Final Exam

Math 332
Introduction to Partial Differential Equations

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Contents

1 Problem 1

Solve the heat equation

$$u_t = 9u_{xx}$$

For $0 \leq x \leq 1, t \geq 0$ with boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 1$$

And initial conditions

$$u(x,0) = x^2$$

solution

Since the one of the boundary conditions are inhomogeneous, the solution is broken into two parts. Let the solution be

$$u(x,t) = w(x,t) + v(x) \tag{1}$$

Where $w(x,t)$ is the solution to the PDE with homogeneous boundary conditions and $v(x)$ is a reference solution which is only required to satisfy the inhomogeneous boundary condition[1].

Let $v(x) = Ax + B$. At $x = 0$ then $v(0) = 0$ which gives $B = 0$. Hence $v(x) = Ax$. At $x = 1$, $v(1) = 1$ or $A = 1$. Therefore

$$v(x) = x$$

And (1) becomes

$$u(x,t) = w(x,t) + x \tag{1A}$$

Where now $w(x,t)$ satisfies the PDE

$$w_t = 9w_{xx} \tag{1B}$$

For $0 \leq x \leq 1, t \geq 0$ but with the following homogeneous boundary conditions

$$w(0,t) = 0$$
$$w(1,t) = 0$$

And initial conditions given by

$$\begin{aligned} w(x,0) &= u(x,0) - v(x) \\ &= x^2 - x \end{aligned} \tag{2}$$

The PDE (1B) is the heat PDE with homogeneous boundary conditions. This was solved before. It has the solution

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n \alpha^2 t} \sin\left(\sqrt{\lambda_n} x\right)$$

Where in this problem $\alpha^2 = 9$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3\cdots$. But $L = 1$, therefore the above solution reduces to

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-9n^2\pi^2 t} \sin(n\pi x) \tag{3}$$

c_n is now found from the initial conditions (2). At $t = 0$ the above becomes

$$x^2 - x = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$$

Applying orthogonality gives

$$\begin{aligned} \int_0^1 \left(x^2 - x\right) \sin(n\pi x)\, dx &= c_n \int_0^1 \sin^2(n\pi x)\, dx \\ &= c_n \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} c_n &= 2\int_0^1 \left(x^2 - x\right) \sin(n\pi x)\, dx \\ &= 2\left(\int_0^1 x^2 \sin(n\pi x) - \int_0^1 x \sin(n\pi x)\, dx\right) \end{aligned} \tag{3A}$$

[1] $w(x,t)$ is called the transient solution with homogeneous boundary conditions, and $v(x)$ the steady state solution with the inhomogeneous boundary conditions.

Applying the following rule based on integration by parts $\int x^2 \sin(ax) = \frac{2x \sin ax}{a^2} + \left(\frac{2}{a^3} - \frac{x^2}{a}\right) \cos ax$, the first integral above becomes (where $a = n\pi$)

$$\begin{aligned}
\int_0^1 x^2 \sin(n\pi x) &= \left[\frac{2x \sin n\pi x}{(n\pi)^2} + \left(\frac{2}{(n\pi)^3} - \frac{x^2}{n\pi}\right) \cos n\pi x\right]_0^1 \\
&= \left[\frac{2 \sin n\pi}{(n\pi)^2} + \left(\frac{2}{(n\pi)^3} - \frac{1}{n\pi}\right) \cos n\pi - \frac{2}{(n\pi)^3}\right]
\end{aligned}$$

But $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, therefore the above becomes

$$\begin{aligned}
\int_0^1 x^2 \sin(n\pi x) &= \left(\frac{2}{n^3\pi^3} - \frac{1}{n\pi}\right)(-1)^n - \frac{2}{n^3\pi^3} \\
&= \frac{2(-1)^n}{n^3\pi^3} - \frac{(-1)^n}{n\pi} - \frac{2}{n^3\pi^3} \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3\pi^3}\left(-1 + (-1)^n\right) \qquad \text{(3A1)}
\end{aligned}$$

Applying the following rule based on integration by parts $\int x \sin(ax) = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$, then the second integral in (3A) becomes (where $a = n\pi$)

$$\begin{aligned}
\int_0^1 x \sin(n\pi x)\, dx &= \left[\frac{\sin n\pi x}{n^2\pi^2} - \frac{x \cos n\pi x}{n\pi}\right]_0^1 \\
&= \frac{\sin n\pi}{n^2\pi^2} - \frac{\cos n\pi}{n\pi} \\
&= \frac{(-1)^{n+1}}{n\pi} \qquad \text{(3A2)}
\end{aligned}$$

Substituting (3A1) and (3A2) into (3A) gives

$$\begin{aligned}
c_n &= 2\left(\frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3\pi^3}\left(-1 + (-1)^n\right) - \frac{(-1)^{n+1}}{n\pi}\right) \\
&= \frac{4}{n^3\pi^3}\left(-1 + (-1)^n\right)
\end{aligned}$$

Therefore, the solution $w(x,t)$ from (3) becomes

$$w(x,t) = \sum_{n=1}^{\infty} 4 \frac{\left(-1 + (-1)^n\right)}{n^3\pi^3} e^{-9n^2\pi^2 t} \sin(n\pi x)$$

And the solution $u(x,t)$ from (1A) is

$$u(x,t) = x + \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{\left(-1 + (-1)^n\right)}{n^3} e^{-9n^2\pi^2 t} \sin(n\pi x)$$

Only few terms are needed to obtain a very good approximation, since the convergence is of order $O\left(\frac{1}{n^3}\right)$.

2 Problem 2

Solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ on the unit disk $x^2 + y^2 \leq 1$ with boundary conditions $u(\cos\theta, \sin\theta) = \pi^2 - \theta^2$ and $-\pi < \theta \leq \pi$

solution

This is Laplace PDE on disk. Where $a = 1$ is the radius and r, θ are polar coordinates. The Laplacian in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

With boundary conditions on r being

$$\begin{aligned} u(a,\theta) &= f(\theta) = \pi^2 - \theta^2 \\ u(0,\theta) &< \infty \end{aligned}$$

And with standard periodic boundary conditions on θ

$$\begin{aligned} u(r,-\pi) &= u(r,\pi) \\ \frac{\partial u}{\partial\theta}(r,-\pi) &= \frac{\partial u}{\partial\theta}(r,\pi) \end{aligned}$$

This PDE was solved before and the solution to the Laplace PDE inside a disk is known to be

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (c_n \cos n\theta + d_n \sin n\theta) \tag{1}$$

With Fourier coefficients given by

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \\ c_n &= \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \\ d_n &= \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta \end{aligned}$$

Since the radius $a = 1$ in this problem, then the above become

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \\ c_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \\ d_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta \end{aligned}$$

The coefficients are now calculated[2].

