Quizz 2

Math 332
Introduction to Partial Differential Equations

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Contents

1 Problem 1

Problem Solve the wave equation $u_{tt} = u_{xx}$ for infinite domain $-\infty < x < \infty$ with initial position $u(x,0) = f(x) = \frac{1}{1+x^2}$ and zero initial velocity $g(x) = 0$. Plot the solution for $t = 0, 1, 2$ seconds.

solution

The solution for wave PDE $u_{tt} = a^2 u_{xx}$ on infinite domain can be written as a series solution or as general solution using D'Alembert form. Using D'Alembert, the solution is

$$u(x,t) = \frac{1}{2}\left(f(x-at) + f(x+at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$

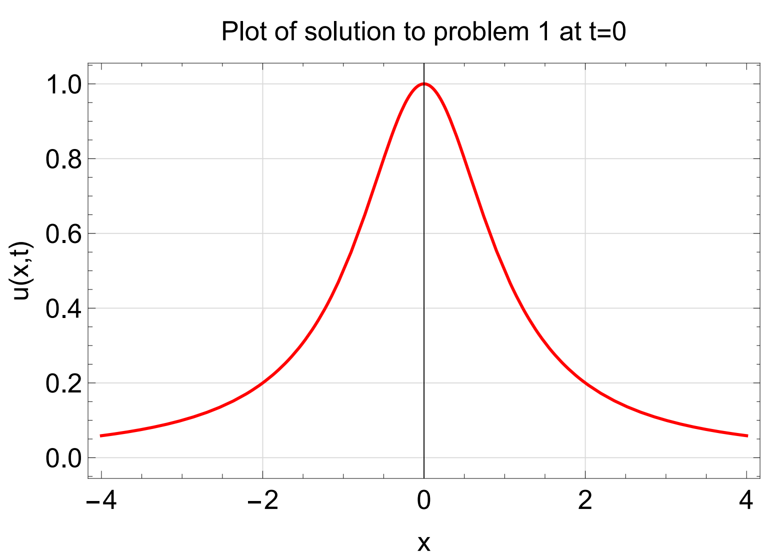
Where in this problem $a = 1$ and $g(x) = 0$. Therefore the above simplifies to

$$u(x,t) = \frac{1}{2}\left(f(x-t) + f(x+t)\right)$$

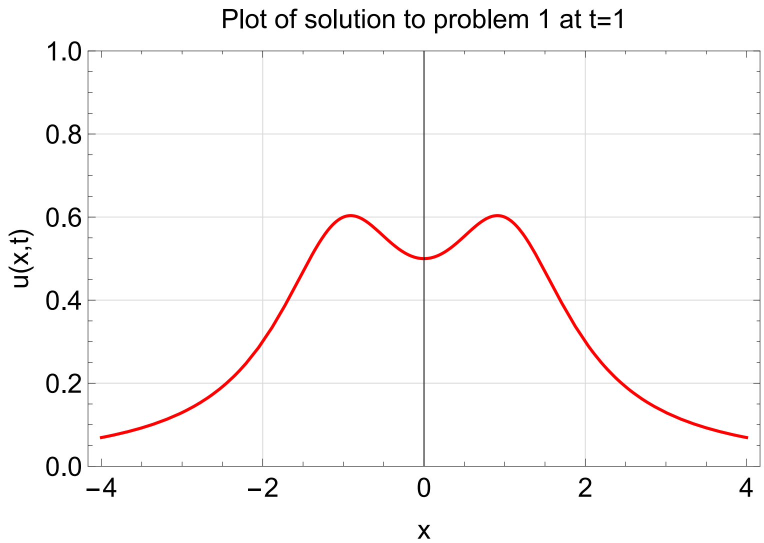
$f(x-t)$ is the initial position shifted to the right by t and $f(x+t)$ is the initial position shifted to the left by t. Since $f(x) = \frac{1}{1+x^2}$, the above solution becomes

$$u(x,t) = \frac{1}{2}\left(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2}\right)$$

