my solution to first exam practice questions
Math 332 Introduction to Partial differential equations
Spring 2018

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Contents

0.0.1 Problem 1

Find the solution to $y'' - y = e^x, y(0) = 1, y(1) = 0$

solution

The solution to the homogeneous ODE is $y_h = Ae^x + Be^{-x}$. Let the particular be $y_p = Cxe^x$. Hence $y_p' = Ce^x + Cxe^x$ and $y_p'' = Ce^x + Ce^x + Cxe^x$. Substituting into the ODE gives

$$
\begin{aligned}
2Ce^x + Cxe^x - Cxe^x &= e^x \\
2C &= 1 \\
C &= \frac{1}{2}
\end{aligned}
$$

Hence $y_p = \frac{1}{2}xe^x$ and the complete solution is

$$y = Ae^x + Be^{-x} + \frac{1}{2}xe^x$$

A, B are now found from boundary conditions. At $x = 0$

$$1 = A + B \tag{1}$$

And at $x = 1$

$$0 = Ae + Be^{-1} + \frac{1}{2}e \tag{2}$$

(1,2) are now solved for A, B. From (1), $A = 1 - B$. (2) becomes

$$\begin{aligned}
0 &= (1-B)\, e + Be^{-1} + \frac{1}{2}e \\
&= e - Be + Be^{-1} + \frac{1}{2}e \\
&= B\left(e^{-1} - e\right) + \frac{3}{2}e \\
B &= -\frac{3}{2}\frac{e}{e^{-1} - e} \\
&= \frac{3}{2}\frac{e}{e - e^{-1}}
\end{aligned}$$

Hence

$$\begin{aligned}
A &= 1 - \frac{3e}{2\left(e - e^{-1}\right)} = \frac{2\left(e - e^{-1}\right) - 3e}{2\left(e - e^{-1}\right)} \\
&= \frac{2e - 2e^{-1} - 3e}{2\left(e - e^{-1}\right)} \\
&= \frac{-e - 2e^{-1}}{2\left(e - e^{-1}\right)} \\
&= \frac{e + 2e^{-1}}{2\left(e^{-1} - e\right)}
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= Ae^{x} + Be^{-x} + \frac{1}{2}xe^{x} \\
&= \frac{e + 2e^{-1}}{2\left(e^{-1} - e\right)}e^{x} + \frac{3}{2}\frac{e}{e - e^{-1}}e^{-x} + \frac{1}{2}xe^{x}
\end{aligned}$$

0.0.2 Problem 2

Find Fourier cosine series for

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 1 & 1 < x < 2 \end{cases}$$

Choose $L = 2$. Apply the Fourier convergence theorem. What do we get at $x = 1$?

solution

For cosine series, the function is even extended from $x = -2 \cdots 2$. Therefore only a_n terms exist.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

Where $L = 2$. But $\frac{a_0}{2}$ is average value. Since the area is $2\left(\frac{1}{2} + 1\right) = 3$, then the average is $\frac{3}{4}$, since the extent is 4. Therefore $a_0 = \frac{3}{2}$. To find a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right) dx$$

But $f(x)$ and cosine are even. Hence the above simplifies to

$$\begin{aligned}
a_n &= \int_0^2 f(x)\cos\left(\frac{n\pi}{2}x\right) dx \\
&= \left(\int_0^1 x\cos\left(\frac{n\pi}{2}x\right) dx + \int_1^2 \cos\left(\frac{n\pi}{2}x\right) dx\right)
\end{aligned}$$

But $\int x\cos axdx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$, therefore

$$\begin{aligned}\int_0^1 x\cos\left(\frac{n\pi}{2}x\right)dx &= \left(\frac{\cos\left(\frac{n\pi}{2}x\right)}{\left(\frac{n\pi}{2}\right)^2} + \frac{x\sin\left(\frac{n\pi}{2}x\right)}{\frac{n\pi}{2}}\right)_0^1 \\ &= \left(\frac{2}{n\pi}\right)^2\left(\cos\left(\frac{n\pi}{2}x\right) + \frac{n\pi}{2}x\sin\left(\frac{n\pi}{2}x\right)\right)_0^1 \\ &= \left(\frac{2}{n\pi}\right)^2\left(\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right)\end{aligned}$$