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\pi^2 - \theta^2\right) d\theta$$

But $\int_{-\pi}^{\pi}\left(\pi^2-\theta^2\right) d\theta = \int_{-\pi}^{\pi}\pi^2 d\theta - \int_{-\pi}^{\pi}\theta^2 d\theta = 2\pi^3 - \left[\frac{\theta^3}{3}\right]_{-\pi}^{\pi} = 2\pi^3 - \frac{1}{3}\left[\pi^3+\pi^3\right] = 2\pi^3 - \frac{2}{3}\pi^3 = \frac{4}{3}\pi^3$. Therefore

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\left(\frac{4}{3}\pi^3\right) \\ &= \frac{2\pi^2}{3} \end{aligned}$$

And

$$\begin{aligned} c_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\pi^2-\theta^2\right)\cos(n\theta)\, d\theta \\ &= \pi\int_{-\pi}^{\pi}\cos(n\theta)\, d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi}\theta^2\cos(n\theta)\, d\theta \end{aligned}$$

[2] It is important to use integration limit $-\pi \cdots \pi$ and not $0 \cdots 2\pi$.

But $\int_{-\pi}^{\pi} \cos(n\theta)\, d\theta = 0$ and by integration by parts as was done earlier $\int_{-\pi}^{\pi} \theta^2 \cos(n\theta)\, d\theta = \frac{4(-1)^n \pi}{n^2}$, hence the above simplifies to

$$c_n = -\frac{4(-1)^n}{n^2}$$

And

$$\begin{aligned} d_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\pi^2 - \theta^2\right)\sin(n\theta)\, d\theta \\ &= \pi\int_{-\pi}^{\pi} \sin(n\theta)\, d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi} \theta^2 \sin(n\theta)\, d\theta \end{aligned}$$

But $\int_{-\pi}^{\pi} \sin(n\theta)\, d\theta = 0$, and by integration by parts as was done earlier, $\int_{-\pi}^{\pi} \theta^2 \sin(n\theta)\, d\theta = 0$, hence

$$d_n = 0$$

Using the value of A_0, c_n, d_n found above the solution (1) becomes

$$\begin{aligned} u(r,\theta) &= A_0 + \sum_{n=1}^{\infty} r^n \left(c_n \cos n\theta + d_n \sin n\theta\right) \\ &= \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} r^n \cos n\theta \end{aligned}$$

3 Problem 3

Solve the inhomogeneous wave equation

$$u_{tt} = u_{xx} + x \sin t$$

For $0 \leq x \leq 1, t \geq 0$ with boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 0$$

And initial conditions

$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

solution

Since the inhomogeneity is in the PDE itself (rather than in the boundary conditions), then the method of eigenfunction expansion is used to obtain the solution. Let the solution be

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \Phi_n(x) \tag{1}$$

Where $\Phi_n(x)$ are the eigenfunctions of the spatial eigenvalue ODE problem that comes from solving the homogeneous wave equation with the given homogeneous boundary conditions, which is $u_{tt} = u_{xx}$. This wave PDE with the given homogeneous boundary conditions was solved before using separation of variables. The eigenfunctions were found to be

$$\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right) \qquad n = 1,2,3,\cdots$$

With eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots$$

But $L = 1$ here, therefore

$$\lambda_n = n^2\pi^2 \qquad n = 1,2,3,\cdots$$

Now that the eigenvalues and eigenfunctions are found, equation (1) is substituted back into the PDE resulting in

$$\sum_{n=1}^{\infty} b_n''(t) \Phi_n(x) = \sum_{n=1}^{\infty} b_n(t) \Phi_n''(x) + x \sin t$$

Since $x \sin t$ is a piecewise continuous function in x, it can be represented using the same eigenfunctions[3] and the above equation becomes

$$\sum_{n=1}^{\infty} b_n''(t) \Phi_n(x) = \sum_{n=1}^{\infty} b_n(t) \Phi_n''(x) + \sum_{n=1}^{\infty} \gamma_n(t) \Phi_n(x)$$

Since $\Phi_n''(x) = -\lambda_n \Phi_n(x)$, which comes from the eigenvalue ODE, the above simplifies to

$$\sum_{n=1}^{\infty} b_n''(t) \Phi_n(x) = \sum_{n=1}^{\infty} -b_n(t) \lambda_n \Phi_n(x) + \sum_{n=1}^{\infty} \gamma_n(t) \Phi_n(x)$$
$$b_n''(t) + b_n(t) \lambda_n = \gamma_n(t) \tag{2}$$

To solve the above ODE for $b_n(t)$, $\gamma_n(t)$ needs to be found first. Using

$$x \sin t = \sum_{n=1}^{\infty} \gamma_n(t) \Phi_n(x)$$

Applying orthogonality gives

$$\sin(t) \int_0^1 x \Phi_n(x)\, dx = \gamma_n(t) \int_0^1 \Phi_n^2(x)\, dx$$

Since $\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right)$, then $\int_0^1 \Phi_n^2(x)\, dx = \frac{1}{2}$ and the above reduces to

$$\sin(t) \int_0^1 x \sin(n\pi x)\, dx = \frac{1}{2} \gamma_n(t)$$

[3] This is the same as saying the eigenfunctions are complete.

