my post-mortem solution to first midterm exam
Math 332 Introduction to Partial differential equations
Spring 2018

Nasser M. Abbasi

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0.0.1 Problem 1

Problem

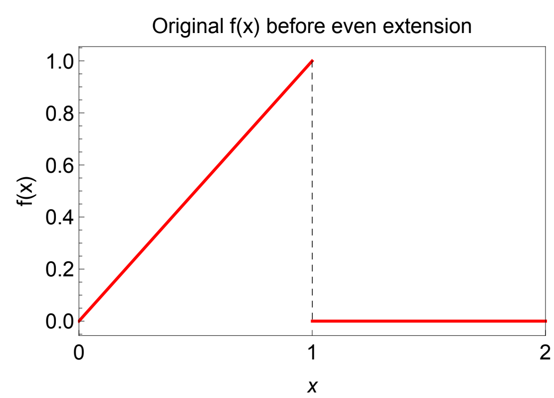
Find the Fourier cosine series of

$$f(x) = \begin{cases} x & 0 < x \leq 1 \\ 0 & 1 < x \leq 2 \end{cases}$$

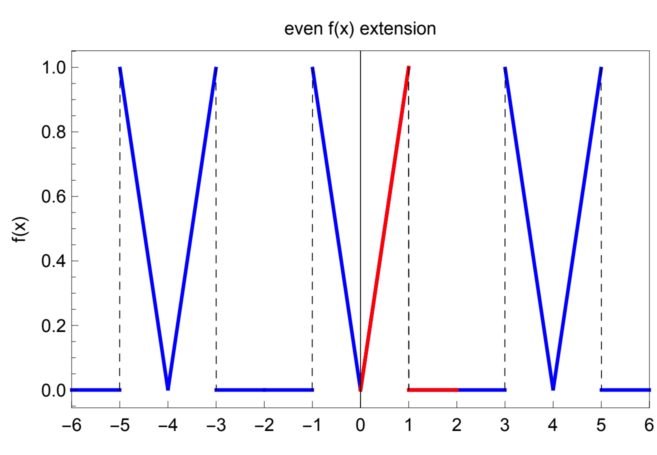
Take $L = 2$.

solution

To obtain the Fourier cosine series, the function $f(x)$ is first even extended to $-2 < x < 2$ with period $2L$ or 4. Then repeated again with period $2L$ over the whole x domain. The following plot shows the original $f(x)$



The following plot shows then even extended $f_e(x)$ over 3 periods for illustrations



The Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

Where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

Since extension is even, then the above simplifies to

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

But $L = 2$, therefore

$$\begin{aligned}
a_0 &= \int_0^2 f(x)\,dx = \int_0^1 x dx + \int_1^2 0 dx \\
&= \frac{1}{2}\left(x^2\right)_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

And

$$a_n = \frac{1}{L}\int_{-L}^L f(x)\cos\left(\frac{n\pi}{L}\right)dx$$

Since cosine is even, and $f(x)$ extension is even, then the product is even and the above simplifies to

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right)dx$$

Since $L = 2$

$$\begin{aligned}
a_n &= \int_0^2 f(x)\cos\left(\frac{n\pi}{2}\right)dx \\
&= \int_0^1 x\cos\left(\frac{n\pi}{2}\right)dx + \int_1^2 0\cos\left(\frac{n\pi}{2}\right)dx \\
&= \int_0^1 x\cos\left(\frac{n\pi}{2}\right)dx
\end{aligned}$$