And

$$\begin{aligned}\int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx &= \left(\frac{\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right)_1^2 \\ &= \frac{2}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) \\ &= -\frac{2}{n\pi}\sin\frac{n\pi}{2}\end{aligned}$$

Hence

$$\begin{aligned}a_n &= \left(\frac{2}{n\pi}\right)^2\left(\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right) - \frac{2}{n\pi}\sin\frac{n\pi}{2} \\ &= \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)^2 - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) \\ &= \frac{2}{n^2\pi^2}\left(-2 + 2\cos\left(\frac{n\pi}{2}\right) + n\pi\sin\left(\frac{n\pi}{2}\right)\right)\end{aligned}$$

Which simplifies to $a_n = -\frac{8\sin\left(\frac{n\pi}{4}\right)^2}{n^2\pi^2}$. Therefore

$$\begin{aligned}f(x) &= \frac{3}{4} - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right)^2\cos\left(\frac{n\pi}{2}x\right) \\ &= \frac{3}{4} - \frac{8}{\pi^2}\sin\left(\frac{\pi}{4}\right)^2\cos\left(\frac{\pi}{2}x\right) - \frac{8}{\pi^2}\frac{1}{4}\sin\left(\frac{2\pi}{4}\right)^2\cos(\pi x) - \cdots \\ &= \frac{3}{4} - \frac{4}{\pi^2}\cos\left(\frac{\pi}{2}x\right) - \frac{2}{\pi^2}\cos(\pi x) - \cdots\end{aligned}$$

At $x = 1$

$$\begin{aligned}f(1) &= \frac{3}{4} - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right)^2\cos\left(\frac{n\pi}{2}\right) \\ &= \frac{3}{4} - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right)^2\cos\left(\frac{n\pi}{2}\right)\end{aligned}$$

In the limit, $\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right)^2\cos\left(\frac{n\pi}{2}\right) = -\frac{\pi^2}{32}$. Therefore the above becomes

$$\begin{aligned}f(1) &= \frac{3}{4} + \frac{8}{\pi^2}\frac{\pi^2}{32} \\ &= \frac{3}{4} + \frac{1}{4} \\ &= 1\end{aligned}$$

Which is the value of original $f(x)$ at 1 as expected.

To apply Fourier convergence theorem. The function $f(x)$ is piecewise continuous over $-2 < x < 2$.

$$f'(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

$f'(x)$ is also piecewise continuous. Therefore, the Fourier series of $f(x)$ will converge to the average of $f(x)$ at each point.

0.0.3 Problem 3

Find Fourier sine series for

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 1 & 1 < x < 2 \end{cases}$$

Choose $L = 2$.

solution

For sine series, the function is odd extended from $x = -2 \cdots 2$. Therefore only b_n terms exist.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

Where $L = 2$. To find b_n

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{L}x\right) dx$$

But $f(x)$ is now odd, and sine is odd, hence the product is even and the above simplifies to

$$\begin{aligned} b_n &= \int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right) dx \\ &= \left(\int_0^1 x\sin\left(\frac{n\pi}{2}x\right) dx + \int_1^2 \sin\left(\frac{n\pi}{2}x\right) dx\right) \end{aligned}$$

But $\int x \sin ax dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$, therefore

$$\begin{aligned} \int_0^1 x\sin\left(\frac{n\pi}{2}x\right) dx &= \left(\frac{\sin\left(\frac{n\pi}{2}x\right)}{\left(\frac{n\pi}{2}\right)^2} - \frac{x\cos\left(\frac{n\pi}{2}x\right)}{\frac{n\pi}{2}}\right)_0^1 \\ &= \left(\frac{2}{n\pi}\right)^2\left(\sin\left(\frac{n\pi}{2}x\right) - \frac{n\pi}{2}x\cos\left(\frac{n\pi}{2}x\right)\right)_0^1 \\ &= \left(\frac{2}{n\pi}\right)^2\left(\sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right) \end{aligned}$$