But $\int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^{n+1}}{n\pi}$ by integration by part as was done before, and the above becomes

$$\begin{aligned}\frac{(-1)^{n+1}}{n\pi}\sin(t) &= \frac{1}{2}\gamma_n(t)\\ \gamma_n(t) &= 2\frac{(-1)^{n+1}}{n\pi}\sin(t)\end{aligned}$$

Using the above back in (2), ODE (2) now becomes

$$b_n''(t) + b_n(t)\,n^2\pi^2 = 2\frac{(-1)^{n+1}}{n\pi}\sin(t) \tag{3}$$

This is a second order, inhomogeneous, linear, with constant coefficients ODE. The solution is the sum of the homogeneous and particular solutions (the subscript n is removed for now from $b_n(t)$, to simplify the notations, then added back after the solution is obtained). Let the solution to (3) be

$$b(t) = b_h(t) + b_p(t)$$

The homogeneous solution is seen to be (since $n^2\pi^2$ is always positive)

$$b_h(t) = A\cos(n\pi t) + B\sin(n\pi t)$$

To find the particular solution, the method of undetermined coefficients is used. let

$$b_p(t) = C\cos(t) + D\sin(t) \tag{4}$$

Hence $b_p' = -C\sin(t) + D\cos(t)\,,\, b_p'' = -C\cos(t) - D\sin(t)$. Substituting these into (3) gives

$$\begin{aligned}-C\cos(t) - D\sin(t) + (C\cos(t) + D\sin(t))\,n^2\pi^2 &= 2\frac{(-1)^{n+1}}{n\pi}\sin(t)\\ \cos(t)\left[-C + Cn^2\pi^2\right] + \sin(t)\left[-D + Dn^2\pi^2\right] &= 2\frac{(-1)^{n+1}}{n\pi}\sin(t)\end{aligned}$$

Therefore, by comparing coefficients

$$\begin{aligned}-C + Cn^2\pi^2 &= 0\\ C\left(n^2\pi^2 - 1\right) &= 0\\ C &= 0\end{aligned}$$

And

$$\begin{aligned}-D + Dn^2\pi^2 &= 2\frac{(-1)^{n+1}}{n\pi}\\ D\left(n^2\pi^2 - 1\right) &= 2\frac{(-1)^{n+1}}{n\pi}\\ D &= 2\frac{(-1)^{n+1}}{n\pi\,(n^2\pi^2 - 1)}\end{aligned}$$

Hence the particular solution (4) is

$$\begin{aligned}b_p(t) &= C\cos(t) + D\sin(t)\\ &= 2\frac{(-1)^{n+1}}{n\pi\,(n^2\pi^2 - 1)}\sin(t)\end{aligned}$$

Now that the particular solution is found, the final solution to (3) becomes

$$b_n(t) = A_n\cos(n\pi t) + B_n\sin(n\pi t) - 2\frac{(-1)^n}{n\pi\,(n^2\pi^2 - 1)}\sin(t) \tag{4}$$

Using the above in the solution (1) gives

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} b_n(t)\,\Phi_n(x)\\ &= \sum_{n=1}^{\infty}\left(A_n\cos(n\pi t) + B_n\sin(n\pi t) - 2\frac{(-1)^n}{n\pi\,(n^2\pi^2 - 1)}\sin(t)\right)\sin(n\pi x)\end{aligned}$$

A_n, B_n are found from initial conditions. At $t = 0$ the above simplifies to

$$0 = \sum_{n=1}^{\infty} A_n\sin(n\pi x)$$

Therefore $A_n = 0$ and the solution above reduces to

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \sin(n\pi t) - 2\frac{(-1)^n}{n\pi(n^2\pi^2 - 1)} \sin(t) \right) \sin(n\pi x) \tag{5}$$

Taking time derivative gives

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(B_n n\pi \cos(n\pi t) - 2\frac{(-1)^n}{n\pi(n^2\pi^2 - 1)} \cos(t) \right) \sin(n\pi x)$$

At $t = 0$ the above simplifies to

$$0 = \sum_{n=1}^{\infty} \left(B_n n\pi - 2\frac{(-1)^n}{n\pi(n^2\pi^2 - 1)} \right) \sin(n\pi x)$$

Since this is valid for each n, then $\left(B_n n\pi - 2\frac{(-1)^n}{n\pi(n^2\pi^2-1)} \right) = 0$ or

$$B_n = 2\frac{(-1)^n}{n^2\pi^2(n^2\pi^2 - 1)}$$

Using the above in (5), the final solution becomes

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} 2 \left(\frac{(-1)^n}{n^2\pi^2(n^2\pi^2 - 1)} \sin(n\pi t) - \frac{(-1)^n}{n\pi(n^2\pi^2 - 1)} \sin(t) \right) \sin(n\pi x) \\ &= 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi(n^2\pi^2 - 1)} \left(\frac{\sin(n\pi t)}{n\pi} - \sin(t) \right) \sin(n\pi x) \end{aligned}$$

4 Problem 4

Solve the wave equation

$$u_{tt} = u_{xx} + u_{yy}$$

On the unit disk $x^2 + y^2 \leq 1$ with with boundary conditions

$$u(x, y) = 0 \qquad \text{if } x^2 + y^2 = 1$$

And initial conditions

$$\begin{aligned} u(x, y, 0) &= 0 \\ u_t(x, y, 0) &= \begin{cases} \frac{1}{\pi\epsilon^2} & \text{if } \sqrt{x^2 + y^2} \leq \epsilon \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Where $0 < \epsilon < 1$. Hint: The formula $\frac{d}{dx}(xJ_1(x)) = xJ_0(x)$ may be used. Extra credit: Plot the solution $u(r, t)$ for $\epsilon = \frac{1}{2}$, $t = 1$ and $t = 2$.

solution

The PDE and initial and boundary conditions are converted to polar coordinates to become

$$u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \tag{1}$$

On the unit disk with radius 1. The boundary conditions are

$$\begin{aligned} u(1, \theta, t) &= 0 \\ u(0, \theta, t) &< \infty \end{aligned}$$

Where $u(0, \theta, t) < \infty$ means the solution is bounded at center of disk $r = 0$. The boundary conditions on θ are the standard periodic boundary conditions

$$\begin{aligned} u(r, -\pi, t) &= u(r, \pi, t) \\ u_\theta(r, -\pi, t) &= u_\theta(r, \pi, t) \end{aligned}$$

