THE DIRICHLET PROBLEM ON AN ELLIPSE

We want to solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0 \text{ for } (x, y) \text{ in } D,$$
$$u(x, y) = f(x, y) \text{ for } (x, y) \text{ on the boundary of } D$$

when D is the region inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We assume that $a > b > 0$. The focal points of the ellipse are $(\pm c, 0)$. We introduce elliptic coordinates

$$x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta.$$

Usually $\xi > 0$ and $0 \le \eta < 2\pi$ or $-\pi < \eta \le \pi$.

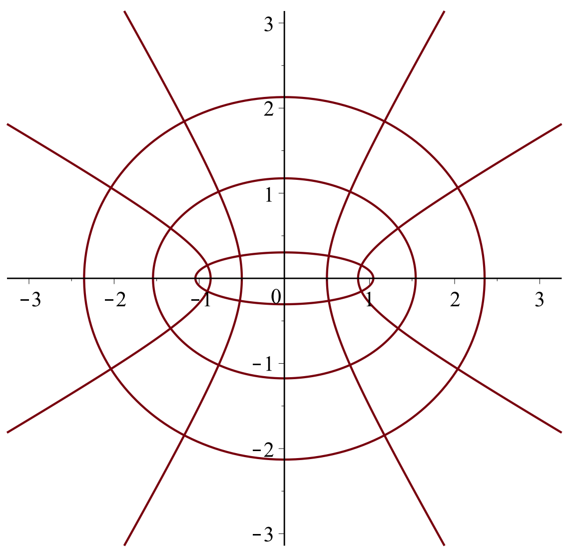

Figure 1. Elliptic Coordinates

We set $v(\xi, \eta) = u(c \cosh \xi \cos \eta, c \sinh \xi \sin \eta)$. Then, by the chain rule,

$$\begin{aligned} v_{\xi\xi} &= (c \sinh \xi \cos \eta)^2 u_{xx} + 2c^2 \cosh \xi \sinh \xi \cos \eta \sin \eta \, u_{xy} \\ &\quad + (c \cosh \xi \sin \eta)^2 u_{yy} + c \cosh \xi \cos \eta \, u_x + c \sinh \xi \sin \eta \, u_y, \\ v_{\eta\eta} &= (c \cosh \xi \sin \eta)^2 u_{xx} - 2c^2 \cosh \xi \sinh \xi \cos \eta \sin \eta \, u_{xy} \\ &\quad + (c \sinh \xi \cos \eta)^2 u_{yy} - c \cosh \xi \cos \eta \, u_x - c \sinh \xi \sin \eta \, u_y. \end{aligned}$$

so

$$v_{\xi\xi} + v_{\eta\eta} = c^2(\cosh^2\xi - \cos^2\eta)(u_{xx} + u_{yy}).$$

Therefore, the equation $u_{xx} + u_{yy} = 0$ is equivalent to $v_{\xi\xi} + v_{\eta\eta} = 0$. We use separation of variables

$$v(\xi, \eta) = \Xi(\xi)E(\eta).$$

Then we find

$$\Xi'' - \lambda\Xi = 0, \quad E'' + \lambda E = 0.$$

The equation for E has to have nontrivial 2π periodic solutions. Therefore, $\lambda = n^2$, $n = 0, 1, 2, \ldots$ and

$$E_n(\eta) = c_n \cos(n\eta) + d_n \sin(n\eta).$$

The general solution of the differential equation for Ξ with $\lambda = n^2$ is

$$\Xi(\xi) = a_n \cosh(n\xi) + d_n \sinh(n\xi).$$

If we consider the function

$$v(\xi, \eta) = \cosh n\xi \sin n\eta,$$

then we notice that $v(\xi, -\eta) = -v(\xi, \eta)$ so $u(x, -y) = -u(x, y)$. But then $u(x, 0)$ should be zero on the focal line which is not true. Therefore, $u(x, y)$ is discontinuous at the focal line $[-c, c]$. Similarly, the function $v(\xi, \eta) = \sinh n\xi \cos n\eta$ has a discontinuous derivative u_y. Therefore, we consider only

$$v_n(\xi, \eta) = c_n \cosh n\xi \cos n\eta + d_n \sinh n\xi \sin n\eta. \tag{1}$$