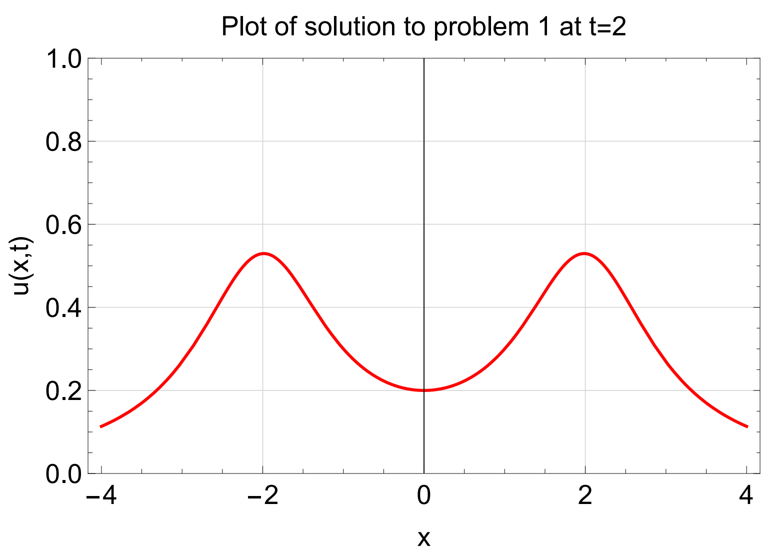
This is a plot of the solution at time $t = 0$ (which is just $\frac{1}{1+x^2}$)



This is a plot of the solution at time $t = 1$



This is a plot of the solution at time $t = 2$



The above shows that, eventually, the initial position splits into two halves, where one half moves to the right and one half moves to the left, but the sum (energy) of the parts remain equal to that at $t = 0$ since there is no damping. An Animation was also made of this solution for better illustration.

2 Problem 2

Problem Apply the method of separation of variables to the damped wave equation $u_{tt}+2u_t = u_{xx}$ on finite domain with fixed ends $u(0,t) = 0$ and $u(\pi,t) = 0$. Let initial position be $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = 0$. Determine the first term in the series solution.

solution

Let the solution be $u(x,t) = X(x)T(t)$. Substituting this back into the PDE gives

$$T''X + 2T'X = X''T$$

Dividing throughout by $XT \neq 0$ and simplifying gives

$$\frac{T''}{T} + 2\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

Hence the eigenvalue ODE is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X(\pi) &= 0 \end{aligned} \tag{1}$$

And the corresponding time ODE

$$T'' + 2T' + \lambda T = 0 \tag{2}$$

The eigenvalue ODE for the homogeneous boundary condition was solved before. The eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Since $L = \pi$, the above becomes

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots \tag{3}$$

The corresponding eigenfunctions are

$$X_n(x) = c_n \sin(nx)$$

Now that the eigenvalues are found, the time ODE (2) is solved.

$$T_n'' + 2T_n' + n^2 T_n = 0$$

This is constant coefficient ODE. The characteristic equation is

$$r^2 + 2r + n^2 = 0$$

The roots are

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{4 - 4n^2}}{2} \\ &= -1 \pm \sqrt{1 - n^2} \end{aligned}$$

For $n = 1$ the root becomes $r = -1$ (double root), hence the solution is

$$T_1(t) = A_1 e^{-t} + B_1 t e^{-t} \tag{4}$$

And for the remaining value $n = 2, 3, \cdots$, the term $\sqrt{1-n^2}$ becomes complex. Therefore the roots can now be written as $r = -1 \pm i\sqrt{n^2-1}$. This implies that the solution can be expressed using trigonometric functions as

$$T_n(t) = e^{-t}\left(A_n \cos\left(t\sqrt{n^2-1}\right) + B_n \sin\left(t\sqrt{n^2-1}\right)\right) \qquad n = 2, 3, \cdots \tag{5}$$

Since initial velocity is zero at $t = 0$, then (4) leads to $T_1' = -A_1 e^{-t} + B_1 e^{-t} - tB_1 e^{-t}$. At $t = 0$ this gives $0 = -A_1 + B_1$. Therefore solution (4) becomes

$$T_1(t) = A_1\left(e^{-t} + te^{-t}\right) \tag{4A}$$

Taking time derivative for (5) gives

$$\begin{aligned} T_n'(t) = -e^{-t}\left(A_n \cos\left(\sqrt{n^2-1}t\right) + B_n \sin\left(\sqrt{n^2-1}t\right)\right) + \\ e^{-t}\left(-A_n\sqrt{n^2-1}\sin\left(\sqrt{n^2-1}t\right) + B_n\sqrt{n^2-1}\cos\left(\sqrt{n^2-1}t\right)\right) \end{aligned}$$