But

$$\int x\cos(ax)\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$$

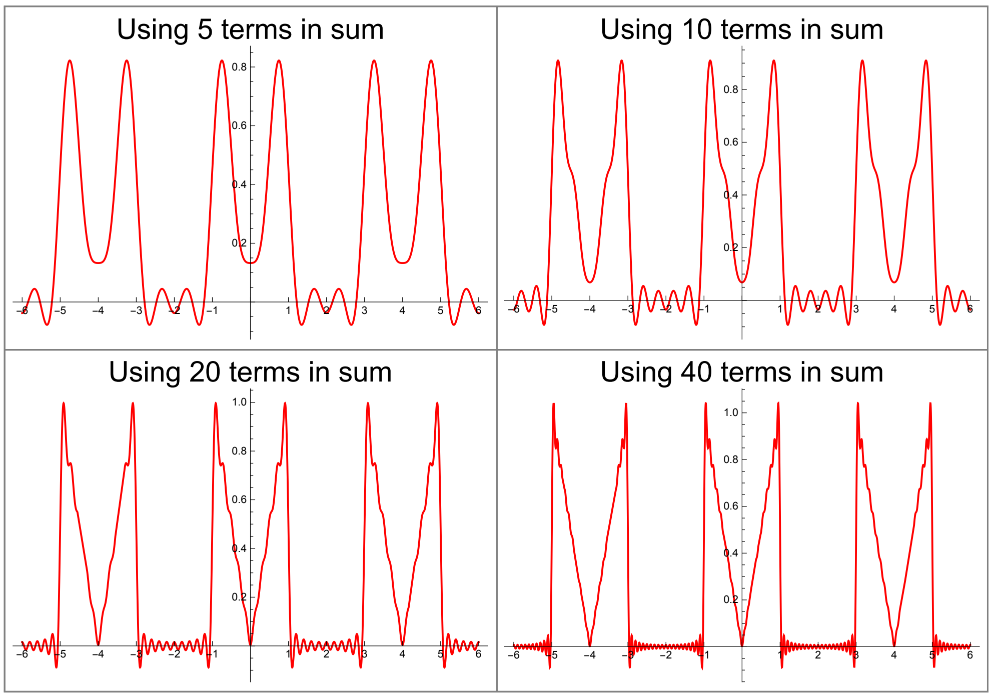
Where $a = \frac{n\pi}{2}$ here. Therefore the integral becomes

$$\begin{aligned}
a_n &= \int_0^1 x\cos\left(\frac{n\pi}{2}\right)dx \\
&= \left(\frac{\cos\left(\frac{n\pi}{2}x\right)}{\left(\frac{n\pi}{2}\right)^2} + \frac{x\sin\left(\frac{n\pi}{2}x\right)}{\left(\frac{n\pi}{2}\right)}\right)_0^1 \\
&= \frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} - \frac{1}{\left(\frac{n\pi}{2}\right)^2} \\
&= \frac{4\cos\left(\frac{n\pi}{2}\right)}{(n\pi)^2} + \frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi} - \frac{4}{(n\pi)^2} \\
&= \frac{4\cos\left(\frac{n\pi}{2}\right) + 2n\pi\sin\left(\frac{n\pi}{2}\right) - 4}{n^2\pi^2} \\
&= \frac{2}{n^2\pi^2}\left(2\cos\left(\frac{n\pi}{2}\right) + n\pi\sin\left(\frac{n\pi}{2}\right) - 2\right)
\end{aligned}$$

Therefore the Fourier series is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left(2\cos\left(\frac{n\pi}{2}\right) + n\pi\sin\left(\frac{n\pi}{2}\right) - 2\right)\cos\left(\frac{n\pi}{2}x\right)$$

By Fourier convergence theorem, since $f(x)$ and $f'(x)$ are piecewise contiguous, the Fourier series will converge to each point of $f(x)$ where there is no jump discontinuity, and will converge to the average of $f(x)$ at the point where there is a jump. In this example, it will converge to $\frac{1}{2}$ at the points where is a jump discontinuity There are $x = 1, 3, 5, \cdots$ and at $x = -1, -3, -5, \cdots$. At all other points, Fourier series will converge to $f(x)$. This is a plot of the above Fourier series for increasing number of terms