In fact, we show below that the corresponding function $u_n(x, y)$ is a polynomial in x, y. Therefore, by superposition, we find the solution

$$v(\xi, \eta) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cosh n\xi \cos n\eta + d_n \sinh n\xi \sin n\eta). \tag{2}$$

The boundary of D is given by $\xi = \xi_0$, where $\xi_0 > 0$ is determined from $c \cosh \xi_0 = a$. Therefore, in order to satisfy the boundary condition

$$F(\eta) := f(c \cosh \xi_0 \cos \eta, c \sinh \xi_0 \sin \eta) = v(\xi_0, \eta)$$

we set

$$c_n \cosh n\xi_0 = \frac{1}{\pi}\int_0^{2\pi} F(\eta) \cos n\eta \, d\eta, \quad n \geq 0$$

and

$$d_n \sinh n\xi_0 = \frac{1}{\pi}\int_0^{2\pi} F(\eta) \sin n\eta \, d\eta, \quad n \geq 1.$$

Substituting these values of c_n, d_n in (2) we find the solution of the Dirichlet problem for the ellipse. We see that the series in (2) converges very well for $\xi < \xi_0$. The quality of convergence on the boundary ellipse $\xi = \xi_0$ is the same as that of the Fourier series for $F(\eta)$.

The function v_n defined in (1) ic called an ellipsoidal harmonic of degree n. These functions are polynomials in x, y as we show below. We use the Chebyshev polynomials T_n defined by $\cos n\theta = T_n(\cos\theta)$. They also satisfy

$\cosh nz = T_n(\cosh z)$. The Chebyshev polynomials can be calculated from the recursion

$$T_0(z) = 1,\, T_1(z) = z,\, T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$$

so

$$T_2(z) = 2z^2 - 1, \quad T_3(z) = 4z^3 - 3z, \quad T_4(z) = 8z^4 - 8z^2 + 1.$$

Then

$$\begin{aligned}\cosh n\xi \cos n\eta + i \sinh n\xi \sin n\eta &= \cosh n(\xi + i\eta)\\ &= T_n(\cosh(\xi + i\eta))\\ &= T_n(\cosh \xi \cos \eta + i \sinh \xi \sin \eta)\\ &= T_n(c^{-1}(x + iy)).\end{aligned}$$

For example,

$$\begin{aligned}\cosh 2\xi \cos 2\eta &= \operatorname{Re}(2c^{-2}(x+iy)^2 - 1) = 2\left(\frac{x}{c}\right)^2 - 2\left(\frac{y}{c}\right)^2 - 1,\\ \sinh 2\xi \sin 2\eta &= \operatorname{Im}(2c^{-2}(x+iy)^2 - 1) = 2\frac{xy}{c^2}.\end{aligned}$$

Example: Solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ inside the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with boundary condition $u(x, y) = \frac{1}{3}x^2$.

Solution: We have $a = 3$, $b = 2$ and $c = \sqrt{5}$. The ellipse is given by $\xi = \xi_0$ where $c \cosh \xi_0 = a$, $c \sinh \xi_0 = b$. The boundary condition is given by the function $F(\eta) = \frac{1}{3}a^2 \cos^2 \eta = 3 \cos^2 \eta$. Its Fourier expansion is

$$f(\eta) = 3 \cos^2 \eta = \frac{3}{2} + \frac{3}{2} \cos 2\eta.$$

The solution of the Dirichlet problem in elliptic coordinates is

$$v(\xi, \eta) = \frac{3}{2} + \frac{3}{2} \frac{\cosh 2\xi}{\cosh 2\xi_0} \cos 2\eta.$$

Transforming to cartesian coordinates we get

$$u(x, y) = \frac{3}{2} + \frac{3}{2}\frac{5}{13}\left(2\frac{x^2}{5} - 2\frac{y^2}{5} - 1\right) = \frac{12}{13} + \frac{3}{13}(x^2 - y^2).$$