And

$$\begin{aligned} \int_1^2 \sin\left(\frac{n\pi}{2}x\right) dx &= -\left(\frac{\cos\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right)_1^2 \\ &= -\frac{2}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) \end{aligned}$$

Therefore

$$\begin{aligned} b_n &= \left(\frac{2}{n\pi}\right)^2\left(\sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right) - \frac{2}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) \\ &= -\frac{2\left(n\pi\cos n\pi - 2\sin\frac{n\pi}{2}\right)}{n^2\pi^2} \end{aligned}$$

Therefore

$$f(x) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{2\sin\frac{n\pi}{2} - n\pi\cos n\pi}{n^2} \sin\left(\frac{n\pi}{L}x\right)$$

As in problem 2, both $f(x)$ and $f'(x)$ are P.W.C. So F.S. converges to average of $f(x)$ at all points.

0.0.4 Problem 4

Solve heat PDE $u_t = u_{xx}$ with boundary conditions $u_x(0,t) = 0, u_x(2,t) = 0$ and initial conditions $u(x,0) = f(x)$ with $f(x)$ from problem 2. Find steady state solution.

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 1 & 1 < x < 2 \end{cases}$$

solution

When both ends are insulated the solution to the heat PDE is

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \cos\left(\sqrt{\lambda_n} x\right)$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with $n = 1, 2, 3, \cdots$. Since $L = 2$, then

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{2}\right)^2 t} \cos\left(\frac{n\pi}{2}x\right)$$

At $t = 0$

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{2}x\right) \tag{1}$$

But the F.S. of $f(x)$ was found in problem 2, with even extension. It is

$$f(x) = \frac{3}{4} - \sum_{n=1}^{\infty} \frac{8}{\pi^2 n^2} \sin\left(\frac{n\pi}{4}\right)^2 \cos\left(\frac{n\pi}{2}x\right) \tag{2}$$

Comparing (1) and (2) gives

$$\frac{c_0}{2} = \frac{3}{4}$$
$$c_n = \frac{8}{\pi^2 n^2} \sin\left(\frac{n\pi}{4}\right)^2$$

Hence solution is

$$u(x,t) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{8}{\pi^2 n^2} \sin\left(\frac{n\pi}{4}\right)^2 e^{-\left(\frac{n\pi}{2}\right)^2 t} \cos\left(\frac{n\pi}{2}x\right)$$

At steady state, the solution is

$$u(x,\infty) = \frac{3}{4}$$

Since as $t \to \infty$, the term $e^{-\left(\frac{n\pi}{2}\right)^2 t} \to 0$.

0.0.5 Problem 5

Solve heat PDE $u_t = u_{xx}$ with boundary conditions $u(0,t) = t, u(\pi,t) = 0$ and initial conditions $u(x,0) = 0$

solution

Since boundary conditions are nonhomogeneous, the PDE is converted to one with homogenous BC using a reference function. The reference function needs to only satisfy the nonhomogeneous B.C.

In this case, it is clear that the following function satisfies the nonhomogeneous B.C.

$$r(x,t) = t\left(1 - \frac{x}{\pi}\right)$$

Therefore

$$u(x,t) = w(x,t) + r(x,t)$$

Substituting this back into $u_t = u_{xx}$ gives

$$w_t + r_t = w_{xx} + r_{xx}$$

but $r_t = 1 - \frac{x}{\pi}$ and $r_{xx} = 0$, therefore the above simplifies to

$$\begin{aligned} w_t &= w_{xx} + \frac{x}{\pi} - 1 \\ w_t &= w_{xx} + Q(x) \end{aligned} \tag{1}$$