0.0.2 Problem 2

Problem Solve heat PDE $u_t = 9u_{xx}$ on $0 < x < \pi, t > 0$ with boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and initial conditions $u(x,0) = f(x) = 5\sin^2 x$

solution

The solution to the heat PDE with isolated end points is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} c_n e^{-\lambda_n a^2 t} \cos\left(\sqrt{\lambda_n x}\right)$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1,2,3\cdots$. But $L = \pi$ here. Hence $\lambda_n = n^2$ and $a = 3$. Therefore the above solution becomes

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} c_n e^{-9n^2 t} \cos(nx) \tag{1}$$

At $t = 0$ the above becomes

$$f(x) = A_0 + \sum_{n=1}^{\infty} c_n \cos(nx)$$
$$5\sin^2 x = A_0 + \sum_{n=1}^{\infty} c_n \cos(nx)$$

But $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$, therefore the above becomes

$$\frac{5}{2} - \frac{5}{2}\cos(2x) = A_0 + \sum_{n=1}^{\infty} c_n \cos(nx)$$

Hence $A_0 = \frac{5}{2}$ and $c_2 = -\frac{5}{2}$ and all other $c_n = 0$. Therefore the solution (1) becomes

$$u(x,t) = \frac{5}{2} - \frac{5}{2} e^{-36t}\cos(2x)$$

At steady state when $t \to \infty$, the solution becomes $u(x) = \frac{5}{2}$. The solution $u\left(\frac{\pi}{2}, t\right)$ becomes

$$\begin{aligned} u\left(\frac{\pi}{2}, t\right) &= \frac{5}{2} - \frac{5}{2} e^{-36t}\cos\left(2\frac{\pi}{2}\right) \\ &= \frac{5}{2} - \frac{5}{2} e^{-36t}\cos(\pi) \\ &= \frac{5}{2} + \frac{5}{2} e^{-36t} \\ &= \frac{5}{2}\left(1 + e^{-36t}\right) \end{aligned}$$

0.0.3 Problem 3

Problem

Solve the wave equation $u_{tt} = u_{xx}$ on string, where initial position $f(x) = 0$ and initial velocity is $g(x) = \sin(x) + \sin(2x)$. The string is fixed at both ends.

solution

$a = 1$ in this problem. Using D'Alembert method

$$u(x,t) = \frac{1}{2}(f(x+at) + f(x-at)) + \frac{1}{2}\int_{x-at}^{x+at} g(s)\,ds$$

Where f, g above are the odd extensions. Since $f(x)$ is zero and $a = 1$, the above simplifies to

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\
&= \frac{1}{2}\int_{x-t}^{x+t} \sin(s) + \sin(2s)\,ds \\
&= \frac{1}{2}\left(-\cos(s) - \frac{1}{2}\cos(2s)\right)_{x-t}^{x+t} \\
&= -\frac{1}{2}\left(\cos(s) + \frac{1}{2}\cos(2s)\right)_{x-t}^{x+t} \\
&= -\frac{1}{2}\left(\cos(x+t) + \frac{1}{2}\cos(2(x+t)) - \cos(x-t) - \frac{1}{2}\cos(2(x-t))\right) \\
&= -\frac{1}{2}\cos(x+t) - \frac{1}{4}\cos(2(x+t)) + \frac{1}{2}\cos(x-t) + \frac{1}{4}\cos(2(x-t)) \\
&= \frac{1}{2}(\cos(x-t) - \cos(x+t)) + \frac{1}{4}(\cos(2(x-t)) - \cos(2(x+t)))
\end{aligned}$$

Using Fourier series method. The solution with initial position zero is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n}at\right)\sin\left(\sqrt{\lambda_n}x\right)$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with $n = 1, 2, 3, \cdots$. Since $L = \pi$ and $a = 1$, the above solution simplifies to