And initial conditions are[4]

$$\begin{aligned} u(r, \theta, 0) &= 0 \\ u_t(r, \theta, 0) &= \begin{cases} \frac{1}{\pi\epsilon^2} & \text{if } r \leq \epsilon \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The above PDE is solved by separation of variables. Let $u(r, \theta, t) = T(t)R(r)\Theta(\theta)$. Substituting this in the PDE (1) gives

$$T''R\Theta = R''T\Theta + \frac{1}{r}R'T\Theta + \frac{1}{r^2}\Theta''RT$$

Dividing by $RT\Theta$

$$\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2$$

Where λ is the first separation variable. This results in two equations

$$\frac{T''}{T} = -\lambda^2 \tag{1}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2 \tag{2}$$

The time ODE (1) is

$$T'' + \lambda^2 T = 0 \tag{1A}$$

Multiplying (2) by r^2 and rearranging

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = -\frac{\Theta''}{\Theta} = \mu^2$$

Where μ is the second separation constant. This gives the R ODE as

$$r^2R'' + rR' + \left(r^2\lambda^2 - \mu^2\right)R = 0 \tag{3}$$

And the Θ ODE as

$$\Theta'' + \mu^2\Theta = 0 \tag{4}$$

[4] The original $r^2 \leq \epsilon$ was changed to $r \leq \epsilon$

The eigenvalues for (4) determine the Bessel equation (3) order. Therefore (4) needs to be solved first to determined the order. The ODE boundary conditions for (4) are periodic

$$\begin{aligned}\Theta(-\pi) &= \Theta(\pi)\\ \Theta'(-\pi) &= \Theta'(\pi)\end{aligned}$$

case $\mu = 0$. This leads to solution

$$\begin{aligned}\Theta &= c_1\theta + c_2\\ \Theta' &= c_1\end{aligned}$$

First BC gives

$$\begin{aligned}-c_1\pi + c_2 &= c_1\pi + c_2\\ c_1 &= 0\end{aligned}$$

And since second BC $\Theta' = c_1$, this implies $\Theta(\theta)$ is constant. So $\mu = 0$ is an eigenvalue, with $\Theta_0(\theta) = 1$ being the eigenfunction.

Case $\mu > 0$ The solution to (4) becomes

$$\Theta(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta)$$

To satisfy the periodic boundary conditions, μ must be an integer, and since $\mu > 0$, then $\mu = n$ for $n = 1, 2, 3, \cdots$. Therefore

$$\Theta_0(\theta) = 1 \qquad n = 0 \tag{5A}$$

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta \qquad n = 1, 2, 3, \cdots \tag{5B}$$

The above solution can be combined to one

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta \qquad n = 0, 1, 2, \cdots \tag{5}$$

Because when $n = 0$ the above solution gives $\Theta_0(\theta) = A_0$ which is the constant eigenfunction. Now that μ is found, Bessel ODE (3) can be solved.

$$\begin{aligned}r^2R''(r) + rR'(r) + \left(r^2\lambda^2 - n^2\right)R(r) &= 0 \qquad n = 0, 1, 2, 3, \cdots &\text{(5C)}\\ R(1) &= 0\\ R(0) &< \infty\end{aligned}$$

$\lambda = 0$ is not a possible eigenvalue. This can be shown as follows. When $\lambda = 0$ equation (5C) becomes the Euler ODE

$$r^2R''(r) + rR'(r) + n^2R(r) = 0 \qquad n = 0, 1, 2, 3, \cdots$$

Now, when $n = 0$, then the ODE becomes $r^2R''(r) + rR'(r) = 0$ whose solution is $R(r) = c_1 + c_2\ln(r)$. Since solution is bounded at $r = 0$, then $R(r) = c_1$. And since $R(1) = 0$ then $c_1 = 0$ also, leading to trivial solution. When $n > 0$, the ODE becomes $r^2R''(r) + rR'(r) + n^2R(r) = 0$ whose solution is $R(r) = c_1r^n + c_2\frac{1}{r^n}$. Since solution is bounded at $r = 0$, then $c_2 = 0$ and the solution now becomes $R(r) = c_1r^n$. Using BC $R(1) = 0$ gives $c_1 = 0$ leading again to trivial solution. This shows that $\lambda = 0$ is not eigenvalue.

Now that λ is is shown not to be zero, the Bessel ODE (5C) is solved . The first step is to convert the ODE to a Bessel ODE in the classical form in order to use the standard solution. Let $t = \lambda r$, then $R'(r) = R'(t)\lambda$ and $R''(r) = R''(t)\lambda^2$. ODE (5C) becomes

$$\begin{aligned}\frac{t^2}{\lambda^2}\lambda^2R''(t) + \frac{t}{\lambda}\lambda R'(t) + \left(\frac{t^2}{\lambda^2}\lambda^2 - n^2\right)R(t) &= 0\\ t^2R''(t) + tR'(t) + \left(t^2 - n^2\right)R(t) &= 0\end{aligned}$$

This is now in standard Bessel ODE form. This is of order n, where n is $n = 0, 1, 2, 3, \cdots$. Since the order is integer, then the solution is given by

$$R_n(t) = C_nJ_n(t) + D_nY_n(t)$$

Where $J_n(t)$ is the Bessel function of order n and $Y_n(t)$ is the Bessel function of second kind of order n. In terms of r the above solution becomes

$$R_n(r) = C_nJ_n(\lambda r) + D_nY_n(\lambda r)$$

Because the solution is bounded at $r = 0$ and since $Y_n(0)$ blows up, then $D_n = 0$. The above solution simplifies to

$$R_n(r) = C_nJ_n(\lambda r)$$

Applying the second boundary conditions, when $r = 1$ then

$$0 = C_n J_n(\lambda)$$

For non-trivial solution $J_n(\lambda) = 0$. Hence λ are the positive zeros of $J_n(z)$. Let the positive zeros of $J_n(z)$ be j_{nm}. For $m = 1, 2, 3, \cdots$. Therefore

$$\lambda_{nm} = j_{nm} \qquad n = 0, 1, 2, \cdots, m = 1, 2, 3, \cdots$$

This means that j_{nm} is the m^{th} eigenvalue for the n^{th} order Bessel function $J_n(z)$. So there are two indices to handle in this problem. The order of the Bessel function is determined from the $\Theta_n(\theta)$ eigenvalues, and then once this order n is fixed, the second eigenvalue λ_{nm} is determined from the zeros of the Bessel function $J_n(z)$. Hence the $R_{nm}(r)$ solution is

$$R_{nm}(r) = C_{nm} J_n(\lambda_{nm} r) \qquad n = 0, 1, 2, 3, \cdots, m = 1, 2, 3, \cdots$$

