Homework 1, Math 322

1. Solve the boundary value problem

$$y'' - y = x, \quad y(0) = 0,\, y(1) = 1.$$

Solution: The general solution is

$$y = c_1 \cosh x + c_2 \sinh x - x.$$

The boundary condition give $c_1 = 0$, $c_2 \sinh 1 = 2$. The solution of the BVP is

$$y = \frac{2}{\sinh 1} \sinh x - x.$$

2. Find the Fourier sine series for the function $f(x) = x(1-x)$, $0 \leq x \leq 1$. Use the result to evaluate the infinite series

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \pm \dots.$$

Solution: The Fourier coefficients are

$$\begin{aligned}
c_n &= 2 \int_0^1 x(1-x) \sin n\pi x \, dx \\
&= -\frac{2}{n\pi} \, x(1-x) \cos n\pi x|_{x=0}^{x=1} + \frac{2}{n\pi} \int_0^1 (1-2x) \cos n\pi x \, dx \\
&= \frac{2}{n\pi} \int_0^1 (1-2x) \cos n\pi x \, dx \\
&= \frac{2}{n^2\pi^2} \, (1-2x) \sin n\pi x \, dx|_{x=0}^{x=1} + \frac{2}{n^2\pi^2} \int_0^1 (-2) \sin n\pi x \, dx \\
&= -\frac{4}{n^2\pi^2} \int_0^1 \sin n\pi x \, dx \\
&= \begin{cases} \frac{8}{n^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

By the convergence theorem, we have, for all $0 \leq x \leq 1$,

$$x(1-x) = \frac{8}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \sin(2k-1)\pi x.$$

If we choose $x = \frac{1}{2}$, we get

$$\frac{1}{4} = \frac{8}{\pi^3} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3},$$

so

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}.$$

3. Find the solution to the heat equation $u_t = u_{xx}$ with initial condition $u(x,0) = f(x)$ with $f(x)$ as in problem 2, and boundary conditions $u(0,t) = u(1,t) = 0$. Approximate $u(\frac{1}{2}, 1)$ to 10 decimal places.
Solution: The solution is

$$u(x,t) = \frac{8}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} e^{-(2k-1)^2\pi^2 t} \sin((2k-1)\pi x).$$

Then

$$u(\tfrac{1}{2}, 1) = \frac{8}{\pi^3} \sum_{k=1}^{\infty} e^{-(2k-1)^3\pi^2} \frac{(-1)^{k+1}}{(2k-1)^3}.$$

This is an alternating series $s = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$ with $a_k \geq 0$, $a_k \geq a_{k+1}$, $a_k \to 0$. Then $\left| s - \sum_{k=1}^{K} (-1)^{k+1} a_k \right| \leq a_{K+1}$. Therefore,

$$\left| u(\tfrac{1}{2}, 1) - \frac{8}{\pi^3} \sum_{k=1}^{K} e^{-(2k-1)^3\pi^2} \frac{(-1)^{k+1}}{(2k-1)^3} \right| < \frac{8}{\pi^3} e^{-(2K+1)^3\pi^2} \frac{1}{(2K+1)^3}.$$

When we choose $K = 1$, the error is less than 10^{-40}. Therefore, we obtain

$$u(\tfrac{1}{2}, 1) \approx \frac{8}{\pi^3} e^{-\pi^2} = 0.00001334521692\ldots$$

with an error less than 10^{-40}.

4. Solve the partial differential equation $u_t + u = u_{xx}$ with initial condition $u(x,0) = f(x)$ and boundary conditions $u(0,t) = u(L,t) = 0$ using Fourier series.
Solution: Using the method of separation of variables $u(x,t) = X(x)T(t)$ we find

$$\frac{T'(t)}{T(t)} + 1 = \frac{X''(x)}{X(x)} = -\lambda.$$

Therefore, we obtain

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0,$$

and

$$T' + (\lambda + 1)T(t) = 0.$$

This gives

$$u_n(x,t) = e^{-(n^2\pi^2/L^2+1)t} \sin(n\pi x/L).$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n^2\pi^2/L^2+1)t} \sin(n\pi x/L),$$

where

$$c_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L)\, dx.$$