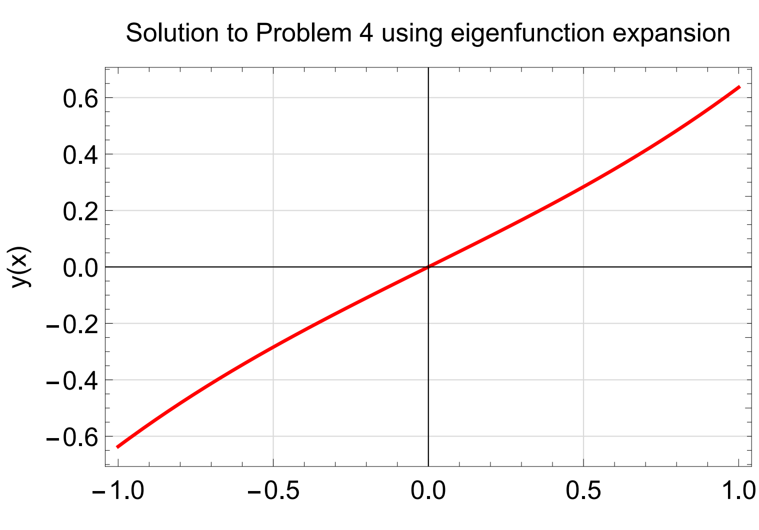
And all other c_n are zero. Hence the final solution from $y(x) = \sum_{n=0}^{\infty} c_n P_n(x)$ reduces to only two terms in the sum

$$\begin{aligned} y(x) &= c_1 P_1(x) + c_3 P_3(x) \\ &= \frac{3}{5}x + \frac{2}{55}\left(\frac{1}{2}\left(5x^3 - 3x\right)\right) \end{aligned}$$

Giving the final solution as

$$\boxed{y(x) = \frac{1}{11}x\left(x^2 + 6\right)}$$

This is a plot of the solution



7 Appendix for problem 4

Initially I did not know we had to use eigenfunction expansion, so solved it directly as follows. Let the solution to

$$\left(1-x^2\right)y''-2xy'+y=x^3$$

Be

$$y\left(x\right)=y_h\left(x\right)+y_p\left(x\right)$$

Where $y_h\left(x\right)$ is the homogeneous solution to $\left(1-x^2\right)y''-2xy'+y=0$ and $y_p\left(x\right)$ is a particular solution. Now, since $\left(1-x^2\right)y''-2xy'+y=0$ is a Legendre ODE but with a non-integer order, then its solution is not a terminating polynomials, but instead is given by

$$y_h\left(x\right)=c_1\bar{P}_n\left(x\right)+c_2\bar{Q}_n\left(x\right)$$

Where $\bar{P}_n\left(x\right)$ is called the Legendre function of order n and $\bar{Q}_n\left(x\right)$ is called the Legendre function of the second kind of order n, and $y_p\left(x\right)$ is a particular solution. The particular solution can be found, using method of undetermined coefficients to be $y_p\left(x\right)=\frac{1}{11}x^3+\frac{6}{11}x$. Hence the general solution becomes

$$y\left(x\right)=c_1\bar{P}_n\left(x\right)+c_2\bar{Q}_n\left(x\right)+\frac{1}{11}x\left(x^2+6\right)$$

Now since the solution must be bounded as $x\rightarrow\pm1$, then we must set $c_1=0$ and $c_2=0$, because both $\bar{P}_n\left(x\right)$ and $\bar{Q}_n\left(x\right)$ are <u>unbounded</u> at the end points ($\bar{Q}_n\left(x\right)$ blows up at $x=\pm1$ and $\bar{P}_n\left(x\right)$ blows up at only $x=-1$), therefore the final solution contains only the particular solution

$$y\left(x\right)=\frac{1}{11}x\left(x^2+6\right)$$

Which is the same solution found using eigenfunction expansion. At first I thought I made an error somewhere, since I did not think all of the homogenous solution basis could vanish leaving only a particular solution.