Where $Q(x) = \frac{x}{\pi} - 1$ and where now this PDE now has now homogenous B.C

$$\begin{aligned} w(0,t) &= 0 \\ w(\pi,t) &= 0 \end{aligned}$$

Since a source term exist in the PDE (nonhomogeneous in the PDE itself), then equation (1) is solved using the method of eigenfunction expansion. Let

$$w(x,t) = \sum a_n(t)\Phi_n(x)$$

Where $\Phi_n(x)$ is the eigenfunction of the homogeneous PDE $w_t = w_{xx}$, which is known to be have the eigenfunction $\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right) = \sin nx$ where the eigenvalues are known to be $\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2$ with $n = 1,2,3,\cdots$. Therefore the above becomes

$$w(x,t) = \sum a_n(t)\sin(nx) \tag{1A}$$

Substituting this back into (1) gives

$$\sum a_n'(t)\Phi_n(x) = \sum a_n(t)\Phi_n''(x) + \sum q_n\Phi_n(x)$$

Where $Q(x) = \sum q_n\Phi_n(x)$ is the eigenfunction expansion of the source term. In the above, and after replacing $\Phi_n''(x)$ by $-\lambda_n\Phi_n(x)$ since $\Phi_n(x)$ satisfies the eigenvalue PDE $\Phi_n''(x)+\lambda_n\Phi_n(x) = 0$ the above becomes

$$\begin{aligned} \sum a_n'(t)\Phi_n(x) &= -\sum a_n(t)\lambda_n\Phi_n(x) + \sum q_n\Phi_n(x) \\ a_n'(t) &= -a_n(t)\lambda_n + q_n \\ a_n'(t) + a_n(t)\lambda_n &= q_n \end{aligned} \tag{2}$$

q_n is now found by applying orthogonality on $Q(x) = \sum q_n \Phi_n(x)$ as follows

$$
\begin{aligned}
Q(x) &= \sum_{n=1}^{\infty} q_n \Phi_n(x) \\
\int_0^{\pi} Q(x) \Phi_n(x)\, dx &= \frac{\pi}{2} q_n \\
q_n &= \frac{2}{\pi} \int_0^{\pi} \left(\frac{x}{\pi} - 1\right) \sin(nx)\, dx \\
&= \frac{2}{\pi} \left(\frac{-n\pi + \sin(n\pi)}{n^2\pi}\right) \\
&= \frac{2}{\pi} \left(\frac{-n\pi}{n^2\pi}\right) \\
&= \frac{-2}{n\pi}
\end{aligned}
$$

Equation (2) becomes

$$a_n'(t) + a_n(t)\, n^2 = \frac{-2}{n\pi}$$

The solution to this first order ODE can be easily found as

$$a_n(t) = -\frac{2}{n^3\pi} + a_n(0)\, e^{-n^2 t} \tag{3}$$

Therefore (1A) becomes

$$w(x,t) = \sum_{n=1}^{\infty} \left(-\frac{2}{n^3\pi} + a_n(0)\, e^{-n^2 t}\right) \sin(nx) \tag{4}$$

At time $t = 0$ the above becomes

$$w(x,0) = \sum_{n=1}^{\infty} \left(-\frac{2}{n^3\pi} + a_n(0)\right) \sin(nx) \tag{5}$$

But

$$
\begin{aligned}
w(x,0) &= u(x,0) - r(x,0) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

Therefore (5) becomes

$$0 = \sum_{n=1}^{\infty} \left(-\frac{2}{n^3\pi} + a_n(0)\right) \sin(nx)$$

Which implies

$$a_n(0) = \frac{2}{n^3\pi}$$

Hence from (4)

$$w(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^3\pi} \left(e^{-n^2 t} - 1\right) \sin(nx) \tag{6}$$

The complete solution is therefore

$$
\begin{aligned}
u(x,t) &= w(x,t) + r(x,t) \\
&= t\left(1 - \frac{x}{\pi}\right) + \sum_{n=1}^{\infty} \frac{2}{n^3\pi} \left(e^{-n^2 t} - 1\right) \sin(nx)
\end{aligned}
$$

0.0.6 Problem 6

Solve wave PDE $u_{tt} = 4u_{xx}$ on bounded domain $0 < x < \pi, t > 0$ with boundary conditions $u(0,t) = 0, u(\pi,t) = 0$ and initial conditions $u(x,0) = \sin^2 x, u_t(x,0) = 0$. Find d'Alembert solution and Fourier series solution.