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nt)\sin(nx) \tag{1}$$

To determine c_n, the velocity from the above solution is $\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n n\cos(nt)\sin(nx)$. And at $t = 0$, this becomes

$$f(x) = \sum_{n=1}^{\infty} nc_n \sin(nx)$$

But $f(x) = \sin(x) + \sin(2x)$. Hence the above becomes

$$\sin(x) + \sin(2x) = \sum_{n=1}^{\infty} nc_n \sin(nx)$$

Therefore by inspection $c_1 = 1$ and $2c_2 = 1$ or $c_2 = \frac{1}{2}$. Therefore the solution (1) becomes

$$u(x,t) = \sin(t)\sin(x) + \frac{1}{2}\sin(2t)\sin(2x)$$

Since the Fourier series and the D'Alembert must be the same, then this implies that

$$\sin(t)\sin(x) + \frac{1}{2}\sin(2t)\sin(2x) = \frac{1}{2}(\cos(x-t) - \cos(x+t)) + \frac{1}{4}(\cos(2(x-t)) - \cos(2(x+t)))$$

This was confirmed on the computer as well. In this problem, it turned out that it is easier to use the Fourier method, since the initial velocity was given as a Fourier sine series already.

0.0.4 Problem 4

Problem

Solve Laplace PDE $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ inside annulus $a < r < b$ where $a > 0$. The boundary conditions is $u(a\cos\theta, a\sin\theta) = 0$ and $u(b\cos\theta, b\sin\theta) = f(\theta)$.

solution

Let $u(r,\theta) = R(r)\Theta(\theta)$. Substituting this back into the PDE gives

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

Or

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\Theta''}{\Theta}=\lambda$$

The eigenvalue ODE is

$$\begin{aligned}\Theta''+\lambda\Theta&=0\\ \Theta(0)&=\Theta(2\pi)\\ \Theta'(0)&=\Theta'(2\pi)\end{aligned}$$

The solution to the above is known to be

$$\Theta_n(\theta)=c_n\cos\left(\sqrt{\lambda_n}\theta\right)+k_n\sin\left(\sqrt{\lambda_n}\theta\right)\tag{1}$$

Where $\lambda_n=n^2$ and $n=0,1,2,3,\cdots$. Therefore solution (1) becomes

$$\Theta_n(\theta)=c_n\cos(n\theta)+k_n\sin(n\theta)\qquad n=1,2,3,\cdots\tag{1A}$$
$$\Theta_n(\theta)=c_0\qquad n=0\tag{1B}$$

Therefore the solution to the $\Theta_n(\theta)$ ode is

$$\Theta_n(\theta)=\begin{cases}c_0 & n=0\\ c_n\cos(n\theta)+k_n\sin(n\theta) & n=1,2,3,\cdots\end{cases}$$

The solution to the $R(r)$ ode (this is a Euler ODE) will have <u>two solutions</u>, one when $\lambda_0=0$ when $n=0$ and another solution for $\lambda_n=n^2$ when $n>0$. When eigenvalue is zero, the $R(r)$ ODE becomes

$$\begin{aligned}r^2\frac{R''}{R}+r\frac{R'}{R}&=0\\ r^2R''+rR'&=0\\ rR''+R'&=0\end{aligned}$$

This has the solution

$$R_0(r)=A_0\ln(r)+B_0\tag{2}$$

Applying the boundary conditions $r=a$ to the above gives

$$\begin{aligned}0&=A_0\ln(a)+B_0\\ B_0&=-A_0\ln(a)\end{aligned}$$

Therefore (2) becomes

$$\begin{aligned}R_0(r)&=A_0\ln(r)-A_0\ln(a)\\ &=A_0(\ln(r)-\ln(a))\end{aligned}\tag{3}$$

The above is only for the zero eigenvalue. When $n>0$, the $R(r)$ ode becomes the Euler ODE

$$\begin{aligned}r^2R''+rR'-\lambda_nR&=0\\ r^2R''+rR'-n^2R&=0\end{aligned}$$

The solution to this ODE is

$$R_n(r)=A_nr^n+D_nr^{-n}\tag{4}$$

Here the term D_nr^{-n} does not vanish as the case with the solution to the disk. But using the boundary condition that $u=0$ when $r=a$, the above ODE at $r=a$ becomes

$$\begin{aligned}R_n(a)=0&=A_na^n+D_na^{-n}\\ D_n&=-A_n\frac{a^n}{a^{-n}}\\ &=-A_na^{2n}\end{aligned}$$

