THE FOURIER CONVERGENCE THEOREM

Before we can prove the Fourier convergence theorem we need some preparations.

Lemma 1. *Let g be a T-periodic function which is integrable on $[0, T]$. Then, for all a,*

$$\int_0^T g(x)\,dx = \int_a^{a+T} g(x)\,dx.$$

Proof. There is an integer k such that $(k-1)T \le a < kT$. Then

$$\int_a^{a+T} g(x)\,dx = \int_a^{kT} g(x)\,dx + \int_{kT}^{a+T} g(x)\,dx.$$

In the first integral on the right-hand side we substitute $x = t - T$ and use $g(t-T) = g(t)$. Then we obtain

$$\int_a^{a+T} g(x)\,dx = \int_{a+T}^{(k+1)T} g(t)\,dt + \int_{kT}^{a+T} g(x)\,dx.$$

Therefore,

$$\int_a^{a+T} g(x)\,dx = \int_{kT}^{(k+1)T} g(x)\,dx = \int_0^T g(s)\,ds,$$

where we substituted $x = s + kT$. □

The Dirichlet kernel D_n, $n = 0, 1, 2, \ldots$, is defined by

$$D_n(t) = \frac{1}{2} + \cos t + \cos(2t) + \cdots + \cos(nt).$$

This is an even function with period 2π. The graph of D_5 is shown in Figure 1.

Lemma 2. *If $t \ne 0, \pm 2\pi, \pm 4\pi, \ldots$ then*

$$D_n(t) = \frac{\sin(2n+1)\frac{1}{2}t}{2\sin\frac{1}{2}t}.$$

Otherwise, $D_n(t) = n + \frac{1}{2}$.

Proof. Using $\cos t = \frac{1}{2}(e^{it} + e^{-it})$, we have

$$D_n(t) = \frac{1}{2}\sum_{m=-n}^{n} e^{imt}.$$

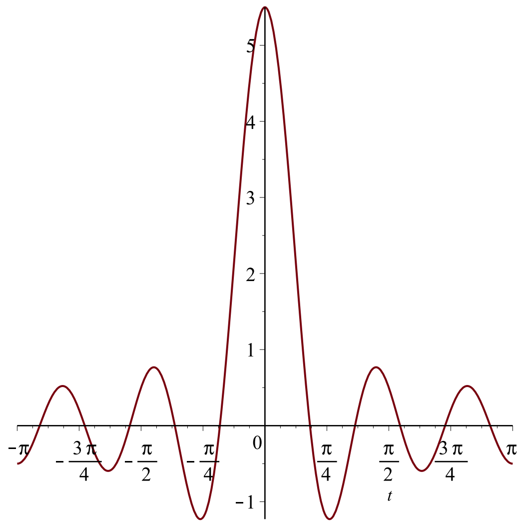

FIGURE 1. Graph of $D_5(t)$

We set $z = e^{it}$. Then

$$\begin{aligned} D_n(t) &= \frac{1}{2} z^{-n}(1 + z + z^2 + \cdots + z^{2n}) \\ &= \frac{1}{2} z^{-n} \frac{z^{2n+1} - 1}{z - 1} \\ &= \frac{1}{2} e^{-int} \frac{e^{(2n+1)it} - 1}{e^{it} - 1} \\ &= \frac{1}{2} \frac{e^{i(2n+1)\frac{1}{2}t} - e^{-i(2n+1)\frac{1}{2}t}}{e^{i\frac{1}{2}t} - e^{-i\frac{1}{2}t}} \\ &= \frac{\sin((2n+1)\frac{1}{2}t}{2 \sin \frac{1}{2}t}, \end{aligned}$$

where we used $\sin t = \frac{1}{2i}(e^{it} - e^{-it})$. □

Lemma 3 (Bessel's inequality)**.** *Let f be a $2L$-periodic function which is integrable on $[-L, L]$ with Fourier coefficients*

$$a_m = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{m\pi t}{L}\, dt, \quad b_m = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{m\pi t}{L}\, dt. \tag{1}$$

Then

$$\frac{1}{2} a_0^2 + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \leq \frac{1}{L} \int_{-L}^{L} f(t)^2\, dt. \tag{2}$$

In particular,

$$\lim_{m \to \infty} a_m = 0, \quad \lim_{m \to \infty} b_m = 0.$$

Proof. Let n be a positive integer, and consider

$$s_n(t) = \frac{1}{2}a_0 + \sum_{m=1}^{n} \left(a_m \cos \frac{m\pi t}{L} + b_m \sin \frac{m\pi t}{L} \right).$$

Then

$$0 \le \frac{1}{L}\int_{-L}^{L} (f(t) - s_n(t))^2\, dt = \frac{1}{L}\int_{-L}^{L} f(t)^2\, dt - \frac{2}{L}\int_{-L}^{L} f(t)s_n(t)\, dt + \frac{1}{L}\int_{-L}^{L} s_n(t)^2\, dt.$$