At $t = 0$ the above becomes

$$0 = -A_n + B_n\sqrt{n^2-1}$$

Hence $A_n = B_n\sqrt{n^2-1}$ and (5) reduces to

$$T_n(t) = B_n e^{-t}\left(\sqrt{n^2-1}\cos\left(t\sqrt{n^2-1}\right) + \sin\left(t\sqrt{n^2-1}\right)\right) \qquad n = 2, 3, \cdots \tag{5A}$$

Therefore the fundamental solution is

$$\begin{aligned} u_n(x,t) &= T_n(t) X_n(x) \\ u(x,t) &= \sum_{n=1}^{\infty} T_n(t) X_n(x) \\ &= T_1(t) X_1(x) + \sum_{n=2}^{\infty} T_n(t) X_n(x) \\ &= c_1\left(\left(e^{-t} + te^{-t}\right)\sin x\right) + \sum_{n=2}^{\infty} c_n e^{-t}\left(\sqrt{n^2-1}\cos\left(t\sqrt{n^2-1}\right) + \sin\left(t\sqrt{n^2-1}\right)\right)\sin(nx) \end{aligned} \tag{6}$$

Where the constant A_1 was combined into c_1 and B_n combined into c_n. The constants c_1 and c_n are now found from initial position. At $t = 0$ (6) becomes

$$f(x) = c_1 \sin x + \sum_{n=2}^{\infty} c_n\sqrt{n^2-1}\sin(nx)$$

Multiplying both sides by $\sin(mx)$ and Integrating gives

$$\int_0^{\pi} f(x)\sin(mx)\,dx = \int_0^{\pi} c_1 \sin x \sin(mx)\,dx + \sum_{n=2}^{\infty} c_n\sqrt{n^2-1}\left(\int_0^{\pi}\sin(nx)\sin(mx)\,dx\right) \tag{7}$$

For $m = 1$ the above reduces to

$$\begin{aligned} \int_0^{\pi} f(x)\sin x dx &= \int_0^{\pi} c_1 \sin^2 x dx \\ \int_0^{\pi} f(x)\sin x dx &= \frac{\pi}{2}c_1 \\ c_1 &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin x dx \end{aligned}$$

And for $m = 2, 3, \cdots$ (7) becomes

$$\int_0^\pi f(x)\sin(mx)\,dx = \overbrace{\int_0^\pi c_1 \sin x \sin(mx)\,dx}^{0} + c_m\sqrt{m^2-1}\left(\int_0^\pi \sin^2(mx)\,dx\right)$$
$$= c_m\sqrt{m^2-1}\left(\int_0^\pi \sin^2(mx)\,dx\right)$$

Hence for $n = 2, 3, \cdots$ the above gives

$$\int_0^\pi f(x)\sin(nx)\,dx = c_n\sqrt{n^2-1}\left(\frac{\pi}{2}\right)$$

Therefore

$$c_n = \frac{2}{\pi\sqrt{n^2-1}}\int_0^\pi f(x)\sin nx dx \qquad n = 2, 3, \cdots$$

This completes the solution. The final solution from (6) becomes

$$u(x,t) = \left(\frac{2}{\pi}\int_0^\pi f(x)\sin x dx\right)\left(e^{-t}+te^{-t}\right)\sin(x)$$
$$+\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\int_0^\pi f(x)\sin(nx)\,dx}{\sqrt{n^2-1}}e^{-t}\left(\sqrt{n^2-1}\cos\left(t\sqrt{n^2-1}\right)+\sin\left(t\sqrt{n^2-1}\right)\right)\sin(nx)$$

To test the solution, it is compared to numerical differential equation solution. Using $f(x) = x(\pi - x)$ as an example. The result showed an exact match. An animation was also made. Therefore the first term is

$$\left(\frac{2}{\pi}\int_0^\pi f(x)\sin x dx\right)\left(e^{-t}+te^{-t}\right)\sin(x)$$