solution

Putting the PDE in standard form $u_{tt} = a^2 u_{xx}$ shows that $a = 2$. Let $f(x) = u(x,0) = \sin^2 x$ and $g(x) = u_t(x,0) = 0$, then the d'Alembert solution is (per key solution, one must use the sign function). Let $F(x) = sign(\sin x)\sin^2 x$, then the solution becomes

$$\begin{aligned} u(x,t) &= \frac{1}{2}(F(x+at) + F(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds \\ &= \frac{1}{2}(F(x+at) + F(x-at)) \end{aligned}$$

Now the Fourier solution is found. Applying separation of variables gives

$$\begin{aligned} T''X &= 4X''T \\ \frac{1}{4}\frac{T''}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

The eigenvalue ODE is $X'' + \lambda X = 0$ with $X(0) = 0, X(\pi) = 0$. This has eigenfunctions $\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$ with $\lambda_n = n^2$ where $n = 1, 2, 3, \cdots$. The time ODE becomes

$$T'' + 4\lambda_n T = 0$$

Since $\lambda_n > 0$, the solution is

$$\begin{aligned} T(t) &= A_n\cos\left(\sqrt{4\lambda_n}t\right) + B_n\sin\left(\sqrt{4\lambda_n}t\right) \\ &= A_n\cos(2nt) + B_n\sin(2nt) \end{aligned}$$

And

$$T' = -2nA_n\sin(2nt) + 2nB_n\cos(2nt)$$

Since $T'(0) = 0$, then the above implies that $B_n = 0$. Therefore the solution simplifies to

$$T_n(t) = A_n\cos(2nt)$$

And the fundamental solution becomes

$$\begin{aligned} u_n &= T_n X_n \\ &= c_n\cos(2nt)\sin(nx) \end{aligned}$$

Hence by superposition, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n\cos(2nt)\sin(nx)$$

At $t = 0$, $u(x,0) = \sin^2 x$, therefore the above becomes

$$\sin^2 x = \sum_{n=1}^{\infty} c_n\sin(nx)$$

Applying orthogonality gives

$$\int_0^{\pi} \sin^2 x \sin(nx)\, dx = c_n \frac{\pi}{2} \tag{1}$$

$$\int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) \sin(nx)\, dx = c_n \frac{\pi}{2}$$

To evaluate $\int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) \sin(nx)\, dx$, it is split into $\int_0^{\pi} \left(\frac{1}{2}\sin(nx) - \frac{1}{2}\cos 2x \sin(nx)\right) dx$. But the first part is

$$\begin{aligned}\int_0^{\pi} \frac{1}{2}\sin(nx)\, dx &= -\frac{1}{2n}\left(\cos(nx)\right)_0^{\pi} \\ &= -\frac{1}{2n}\left(\cos(n\pi) - 1\right)\end{aligned}$$

For even $n = 2, 4, \cdots$ the above vanishes. For odd $n = 1, 3, 5, \cdots$ the above becomes

$$\int_0^{\pi} \frac{1}{2}\sin(nx)\, dx = \frac{1}{n}$$

Now the second integral is evaluated

$$\int_0^{\pi} -\frac{1}{2}\cos 2x \sin(nx)\, dx = -\frac{1}{2}\int_0^{\pi} \cos 2x \sin(nx)\, dx$$

Using $\int_0^{\pi} \sin(px)\cos(qx)\, dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)}$, then the above becomes, where $p = n, q = 2$

$$\begin{aligned}-\frac{1}{2}\int_0^{\pi} \sin(nx)\cos 2x dx &= -\frac{1}{2}\left(-\frac{\cos(n-2)x}{2(n-2)} - \frac{\cos(n+2)x}{2(n+2)}\right)_0^{\pi} \\ &= \frac{1}{2}\left(\frac{\cos(n-2)x}{2(n-2)} + \frac{\cos(n+2)x}{2(n+2)}\right)_0^{\pi} \\ &= \frac{1}{2}\left(\frac{\cos(n-2)\pi}{2(n-2)} + \frac{\cos(n+2)\pi}{2(n+2)} - \frac{1}{2(n-2)} - \frac{1}{2(n+2)}\right)\end{aligned}$$