Substituting the above back in (4) gives

$$\begin{aligned}R_n(r)&=A_nr^n-A_na^{2n}r^{-n}\\ &=A_n\left(r^n-a^{2n}r^{-n}\right)\end{aligned}\tag{4A}$$

Therefore the solution to the $R(r)$ ode is

$$R_n(r)=\begin{cases}A_0(\ln(r)-\ln(a)) & n=0\\ A_n\left(r^n-a^{2n}r^{-n}\right) & n=1,2,3,\cdots\end{cases}$$

The fundamental solution is

$$
\begin{aligned}
u_n(r,\theta) &= R_n(r)\,\Theta_n(\theta) \\
&= \overbrace{c_0 A_0\left(\ln(r)-\ln(a)\right)}^{\text{zero eigenvalue}} + \overbrace{\left(r^n - a^{2n}r^{-n}\right)\left(c_n\cos(n\theta)+k_n\sin(n\theta)\right)}^{n>0 \text{ eigenvalues}}
\end{aligned}
$$

By superposition, the complete solution is

$$
u(r,\theta) = c_0A_0\left(\ln(r)-\ln(a)\right) + \sum_{n=1}^{\infty} A_n\left(r^n - a^{2n}r^{-n}\right)\left(c_n\cos(n\theta)+k_n\sin(n\theta)\right)
$$

Combining c_0A_0 into c_0 and A_nc_n into c_n and A_nk_n into k_n the above simplifies to

$$
u(r,\theta) = c_0\left(\ln(r)-\ln(a)\right) + \sum_{n=1}^{\infty}\left(r^n - a^{2n}r^{-n}\right)\left(c_n\cos(n\theta)+k_n\sin(n\theta)\right) \tag{5}
$$

Now the boundary condition at $r=b$ is used to determined c_0, c_n and k_n. At $r=b$ and for $\underline{n=0 \text{ case}}$, the above becomes, by orthogonality

$$
\begin{aligned}
\int_0^{2\pi} f(\theta)\,d\theta &= (2\pi)\,c_0\left(\ln(b)-\ln(a)\right) \\
c_0 &= \frac{1}{2\pi\left(\ln(b)-\ln(a)\right)}\int_0^{2\pi} f(\theta)\,d\theta
\end{aligned} \tag{6}
$$

And for $n>0$, solution (5) becomes

$$
f(\theta) = \sum_{n=1}^{\infty}\left(b^n - a^{2n}b^{-n}\right)\left(c_n\cos(n\theta)+k_n\sin(n\theta)\right) \tag{7}
$$

By orthogonality with $\cos(n\theta)$ equation (7) becomes

$$
\begin{aligned}
\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta &= \left(b^n - a^{2n}b^{-n}\right)c_n\pi \\
c_n &= \frac{1}{\left(b^n - a^{2n}b^{-n}\right)\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta
\end{aligned}
$$

And by orthogonality with $\sin(n\theta)$ equation (4) becomes

$$
\begin{aligned}
\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta &= \left(b^n - a^{2n}b^{-n}\right)k_n\pi \\
k_n &= \frac{1}{\left(b^n - a^{2n}b^{-n}\right)\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta
\end{aligned}
$$

This completes the solution. Solution (5) becomes

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\frac{\ln(r)-\ln(a)}{\ln(b)-\ln(a)}\int_0^{2\pi} f(\theta)\,d\theta + \sum_{n=1}^{\infty}\left(r^n - a^{2n}r^{-n}\right)\left(c_n\cos(n\theta)+k_n\sin(n\theta)\right) \\
c_n &= \frac{1}{\left(b^n - a^{2n}b^{-n}\right)\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta \\
k_n &= \frac{1}{\left(b^n - a^{2n}b^{-n}\right)\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta
\end{aligned}
$$