THE PRÜFER ANGLE

We consider a regular Sturm-Liouville eigenvalue problem

$$-(p(x)y')' + q(x)y = \lambda\, r(x)y, \quad x \in [a,b] \tag{1}$$

with boundary conditions of the form

$$\cos\alpha\, y(a) = \sin\alpha\, p(a)y'(a), \tag{2}$$

$$\cos\beta\, y(b) = \sin\beta\, p(b)y'(b), \tag{3}$$

where

$$0 \le \alpha < \pi, \quad 0 < \beta \le \pi.$$

Let $y(x)$ be a nontrivial solution of (1). Then we set

$$\xi(x) = p(x)y'(x) = \rho(x)\cos\phi(x), \quad \eta(x) = y(x) = \rho(x)\sin\phi(x).$$

Then

$$\rho(x) = \sqrt{\xi(x)^2 + \eta(x)^2}, \quad \phi(x) = \arctan\frac{\eta(x)}{\xi(x)} = \operatorname{arccot}\frac{\xi(x)}{\eta(x)}.$$

ϕ is called Prüfer angle, and ρ is called Prüfer radius. In order to determine $\phi(x)$ we first choose $\phi(a)$, for example, $-\pi < \phi(a) \le \pi$. Then we use the arctan-formula if $\xi \ne 0$ and the arccot-formula if $\eta \ne 0$. We have to choose the proper branch of the multi-valued arctan, arccot, so that $\phi(x)$ becomes a continuous function (and then also continuously differentiable.)

From the equations

$$\xi' = \rho'\cos\phi - \rho\phi'\sin\phi, \quad \eta' = \rho'\sin\phi + \rho\phi'\cos\phi,$$

we obtain

$$\eta'\cos\phi - \xi'\sin\phi = \rho\phi'.$$

Since $\xi' = (pu')' = (q-\lambda r)\rho\sin\phi$, $\eta' = \frac{\xi}{p} = \frac{\rho}{p}\cos\phi$, it follows that

$$\phi' = \frac{1}{p}\cos^2\phi + (\lambda r - q)\sin^2\phi. \tag{4}$$

A similar calculation shows that

$$\rho' = (\tfrac{1}{p} + q - \lambda r)\rho\cos\phi\sin\phi.$$

It is important to note that (4) is a first order differential equation for the Prüfer angle. In order to satisfy the first boundary condition (2), we choose $\phi(a) = \alpha$. Then $\phi(x,\lambda)$ is uniquely determined by (2). The second boundary condition (3) is satisfied if

$$\phi(b,\lambda) = \beta + n\pi,$$

where n is an integer. One can show that $\lim_{\lambda\to-\infty}\phi(b,\lambda) = 0$, $\lim_{\lambda\to\infty}\phi(b,\lambda) = \infty$ and $\phi(b,\lambda)$ is an increasing function of λ. Therefore, for every $n =$

$0, 1, 2, \dots$, there is a unique solution $\lambda = \lambda_n$ of $\phi(b, \lambda) = \beta + n$ and the sequence $\{\lambda_n\}_{n=0}^{\infty}$ represents all the eigenvalues of the regular Sturm-Liouville problem.

Example: Consider

$$-((1+x)y'(x))' + xy = \lambda(1+x^2)y, \quad y(0) = 0,\ y'(1) = 0.$$

Then $p(x) = 1 + x$, $q(x) = x$, $r(x) = 1 + x^2$, $\alpha = 0$, $\beta = \pi/2$.

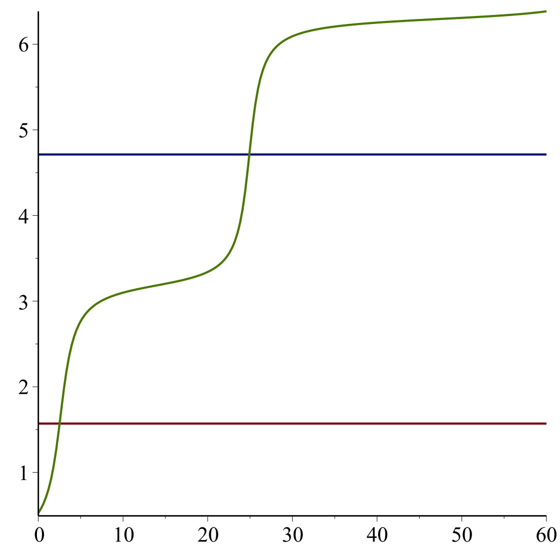

FIGURE 1. Prüfer angle $\phi(1, \lambda)$

The smallest two eigenvalues are

$$\lambda_0 = 2.51173, \quad \lambda_1 = 24.9158$$

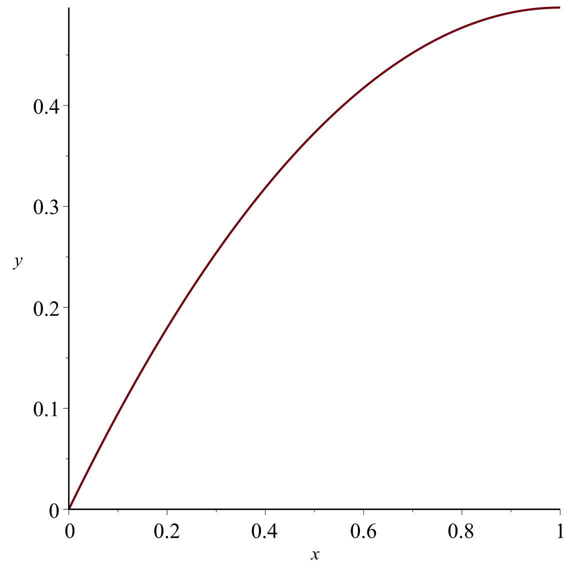

FIGURE 2. Eigenfunction for $\lambda = \lambda_0$

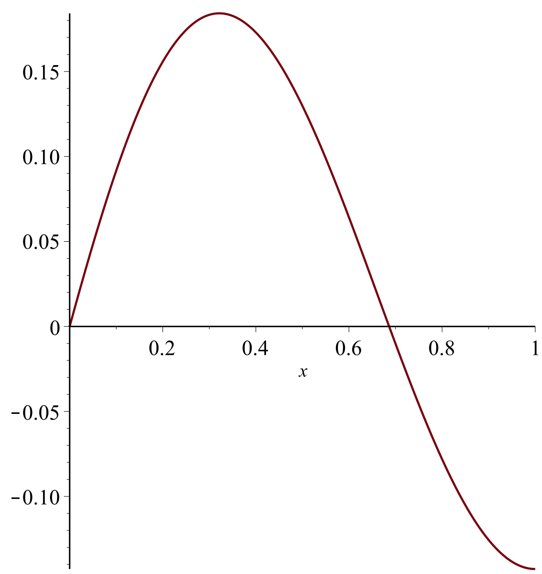

FIGURE 3. Eigenfunction for $\lambda = \lambda_1$