Now that λ_{nm} is known, the time ODE (1) can be solved

$$\begin{aligned} T''_{nm} + \lambda_{nm}^2 T_{nm} &= 0 \\ T_{nm} &= A_{nm}\cos(\lambda_{nm} t) + B_{nm}\sin(\lambda_{nm} t) \qquad n = 0, 1, 2, 3, \cdots, m = 1, 2, 3, \cdots \end{aligned}$$

The fundamental solution is therefore

$$u_{nm}(r, \theta, t) = \Theta_n(\theta) T_{nm}(t) R_{nm}(r)$$

The complete solution is the superposition of the fundamental solutions given by

$$\begin{aligned} u(r, \theta, t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \Theta_n(\theta) T_{nm}(t) R_{nm}(r) \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)\{A_{nm}\cos(\lambda_{nm} t) + B_{nm}\sin(\lambda_{nm} t)\} C_{nm} J_n(\lambda_{nm} r) \end{aligned}$$

The above can now be written as

$$\begin{aligned} u(r, \theta, t) = &\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n \cos n\theta\,((A_{nm}\cos(\lambda_{nm} t) + B_{nm}\sin(\lambda_{nm} t)) C_{nm} J_n(\lambda_{nm} r)) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n \sin n\theta\,((A_{nm}\cos(\lambda_{nm} t) + B_{nm}\sin(\lambda_{nm} t)) C_{nm} J_n(\lambda_{nm} r)) \end{aligned}$$

Or

$$\begin{aligned} u(r, \theta, t) = &\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n \cos n\theta A_{nm}\cos(\lambda_{nm} t) C_{nm} J_n(\lambda_{nm} r) \\ &+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_n \cos n\theta B_{nm}\sin(\lambda_{nm} t) C_{nm} J_n(\lambda_{nm} r) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n \sin n\theta A_{nm}\cos(\lambda_{nm} t) C_{nm} J_n(\lambda_{nm} r) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n \sin n\theta B_{nm}\sin(\lambda_{nm} t) C_{nm} J_n(\lambda_{nm} r) \end{aligned} \tag{6}$$

Constants are now merged and renamed as follows in order to simplify the rest of the solution. Let

$$\begin{aligned} A_n A_{nm} C_{nm} &= \bar{A}_{nm} \\ A_n B_{nm} C_{nm} &= \bar{B}_{nm} \\ B_n A_{nm} C_{nm} &= \bar{C}_{nm} \\ B_n B_{nm} C_{nm} &= \bar{D}_{nm} \end{aligned}$$

Equation (6) can now be written as

$$\begin{aligned} u(r, \theta, t) = &\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{A}_{nm}\cos(n\theta)\cos(\lambda_{nm} t) J_n(\lambda_{nm} r) \\ &+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{B}_{nm}\cos(n\theta)\sin(\lambda_{nm} t) J_n(\lambda_{nm} r) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{nm}\sin(n\theta)\cos(\lambda_{nm} t) J_n(\lambda_{nm} r) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \bar{D}_{nm}\sin(n\theta)\sin(\lambda_{nm} t) J_n(\lambda_{nm} r) \end{aligned} \tag{7}$$

Initial conditions are used to determine the 4 new constants above. Using initial condition at $t=0, u(r,\theta,0)=0$ the above equation becomes

$$0=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\bar{A}_{nm}\cos(n\theta)J_n(\lambda_{nm}r)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{C}_{nm}\sin(n\theta)J_n(\lambda_{nm}r)$$

Applying orthogonality on $\cos(n\theta)$ and $\sin(n\theta)$ in turn shows that $\bar{A}_{nm}=0$ and $\bar{C}_{nm}=0$. Therefore the solution (7) reduces to the following two sums only

$$u(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\bar{B}_{nm}\cos(n\theta)\sin(\lambda_{nm}t)J_n(\lambda_{nm}r)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{D}_{nm}\sin(n\theta)\sin(\lambda_{nm}t)J_n(\lambda_{nm}r) \tag{8}$$

Taking time derivative gives

$$u_t(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\bar{B}_{nm}\cos(n\theta)\lambda_{nm}\cos(\lambda_{nm}t)J_n(\lambda_{nm}r)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{D}_{nm}\sin(n\theta)\lambda_{nm}\cos(\lambda_{nm}t)J_n(\lambda_{nm}r)$$

Applying the second initial condition at $t=0$ gives

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\bar{B}_{nm}\cos(n\theta)\lambda_{nm}J_n(\lambda_{nm}r)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{D}_{nm}\sin(n\theta)\lambda_{nm}J_n(\lambda_{nm}r)=\begin{cases}\frac{1}{\pi\epsilon^2} & \text{if } r\leq\epsilon\\ 0 & \text{otherwise}\end{cases} \tag{9}$$

Case $n=0$ (9) becomes

$$\sum_{m=1}^{\infty}\bar{B}_{0m}\lambda_{0m}J_0(\lambda_{0m}r)=\begin{cases}\frac{1}{\pi\epsilon^2} & \text{if } r\leq\epsilon\\ 0 & \text{otherwise}\end{cases}$$

Applying orthogonality on $J_0(\lambda_{0m}r)$ results in

$$\bar{B}_{0m}\lambda_{0m}\int_0^1 rJ_0^2(\lambda_{0m}r)\,dr=\frac{1}{\pi\epsilon^2}\int_0^{\epsilon}rJ_0(\lambda_{0m}r)\,dr$$

$$\bar{B}_{0m}=\frac{1}{\pi\epsilon^2\lambda_{0m}}\frac{\int_0^{\epsilon}rJ_0(\lambda_{0m}r)\,dr}{\int_0^1 rJ_0^2(\lambda_{0m}r)\,dr} \tag{9A}$$

Case $n>1$ Applying orthogonality on $\cos(n\theta)$, equation (9) becomes

$$\sum_{m=1}^{\infty}\bar{B}_{nm}\left(\int_{-\pi}^{\pi}\cos^2(n\theta)\,d\theta\right)\lambda_{nm}J_n(\lambda_{nm}r)=\begin{cases}\frac{1}{\pi\epsilon^2}\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta & \text{if } r^2\leq\epsilon\\ 0 & \text{otherwise}\end{cases}$$

$$\sum_{m=1}^{\infty}\pi\bar{B}_{nm}\lambda_{nm}J_n(\lambda_{nm}r)=\begin{cases}0 & \text{if } r^2\leq\epsilon\\ 0 & \text{otherwise}\end{cases}$$

Hence $\bar{B}_{nm}=0$ for all $n>0$.