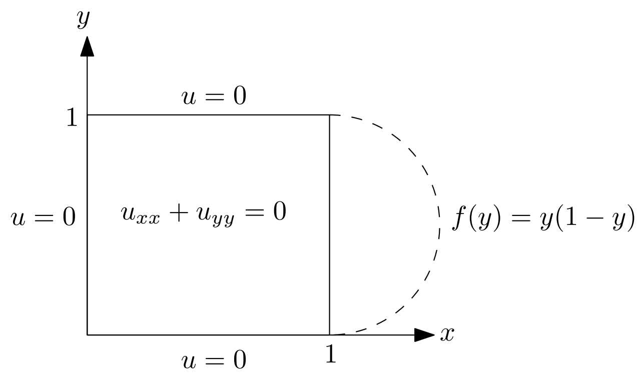
And for $n = 2, 3, \cdots$ the n^{th} term is

$$\left(\frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx\right)\frac{e^{-t}\left(\sqrt{n^2-1}\cos\left(t\sqrt{n^2-1}\right)+\sin\left(t\sqrt{n^2-1}\right)\right)\sin(nx)}{\sqrt{n^2-1}}$$

3 Problem 3

Problem Solve $u_{xx} + u_{yy} = 0$ on the square $0 \leq x, y \leq 1$. If $u(0, y) = u(x, 0) = u(x, 1) = 0$ and $u(1, y) = y - y^2$. Find an approximate value for $u\left(\frac{1}{2}, \frac{1}{2}\right)$

solution To make the solution steps more useful and general, a is used for the length of the x dimension and b for the length of the y dimension, then these are replaced by 1 at the very end. This is a plot of boundary conditions



Let $u(x, y) = X(x) Y(x)$. Substituting this into the PDE gives

$$X''Y + Y''X = 0$$

Dividing throughout by $XY \neq 0$ and simplifying gives

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

This gives the eigenvalue ODE

$$\begin{aligned} Y'' + \lambda Y &= 0 \\ Y(0) &= 0 \\ Y(b) &= 0 \end{aligned} \tag{1}$$

The solution to (1) gives the eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, 3 \cdots$ and since $L = b$, this becomes

$$\lambda_n = \left(\frac{n\pi}{b}\right)^2 \qquad n = 1, 2, \cdots$$

And the corresponding eigenfunction

$$\begin{aligned} Y_n(y) &= c_n \sin\left(\sqrt{\lambda_n} y\right) \\ &= c_n \sin\left(\frac{n\pi}{b} y\right) \end{aligned}$$

Therefore the corresponding nonhomogeneous $X(x)$ ODE

$$\begin{aligned} X_n'' - \lambda_n X_n &= 0 \\ X_n(0) &= 0 \\ X_n(a) &= y - y^2 \end{aligned} \tag{2}$$

The solution to (2), since λ_n is positive is

$$\begin{aligned} X_n(x) &= A_n \cosh\left(\sqrt{\lambda_n} x\right) + B_n \sinh\left(\sqrt{\lambda_n} x\right) \\ &= A_n \cosh\left(\frac{n\pi}{b} x\right) + B_n \sinh\left(\frac{n\pi}{b} x\right) \end{aligned}$$

Boundary conditions $X(0) = 0$ gives

$$0 = A_n$$

The solution (3) now simplifies to

$$X_n(x) = B_n \sinh\left(\frac{n\pi}{b} x\right)$$

Hence the fundamental solution is

$$\begin{aligned} u_n(x, y) &= X_n Y_n \\ &= c_n \sinh\left(\frac{n\pi}{b} x\right) \sin\left(\frac{n\pi}{b} y\right) \end{aligned}$$

Where the constants B_n is merged with c_n. The solution is

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b} x\right) \sin\left(\frac{n\pi}{b} y\right) \tag{3}$$

c_n is now found by applying the boundary condition at $x = a$. The above becomes

$$y - y^2 = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b} a\right) \sin\left(\frac{n\pi}{b} y\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{b} y\right)$ and integrating gives

$$\int_0^b \left(y - y^2\right) \sin\left(\frac{m\pi}{b} y\right) dy = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b} a\right) \left(\int_0^b \sin\left(\frac{m\pi}{b} y\right) \sin\left(\frac{n\pi}{b} y\right) dy\right)$$

By orthogonality the above reduces to

$$\begin{aligned} \int_0^b \left(y - y^2\right) \sin\left(\frac{m\pi}{b} y\right) dy &= c_n \sinh\left(\frac{m\pi}{b} a\right) \int_0^b \sin^2\left(\frac{m\pi}{b} y\right) dy \\ &= \frac{b}{2} c_m \sinh\left(\frac{m\pi}{b} a\right) \end{aligned}$$