For even $n = 2, 4, \cdots$ the above vanishes, since it becomes $\frac{1}{2}\left(\frac{1}{2(n-2)} + \frac{1}{2(n+2)} - \frac{1}{2(n-2)} - \frac{1}{2(n+2)}\right)$, and for odd $n = 1, 3, 5, \cdots$, the above becomes

$$\begin{aligned}-\frac{1}{2}\int_0^{\pi} \sin(nx)\cos 2x dx &= \frac{1}{2}\left(\frac{-1}{2(n-2)} - \frac{1}{2(n+2)} - \frac{1}{2(n-2)} - \frac{1}{2(n+2)}\right) \\ &= \frac{1}{2}\left(\frac{-2}{2(n-2)} + \frac{-2}{2(n+2)}\right) \\ &= \frac{-1}{2(n-2)} + \frac{-1}{2(n+2)} \\ &= -\frac{n}{n^2-4}\end{aligned}$$

Therefore, the final result of integration is

$$\begin{aligned}\int_0^{\pi} \sin^2 x \sin(nx)\, dx &= \frac{1}{n} - \frac{n}{n^2-4} \qquad n = 1, 3, 5, \cdots \\ &= -\frac{4}{n(n^2-4)} \qquad n = 1, 3, 5, \cdots\end{aligned}$$

Hence from (1), this results in

$$\begin{aligned} c_n &= -\frac{2}{\pi} \frac{4}{n(n^2-4)} \\ &= -\frac{8}{\pi n(n^2-4)} \qquad n = 1,3,5,\cdots \end{aligned}$$

Hence the final solution is

$$u(x,t) = \frac{-8}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^3-4n} \cos(2nt)\sin(nx)$$

The above solution was verified against numerical solution. The result gave an exact match (20 terms was used in the sum).

0.0.7 Problem 7

Find d'Alembert solution for wave PDE $u_{tt} = 4u_{xx}$ on infinite domain with initial position $u(x,0) = \sin x$ and initial velocity $u_t(x,0) = \cos x$

solution

Putting the PDE in standard form $u_{tt} = a^2 u_{xx}$ shows that $a = 2$. Let $f(x) = u(x,0) = \sin x$ and $g(x) = u_t(x,0) = \cos x$, then the d'Alembert solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}(f(x+at)+f(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds \\ &= \frac{1}{2}(\sin(x+2t)+\sin(x-2t)) + \frac{1}{4}\int_{x-2t}^{x+2t}\cos(s)\,ds \\ &= \frac{1}{2}\sin(x+2t) + \frac{1}{2}\sin(x-2t) + \frac{1}{4}\sin(s)_{x-2t}^{x+2t} \\ &= \frac{1}{2}\sin(x+2t) + \frac{1}{2}\sin(x-2t) + \frac{1}{4}(\sin(x+2t)-\sin(x-2t)) \\ &= \frac{1}{2}\sin(x+2t) + \frac{1}{2}\sin(x-2t) + \frac{1}{4}\sin(x+2t) - \frac{1}{4}\sin(x-2t) \\ &= \frac{3}{4}\sin(x+2t) + \frac{1}{4}\sin(x-2t) \end{aligned}$$

0.0.8 Problem 8

Solve the Dirichlet problem $u_{xx}+u_{yy} = 0$ inside the disk $x^2+y^2 < 1$ and $u(x,y) = \begin{cases} 20 & y>0 \\ 0 & y<0 \end{cases}$ on the unit circle $x^2+y^2 = 1$. Find $u(0,0)$ and $u\left(0,\frac{1}{2}\right)$

solution

The PDE in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + u_{\theta\theta} = 0 \tag{1}$$

Where r is radial distance and θ the polar angle. The boundary conditions in polar coordinates become

$$f(\theta) = \begin{cases} 20 & 0<\theta<\pi \\ 0 & \pi<\theta<2\pi \end{cases}$$

The solution to (1) is

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (a_n\cos(n\theta) + b_n\sin(n\theta))$$

At $r = 1$ (on the boundary) the above solution become

$$f(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta)$$

By orthogonality on cosine the above becomes

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,d\theta = \int_0^{2\pi} \frac{c_0}{2}\cos(m\theta)\,d\theta + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(m\theta)\cos(n\theta)\,d\theta + b_n \int_0^{2\pi} \cos(m\theta)\sin(n\theta)\,d\theta \tag{2}$$