The same is now done to find $\bar{D}_{nm}$. Applying orthogonality on $\sin(n\theta)$, equation (9) becomes

$$\sum_{m=1}^{\infty}\bar{D}_{nm}\left(\int_{-\pi}^{\pi}\sin^2(n\theta)\,d\theta\right)\lambda_{nm}J_n(\lambda_{nm}r)=\begin{cases}\frac{1}{\pi\epsilon^2}\int_{-\pi}^{\pi}\sin(n\theta)\,d\theta & \text{if } r^2\leq\epsilon\\ 0 & \text{otherwise}\end{cases}$$

$$\sum_{m=1}^{\infty}\bar{D}_{nm}\left(\int_{-\pi}^{\pi}\sin^2(n\theta)\,d\theta\right)\lambda_{nm}J_n(\lambda_{nm}r)=\begin{cases}0 & \text{if } r^2\leq\epsilon\\ 0 & \text{otherwise}\end{cases}$$

Hence all $\bar{D}_{nm}=0$ for all $n>0$.

Therefore the solution (8) reduces to only using $n=0, m=1,2,3,\cdots$. The solution can now be written as

$$u(r,\theta,t)=\sum_{m=1}^{\infty}\bar{B}_{0m}\sin(\lambda_{0m}t)J_0(\lambda_{0m}r) \tag{10}$$

Where $\bar{B}_{0m}=\frac{1}{\pi\epsilon^2\lambda_{0m}}\frac{\int_0^{\epsilon}rJ_0(\lambda_{0m}r)dr}{\int_0^1 rJ_0^2(\lambda_{0m}r)dr}$ And λ_{0m} are all the positive zeros of $J_0(z)$, $m=1,2,3,\cdots$.

$\bar{B}_{0m}$ is now simplified more. Considering first the numerator of $\bar{B}_{0m}$ which is $\int_0^{\epsilon}rJ_0(\lambda_{0m}r)\,dr$. The hint given says that

$$\frac{d}{dr}(rJ_1(r))=rJ_0(r)$$

This is the same as saying

$$rJ_1(r)=\int rJ_0(r)\,dr \tag{10A}$$

However the integral in $\bar{B}_{0m}$ is $\int rJ_0(\lambda_{0m}r)\,dr$ and not $\int rJ_0(r)\,dr$. To transform it so that the hint can be used, let $\lambda_{0m}r=\bar{r}$, then $\frac{dr}{d\bar{r}}=\frac{1}{\lambda_{0m}}$ or $dr=\frac{d\bar{r}}{\lambda_{0m}}$. Now $\int rJ_0(\lambda_{0m}r)\,dr$ becomes $\int\frac{\bar{r}}{\lambda_{0m}}J_0(\bar{r})\frac{d\bar{r}}{\lambda_{0m}}$ or $\frac{1}{\lambda_{0m}^2}\int\bar{r}J_0(\bar{r})\,d\bar{r}$ and now the hint (10A) can be used on this integral giving

$$\frac{1}{\lambda_{0m}^2}\left(\int\bar{r}J_0(\bar{r})\,d\bar{r}\right)=\frac{1}{\lambda_{0m}^2}(\bar{r}J_1(\bar{r}))$$

Replacing $\bar{r}$ back by $\lambda_{0m}r$, gives the result needed

$$\begin{aligned}\frac{1}{\lambda_{0m}^2}\left(\bar{r}J_1\left(\bar{r}\right)\right) &= \frac{1}{\lambda_{0m}^2}\left(\lambda_{0m}rJ_1\left(\lambda_{0m}r\right)\right)\\ &= \frac{1}{\lambda_{0m}}rJ_1\left(\lambda_{0m}r\right)\end{aligned}$$

Now the limits are applied, using the fundamental theory of calculus

$$\begin{aligned}\int_0^{\epsilon} rJ_0\left(\lambda_{0m}r\right)dr &= \frac{1}{\lambda_{0m}}\left[rJ_1\left(\lambda_{0m}r\right)\right]_0^{\epsilon}\\ &= \frac{\epsilon}{\lambda_{0m}}J_1\left(\lambda_{0m}\epsilon\right) \qquad \text{(10B)}\end{aligned}$$

This completes finding the numerator integral in $\bar{B}_{0m}$. The denominator integral in $\bar{B}_{0m}$ is $\int_0^1 rJ_0^2\left(\lambda_{0m}r\right)dr$. This was found in HW4, from problem 3, which is

$$\int_0^1 rJ_0^2\left(\lambda_{0m}r\right)dr = \frac{1}{2}\left[J_0'\left(\lambda_{0m}\right)\right]^2$$

But $J_0'\left(\lambda_{0m}\right) = -J_1\left(\lambda_{0m}\right)$, hence the above becomes

$$\int_0^1 rJ_0^2\left(\lambda_{0m}r\right)dr = \frac{1}{2}J_1^2\left(\lambda_{0m}\right) \qquad \text{(10C)}$$

Applying (10B) and (10C), $\bar{B}_{0m}$ simplifies to the following expression

$$\begin{aligned}\bar{B}_{0m} &= \frac{1}{\pi\epsilon^2\lambda_{0m}}\frac{\frac{\epsilon}{\lambda_{0m}}J_1\left(\lambda_{0m}\epsilon\right)}{\frac{1}{2}J_1^2\left(\lambda_{0m}\right)}\\ &= \frac{2}{\pi\epsilon\lambda_{0m}^2}\frac{J_1\left(\lambda_{0m}\epsilon\right)}{J_1^2\left(\lambda_{0m}\right)}\end{aligned}$$