Therefore

$$c_n = \frac{2}{b} \frac{1}{\sinh\left(\frac{m\pi}{b} a\right)} \int_0^b \left(y - y^2\right) \sin\left(\frac{n\pi}{b} y\right) dy$$

Now replacing $\underline{a = 1, b = 1}$, the above becomes

$$\begin{aligned} c_n &= \frac{2}{\sinh(n\pi)} \int_0^1 \left(y - y^2\right) \sin(n\pi y)\, dy \\ &= \frac{2}{\sinh(n\pi)} \left(\frac{-2\left(-1 + (-1)^n\right)}{n^3 \pi^3}\right) \\ &= \frac{-4}{\sinh(n\pi)} \frac{\left(-1 + (-1)^n\right)}{n^3 \pi^3} \end{aligned}$$

Hence the solution (3) becomes

$$u(x,y) = \frac{-4}{\pi^3}\sum_{n=1}^{\infty}\frac{\left(-1+(-1)^n\right)}{n^3}\frac{\sinh(n\pi x)}{\sinh(n\pi)}\sin(n\pi y)$$

At $x=\frac{1}{2}, y=\frac{1}{2}$ the above becomes

$$u\left(\frac{1}{2},\frac{1}{2}\right) = \frac{-4}{\pi^3}\sum_{n=1}^{\infty}\frac{\left(-1+(-1)^n\right)}{n^3}\frac{\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)}\sin\left(\frac{n\pi}{2}\right)$$

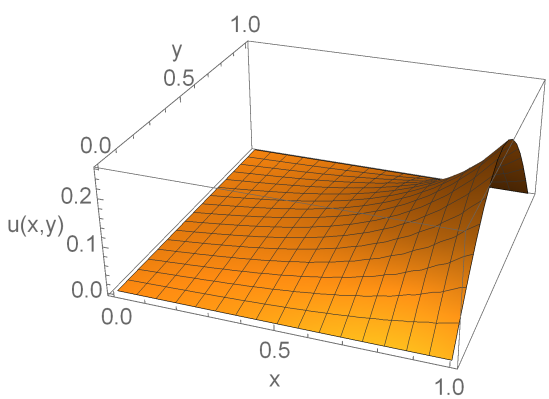
For $n=1$, the above gives 0.0514136952911346 and for $n=2$ the value do not change beyond 16 decimal points. So only need to use one term to get very good approximation value as

$$u\left(\frac{1}{2},\frac{1}{2}\right) = 0.0514136952911346$$

This value is between zero and 0.25, where 0.25 is the maximum value at the boundary and zero is the minimum value at the boundary. This agrees with the min-max principle. This is a 3D plot of the solution over the whole square.

```
In[40]:= mySol[x_, y_] := -4 / Pi^3 Sum[ (-1 + (-1)^n) / n^3 (Sinh[n Pi x] / Sinh[n Pi]) Sin[n Pi y] , {n, 1, 2}]
         Plot3D[mySol[x, y], {x, 0, 1}, {y, 0, 1}, AxesLabel → {"x", "y", "u(x,y)"}, BaseStyle → 14]
```

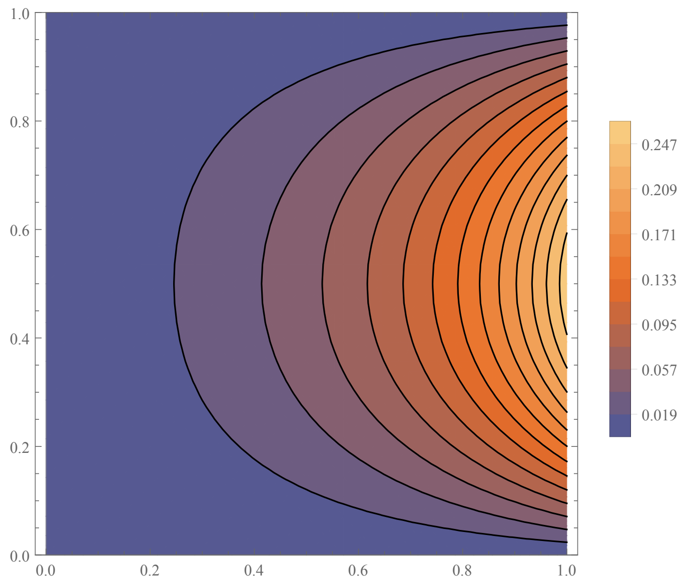
Out[41]=



This is a contour plot

```
ContourPlot[Evaluate[mySol[x, y]], {x, 0, 1}, {y, 0, 1}, AxesLabel → {x, y},
  PlotRange → {-1, 1}, Contours → 100, PlotTheme -> "Scientific", PlotLegends → Automatic]
```