For $n = 0$

$$\begin{aligned}
\int_0^{2\pi} f(\theta)\,d\theta &= \int_0^{2\pi} \frac{c_0}{2} d\theta \\
\int_0^{\pi} 20 d\theta &= \frac{c_0}{2}(2\pi) \\
20\pi &= \frac{c_0}{2}(2\pi) \\
c_0 &= 20
\end{aligned}$$

For $n > 0$ (2) becomes

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,d\theta = \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(m\theta)\cos(n\theta)\,d\theta + b_n \int_0^{2\pi} \cos(m\theta)\sin(n\theta)\,d\theta$$

But $\int_0^{2\pi} \cos(m\theta)\sin(n\theta)\,d\theta = 0$ for all n, m and the above reduces to

$$\begin{aligned}
\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta &= a_n\pi \\
\int_0^{\pi} 20\cos(n\theta)\,d\theta &= a_n\pi \\
\frac{20}{n}\left[\sin(n\theta)\right]_0^{\pi} &= a_n\pi \\
\frac{20}{n}\left(\sin(n\pi) - 0\right) &= a_n\pi
\end{aligned}$$

Hence $a_n = 0$ for all $n > 0$. By orthogonality on sine, for $n > 0$, (2) becomes

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,d\theta = \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \sin(m\theta)\cos(n\theta)\,d\theta + b_n \int_0^{2\pi} \sin(m\theta)\sin(n\theta)\,d\theta$$

But $\int_0^{2\pi} \sin(m\theta)\cos(n\theta)\,d\theta = 0$ for all m, n and the above reduces to

$$\begin{aligned}
\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta &= b_n\pi \\
\int_0^{\pi} 20\sin(n\theta)\,d\theta &= b_n\pi \\
-\frac{20}{n}\left(\cos(n\theta)\right)_0^{\pi} &= b_n\pi \\
-\frac{20}{n}\left(\cos(n\pi) - 1\right) &= b_n\pi \\
\frac{20}{n}\left(1 - \cos(n\pi)\right) &= b_n\pi
\end{aligned}$$

When $n=2,4,6,\cdots$ the above gives $b_n=0$. For $n=1,3,5,\cdots$ the above gives

$$\begin{aligned}\frac{40}{n}&=b_n\pi\\ b_n&=\frac{40}{n\pi}\end{aligned}$$

Therefore the complete solution is

$$u(r,\theta)=10+\frac{40}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{r^n}{n}\sin(n\theta)$$

At $u(0,0)$, which corresponds to $r=0,\theta=0$, the above gives $u(0,0)=10$. At $u\left(0,\frac{1}{2}\right)$ which corresponds to $r=\frac{1}{2},\theta=\frac{\pi}{2}$ the solution gives

$$u(r,\theta)=10+\frac{40}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\left(\frac{1}{2}\right)^n\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

Evaluated numerically, it converges to 15.90381156

To convert to x,y, the solution is first written as

$$u(r,\theta)=10+\frac{40}{\pi}\left(r\sin(\theta)+\frac{1}{3}r^3\sin(3\theta)+\frac{1}{5}r^5\sin(5\theta)+\cdots\right)$$

But

$$r\sin(\theta)=y$$

And

$$\begin{aligned}r^3\sin(3\theta)&=\sum_{\substack{k=1\\ odd}}^{3}\frac{n!}{k!\,(n-k)!}x^{n-k}(-1)^{\frac{k-1}{2}}y^k\\ &=\frac{6}{2}x^2y-y^3\end{aligned}$$

And

$$\begin{aligned}r^5\sin(5\theta)&=\sum_{\substack{k=1\\ odd}}^{5}\frac{n!}{k!\,(n-k)!}x^{n-k}(-1)^{\frac{k-1}{2}}y^k\\ &=\frac{120}{24}x^4y-\frac{120}{12}x^2y^3+xy^5\end{aligned}$$