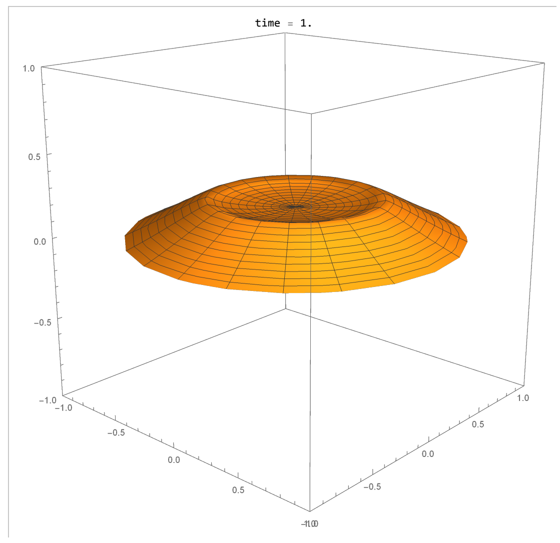
Therefore the final solution becomes

$$\begin{aligned}u\left(r,\theta,t\right) &= \sum_{m=1}^{\infty}\bar{B}_{0m}\sin\left(\lambda_{0m}t\right)J_0\left(\lambda_{0m}r\right)\\ u\left(r,\theta,t\right) &= \frac{2}{\pi\epsilon}\sum_{m=1}^{\infty}\frac{1}{\lambda_{0m}^2}\frac{J_1\left(\lambda_{0m}\epsilon\right)}{J_1^2\left(\lambda_{0m}\right)}J_0\left(\lambda_{0m}r\right)\sin\left(\lambda_{0m}t\right) \qquad \text{(11)}\end{aligned}$$

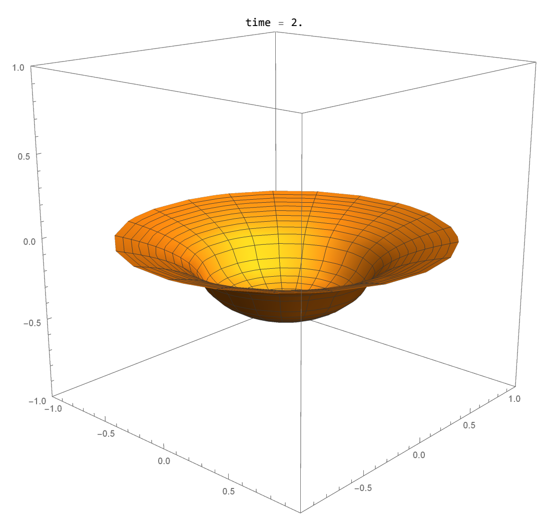
Plotting. When $\epsilon = \frac{1}{2}$, the above solution (11) becomes

$$u(r,\theta,t) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{\lambda_{0m}^2}\frac{J_1\left(\frac{1}{2}\lambda_{0m}\right)}{J_1^2\left(\lambda_{0m}\right)}J_0\left(\lambda_{0m}r\right)\sin\left(\lambda_{0m}t\right) \tag{11A}$$

This is the 3D plot at $t = 1$ second



This is the 3D plot at $t = 2$ seconds



5 Problem 5

Find the radial eigenfunctions and corresponding eigenvalues of the Laplace operator on the unit ball subject to Dirichlet boundary conditions. A radial eigenfunction is one which depends only on $r = \sqrt{x^2 + y^2 + z^2}$. That is, solve

$$u_{xx} + u_{yy} + u_{zz} + \lambda^2 u = 0$$

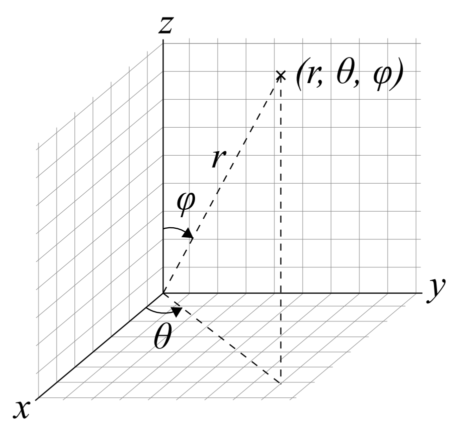
Where $u(x,y,z) = R(r)$ with boundary conditions $u(x,y,z) = 0$ when $x^2 + y^2 + z^2 = 1$.

Hint: The substitution $rR(r) = \bar{R}(r)$ is useful.

solution

This is Helmholtz PDE $\nabla^2 u + \lambda^2 u = 0$ in 3D. (Steady state of the wave equation, or standing waves).

The following spherical coordinates system are used [5]



The Laplace operator in 3D using spherical coordinates (r,θ,ϕ) is given by

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

Therefore $\nabla^2 u + \lambda^2 u = 0$ becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \lambda^2 u = 0$$

The problem says that $u(x,y,z) = R(r)$. This implies that solution depends only on r. This means there is no dependency on θ nor on ϕ. In this case, the PDE above simplifies to an ODE in r only.

$$\begin{aligned}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + \lambda^2 u &= 0\\ \frac{d}{dr}\left(r^2\frac{du}{dr}\right) + \lambda^2 r^2 u &= 0\\ r^2\frac{d^2u}{dr^2} + 2r\frac{du}{dr} + \lambda^2 r^2 u &= 0\end{aligned}$$

And since $u(r,\theta,\phi) \equiv R(r)$, then the above can be written as

$$r^2R''(r) + 2rR'(r) + \lambda^2 r^2 R(r) = 0 \tag{1}$$

With the boundary conditions $R(1) = 0$. Now the eigenvalue will be found.

case $\lambda = 0$

The ODE (1) becomes $r^2R'' + 2rR' = 0$. Let $R'(r) = v(r)$, and the ODE becomes $v' + \frac{2}{r}v = 0$. The integrating factor is $e^{\int \frac{2}{r}dr} = e^{2\ln|r|} = r^2$. $\frac{d}{dr}(r^2 v) = 0$ or $v = \frac{c_1}{r^2}$. Therefore $R'(r) = \frac{c_1}{r^2}$. Integrating again gives

$$R(r) = c_2 - \frac{c_1}{r}$$

At $R(1) = 0$, the above becomes

$$\begin{aligned}0 &= c_2 - c_1\\ c_2 &= c_1\end{aligned}$$

[5]Image obtained from Wikepedia

Hence the solution becomes

$$R(r) = c_1\left(1 - \frac{1}{r}\right)$$

The solution must be bounded as $r \longrightarrow 0$, therefore only choice is $c_1 = 0$, leading to trivial solution. Therefore $\lambda = 0$ is not eigenvalue.