Out[16]=



4 Problem 4

Problem Solve $u_{xx} + u_{yy} = 0$ on disk $x^2 + y^2 < 1$ with boundary condition xy^2 when $x^2 + y^2 = a$. Where $a = 1$ in this problem. Express solution in x, y

solution The first step is to convert the boundary condition to polar coordinates. Since $x = r\cos\theta, y = r\sin\theta$, then at the boundary $u(r,\theta) = r\cos\theta\,(r\sin\theta)^2$. But $r = 1$ (the radius). Hence at the boundary, $u(1,\theta) = f(\theta)$ where

$$
\begin{aligned}
f(\theta) &= \cos\theta\sin^2\theta \\
&= \cos\theta\left(1-\cos^2\theta\right) \\
&= \cos\theta - \cos^3\theta
\end{aligned}
$$

But $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$. Therefore the above becomes

$$
\begin{aligned}
f(\theta) &= \cos\theta - \left(\frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta\right) \\
&= \frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta \qquad (1)
\end{aligned}
$$

The above is also seen as the Fourier series of $f(\theta)$. The PDE in polar coordinates is

$$
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0
$$

The solution is known to be

$$
u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n\left(c_n\cos(n\theta) + k_n\sin(n\theta)\right) \qquad (2)
$$

Since the above solution is the same as $f(\theta)$ when $r = 1$, then equating (2) when $r = 1$ to (1) gives

$$
\frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta = \frac{c_0}{2} + \sum_{n=1}^{\infty}\left(c_n\cos(n\theta) + k_n\sin(n\theta)\right)
$$

By comparing terms on both sides, this shows by inspection that

$$
\begin{aligned}
c_0 &= 0 \\
c_1 &= \frac{1}{4} \\
c_3 &= \frac{-1}{4}
\end{aligned}
$$

And all other c_n, k_n are zero. Using the above result back in (2) gives the solution as

$$
\boxed{u(r,\theta) = \tfrac{r}{4}\cos\theta - \tfrac{r^3}{4}\cos 3\theta} \qquad (3)
$$

This solution is now converted to xy using the formula

$$
\begin{aligned}
r^n\cos n\theta &= \sum_{\substack{k=0\\ even}}^{n} \binom{n}{k} x^{n-k}(-1)^{\frac{k}{2}} y^k \\
&= \sum_{\substack{k=0\\ even}}^{n} \frac{n!}{k!\,(n-k)!} x^{n-k}(-1)^{\frac{k}{2}} y^k
\end{aligned}
$$

For $n=1$ the above gives

$$\begin{aligned} r\cos\theta &= \frac{1!}{0!\,(1-0)!}x^{1-0}\,(-1)^0\,y^0 \\ &= x \end{aligned} \tag{4}$$

And for $n=3$

$$\begin{aligned} r^3\cos 3\theta &= \frac{3!}{0!\,(3-0)!}x^{3-0}\,(-1)^0\,y^0 + \frac{3!}{2!\,(3-2)!}x^{3-2}\,(-1)^1\,y^2 \\ &= x^3 - 3xy^2 \end{aligned} \tag{5}$$

Using (4,5) in (3) gives the solution in x, y

$$\boxed{u(x,y) = \tfrac{1}{4}x - \tfrac{1}{4}\left(x^3 - 3xy^2\right)} \tag{6}$$

This is now verified that is satisfies the PDE $u_{xx} + u_{yy} = 0$.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{4} - \frac{1}{4}\left(3x^2 - 3y^2\right) \\ \frac{\partial^2 u}{\partial x^2} &= -\frac{6}{4}x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{6}{4}xy \\ \frac{\partial^2 u}{\partial y^2} &= \frac{6}{4}x \end{aligned}$$