And so on. Hence the solution in xy is

$$u(x,y)=10+\frac{40}{\pi}\left(y+\frac{1}{3}\left(3x^2y-y^3\right)+\frac{1}{5}\left(5x^4y-10x^2y^3+xy^5\right)+\cdots\right)$$

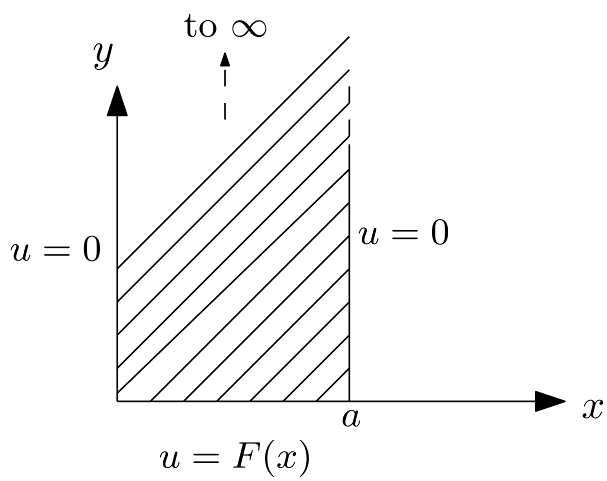
To verify is the above 3 terms give good approximation, the value at $x=0,y=\frac{1}{2}$ is now evaluated from the above, which gives 15.8356812467. Which is very close to the above result. One more term can be added to improve this. I am not sure now if there is a way to obtain closed form expression in x,y as the case was with the solution in polar coordinates.

0.0.9 Problem 9

Solve $u_{xx} + u_{yy} = 0$ inside semi-infinite strip $0 < x < a, y > 0$ with $u(0, y) - 0, u(a, y) = 0, u(x, 0) = F(x)$ and additional conditions that $u(x, y) \longrightarrow 0$ as $y \longrightarrow \infty$

solution

This is a plot of the boundary conditions.



Let $u = X(x)Y(y)$. Substituting this in the PDE gives

$$X''Y + Y''X = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Which gives the eigenvalue ODE

$$X''(x) + \lambda X(x) = 0$$
$$X(0) = 0$$
$$X(a) = 0$$

which gives the eigenfunction $\Phi_n(x) = c_n \sin\left(\sqrt{\lambda_n} x\right)$ where $\lambda_n = \left(\frac{n\pi}{a}\right)^2$ for $n = 1, 2, 3, \cdots$. The corresponding Y ODE is

$$Y'' - \lambda_n Y = 0$$

Since $\lambda_n > 0$, then the solution to this ODE is

$$Y_n = A_n e^{\sqrt{\lambda_n} y} + B_n e^{-\sqrt{\lambda_n} y}$$

Since $\lambda_n > 0$ and the solution goes to zero for large y, then A_n must be zero. Therefore the above simplifies to

$$Y_n(y) = B_n e^{-\sqrt{\lambda_n} y}$$

And the complete solution becomes

$$u(x, y) = \sum_{n=1}^{\infty} c_n e^{-\sqrt{\lambda_n} y} \sin\left(\sqrt{\lambda_n} x\right)$$

Where constants are combined into c_n. Since $\lambda_n = \left(\frac{n\pi}{a}\right)^2$, the above becomes

$$u(x,y) = \sum_{n=1}^{\infty} c_n e^{-\frac{n\pi}{a}y} \sin\left(\frac{n\pi}{a}x\right)$$

At $y = 0$, the above becomes

$$F(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right)$$

Applying orthogonality gives

$$\begin{aligned}\int_0^a F(x) \sin\left(\frac{n\pi}{a}x\right) dx &= c_n \frac{a}{2} \\ c_n &= \frac{2}{a}\int_0^a F(x)\sin\left(\frac{n\pi}{a}x\right) dx\end{aligned}$$

Hence the complete solution is

$$u(x,y) = \frac{2}{a}\sum_{n=1}^{\infty}\left(\int_0^a F(x)\sin\left(\frac{n\pi}{a}x\right) dx\right) e^{-\frac{n\pi}{a}y}\sin\left(\frac{n\pi}{a}x\right)$$