Now, using the definition of s_n,

$$\frac{2}{L}\int_{-L}^{L} f(t)s_n(t)\, dt = 2\left(\frac{1}{2}a_0^2 + \sum_{m=1}^{n} (a_m^2 + b_m^2) \right).$$

By orthogonality,

$$\frac{1}{L}\int_{-L}^{L} s_n(t)^2\, dt = \frac{1}{2}a_0^2 + \sum_{m=1}^{n} (a_m^2 + b_m^2).$$

Therefore,

$$0 \le \frac{1}{L}\int_{-L}^{L} f(t)^2\, dt - \left(\frac{1}{2}a_0^2 + \sum_{m=1}^{n} (a_m^2 + b_m^2) \right).$$

This is true for all n so (2) follows. □

Actually, equality holds in (2) (Parseval's equation) but we do not need this result right now.

A function f is said to be piecewise continuous on the interval $[a, b]$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$ so that

1. f is continuous on the open interval (x_{i-1}, x_i) for $i = 1, 2, \ldots, n$;
2. the one-sided limits $f(x_{i-1}^+) = \lim_{x \to x_{i-1}^+} f(x) =$ and $f(x_i^-) = \lim_{x \to x_i^-} f(x)$ exist and are finite for each $i = 1, 2, \ldots, n$.

Theorem 4 (Fourier convergence theorem)**.** *Let f be a function with period $2L$ such that f and f' are piecewise continuous on $[-L, L]$. Let a_m, b_m be the Fourier coefficients of f as defined in* (1)*. Then, for all real x,*

$$\frac{1}{2}(f(x^+) + f(x^-)) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).$$

In particular, if f is continuous at x,

$$f(x) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).$$

Proof. In order to simplify the writing we assume that $L = \pi$ (consider $f(\frac{L}{\pi}t)$ in place of f.) In the following x denotes a fixed real number. For a positive integer n we define the partial sum of the Fourier series

$$s_n(x) = \frac{1}{2}a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx)\,.$$

Then using (1)

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{m=1}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(\cos mx \cos mt + \sin mx \sin mt)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{m=1}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos m(t-x)\,dt. \end{aligned}$$

By definition of D_n,

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)\,dt.$$

We substitute $t - x = u$. Then

$$s_n(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+u)D_n(u)\,du.$$

By Lemma 1,

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)D_n(u)\,du.$$

We split the integral in two

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{0} f(x+u)D_n(u)\,du + \frac{1}{\pi}\int_{0}^{\pi} f(x+u)D_n(u)\,du.$$

It follows easily from the definition of D_n that

$$\frac{1}{\pi}\int_{-\pi}^{0} D_n(t)\,dt = \frac{1}{\pi}\int_{0}^{\pi} D_n(t)\,dt = \frac{1}{2}.$$

Therefore,

$$s_n(x) - \frac{1}{2}(f(x^+) + f(x^-)) = I_n + J_n,$$

where

$$I_n = \frac{1}{\pi}\int_{-\pi}^{0} (f(x+u) - f(x^-))D_n(u)\,du, \quad J_n = \frac{1}{\pi}\int_{0}^{\pi} (f(x+u) - f(x^+))D_n(u)\,du.$$

We now show that the two integrals I_n, J_n converge to 0 as $n \to \infty$ which completes the proof. We do this only for J_n, I_n is treated similarly. Now, using Lemma 2,

$$J_n = \frac{1}{\pi}\int_{0}^{\pi} (f(x+u) - f(x^+))\frac{\sin(2n+1)\frac{1}{2}u}{2\sin\frac{1}{2}u}\,du.$$

Substituting $u = 2t$ we can write this as

$$J_n = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} g(t) \sin(2n+1)t \, dt,$$

where

$$g(t) = \frac{f(x+2t) - f(x^+)}{2t} \frac{t}{\sin t} \quad \text{for } 0 < t \leq \frac{1}{2}\pi.$$

Since we assumed that f' is piecewise continuous, the limit $\lim_{t\to 0^+} g(t)$ exists as a finite number (to see this one has to apply the mean-value theorem). Therefore, the function g is piecewise continuous and thus integrable on $[0, \frac{1}{2}\pi]$. It follows from Lemma 3 (with $L = \frac{1}{2}\pi$ and $g(t) = 0$ for $-L < t < 0$) that $\lim_{n\to\infty} J_n = 0$. □

Remark: In the proof we did not directly use that f' is piecewise continuous. It would be simpler to just assume that the limits

$$\lim_{t\to 0^+} \frac{f(x+t) - f(x^+)}{t}, \quad \lim_{t\to 0^-} \frac{f(x+t) - f(x^-)}{t}$$

exist and are finite.