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Now the boundary conditions $u(x,y) = xy^2$ are also verified. This condition applies when $x^2 + y^2 = 1$ or $y^2 = 1 - x^2$. Substituting this into (6) gives

$$u(x,y)_{@D} = \frac{1}{4}x - \frac{1}{4}\left(x^3 - 3x\overbrace{\left(1-x^2\right)}^{y2}\right)$$

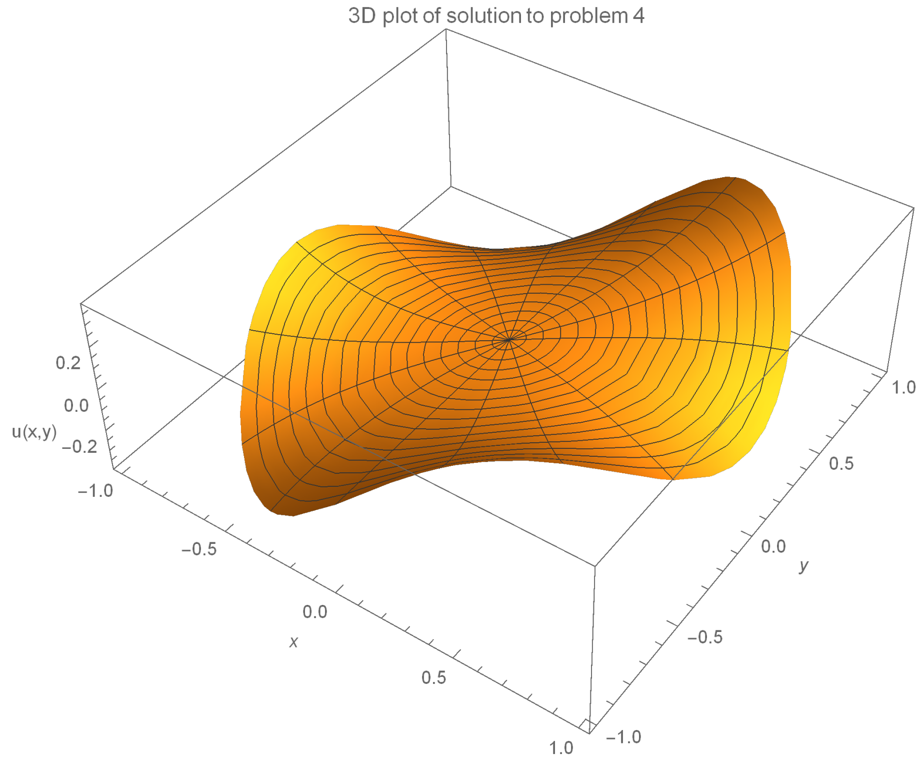
Simplifying gives

$$\begin{aligned} u(x,y)_{@D} &= \frac{1}{4}x - \frac{1}{4}\left(x^3 - \left(3x - 3x^3\right)\right) \\ &= \frac{1}{4}x - \frac{1}{4}x^3 + \frac{1}{4}\left(3x - 3x^3\right) \\ &= \frac{1}{4}x - \frac{1}{4}x^3 + \frac{3}{4}x - \frac{3}{4}x^3 \\ &= x - x^3 \\ &= x\left(1 - x^2\right) \\ &= xy^2 \end{aligned}$$

Verified. This is 3D plot of the solution

In[76]:=
```
ParametricPlot3D[{r Cos[t], r Sin[t], r / 4 Cos[t] - r^3 / 4 Cos[3 t]},
  {r, 0, 1}, {t, 0, 2 Pi}, AxesLabel → {x, y, "u(x,y)"},
  PlotLabel → "3D plot of solution to problem 4", ImageSize → 500]
```

Out[76]=



This is a contour plot

In[96]:=
```
ContourPlot[1 / 4 x - 1 / 4 (x^3 - 3 x y^2), {x, -1, 1}, {y, -1, 1}, AxesLabel → {x, y},
  Contours → 50, PlotLegends → Automatic, ColorFunction → "Pastel",
  Epilog → {Thick, Circle[]},
  PlotRange → {-1, 1},
  RegionFunction → Function[{x, y, z}, Norm[{x, y}] < 1.]]
```

Out[96]=