Case $\lambda \neq 0$ [6]

The ODE is

$$r^2R''(r) + 2rR'(r) + \lambda^2 r^2 R(r) = 0$$

Using standard transformation $t = \lambda r$, then $R'(r) = \lambda R'(t)$ and $R''(r) = \lambda^2 R''(t)$. The above ODE becomes

$$\begin{aligned}\lambda^2 r^2 R''(t) + 2\lambda r R'(t) + \lambda^2 r^2 R(t) &= 0\\ t^2R''(t) + 2tR'(t) + t^2R(t) &= 0 \qquad (2)\end{aligned}$$

This looks like a Bessel ODE of zero order, except Bessel ODE is $t^2R''(t)+tR'(t)+t^2R(t) = 0$. The difference is (2) has $2t$ instead of t. To convert it to Bessel ODE, there is another transformation in the dependent variable to achieve this. Let $R(t) = \frac{Z(t)}{\sqrt{t}}$, then

$$\begin{aligned}R'(t) &= \frac{Z'(t)}{\sqrt{t}} - \frac{1}{2}Z(t)\frac{1}{t^{\frac{3}{2}}} \qquad (3)\\ R''(t) &= \frac{Z''(t)}{\sqrt{t}} - \frac{1}{2}Z'(t)\frac{1}{t^{\frac{3}{2}}} - \frac{1}{2}Z'(t)\frac{1}{t^{\frac{3}{2}}} - \frac{1}{2}\left(-\frac{3}{2}\right)Z(t)\frac{1}{t^{\frac{5}{2}}} \qquad (4)\\ &= \frac{Z''(t)}{\sqrt{t}} - Z'(t)\frac{1}{t^{\frac{3}{2}}} + \frac{3}{4}Z(t)\frac{1}{t^{\frac{5}{2}}}\end{aligned}$$

Substituting (3,4) back in (2) gives

$$t^2\left(\frac{Z''(t)}{\sqrt{t}} - Z'(t)\frac{1}{t^{\frac{3}{2}}} + \frac{3}{4}Z(t)\frac{1}{t^{\frac{5}{2}}}\right) + 2t\left(\frac{Z'(t)}{\sqrt{t}} - \frac{1}{2}Z(t)\frac{1}{t^{\frac{3}{2}}}\right) + t^2\frac{Z(t)}{\sqrt{t}} = 0$$

Multiplying by $\sqrt{t}$ gives

$$\begin{aligned}t^2\left(Z''(t) - Z'(t)\frac{1}{t} + \frac{3}{4}Z(t)\frac{1}{t^2}\right) + 2t\left(Z'(t) - \frac{1}{2}Z(t)\frac{1}{t}\right) + t^2Z(t) &= 0\\ \left(t^2Z''(t) - tZ'(t) + \frac{3}{4}Z(t)\right) + (2tZ'(t) - Z(t)) + t^2Z(t) &= 0\\ t^2Z''(t) + tZ'(t) + \frac{3}{4}Z(t) - Z(t) + t^2Z(t) &= 0\\ t^2Z''(t) + tZ'(t) + \left(\frac{3}{4} - 1 + t^2\right)Z(t) &= 0\\ t^2Z''(t) + tZ'(t) + \left(t^2 - \frac{1}{4}\right)Z(t) &= 0\end{aligned}$$

Or

$$t^2Z''(t) + tZ'(t) + \left(t^2 - \frac{1}{4}\right)Z(t) = 0$$

This is now in standard Bessel ODE form. To find the order, comparing it to $t^2Z''(t) + tZ'(t) + \left(t^2 - n^2\right)Z(t) = 0$ shows that $n^2 = \frac{1}{4}$, hence the order is $\frac{1}{2}$. (the negative root, give Bessel function that blow up at zero. Therefore only $\frac{1}{2}$ root is used as the order. The solution of the above Bessel ODE is known to be

$$Z(t) = c_1J_{\frac{1}{2}}(t) + c_2Y_{\frac{1}{2}}(t)$$

From above, $R(t) = \frac{Z(t)}{\sqrt{t}}$. Therefore the solution now becomes

$$R(t) = c_1\frac{J_{\frac{1}{2}}(t)}{\sqrt{t}} + c_2\frac{Y_{\frac{1}{2}}(t)}{\sqrt{t}}$$

And converting back to $R(r)$ finally gives the radial solution as

$$R(r) = c_1\frac{J_{\frac{1}{2}}(\lambda r)}{\sqrt{\lambda r}} + c_2\frac{Y_{\frac{1}{2}}(\lambda r)}{\sqrt{\lambda r}}$$

Since the solution is bounded at $r = 0$, then $c_2 = 0$ and the solution simplifies to

$$R(r) = c_1\frac{J_{\frac{1}{2}}(\lambda r)}{\sqrt{\lambda r}} \qquad (5)$$

[6] I am assuming λ is real eigenvalue. Not complex.

Using $R(1) = 0$ gives

$$0 = c_1 \frac{J_{\frac{1}{2}}(\lambda)}{\sqrt{\lambda}}$$

For non-trivial solution then

$$J_{\frac{1}{2}}(\lambda) = 0$$

Hence λ are the positive zeros of $J_{\frac{1}{2}}(\lambda)$. These are the eigenvalues. The zeros of $J_{\frac{1}{2}}(\lambda)$ are multiple of π. Hence the first zero is π, the second zero is 2π and so on.

$$\lambda_n = n\pi \qquad n = 1, 2, 3, \cdots$$

Therefore, the eigenfunctions (5) becomes

$$R_n(r) = \sqrt{\frac{1}{n\pi r}} J_{\frac{1}{2}}(n\pi r) \qquad n = 1, 2, 3, \cdots \tag{6}$$

These are also called spherical Bessel functions, since half integer order. There is a known relation between spherical Bessel functions and circular trigonometric functions which says

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$$

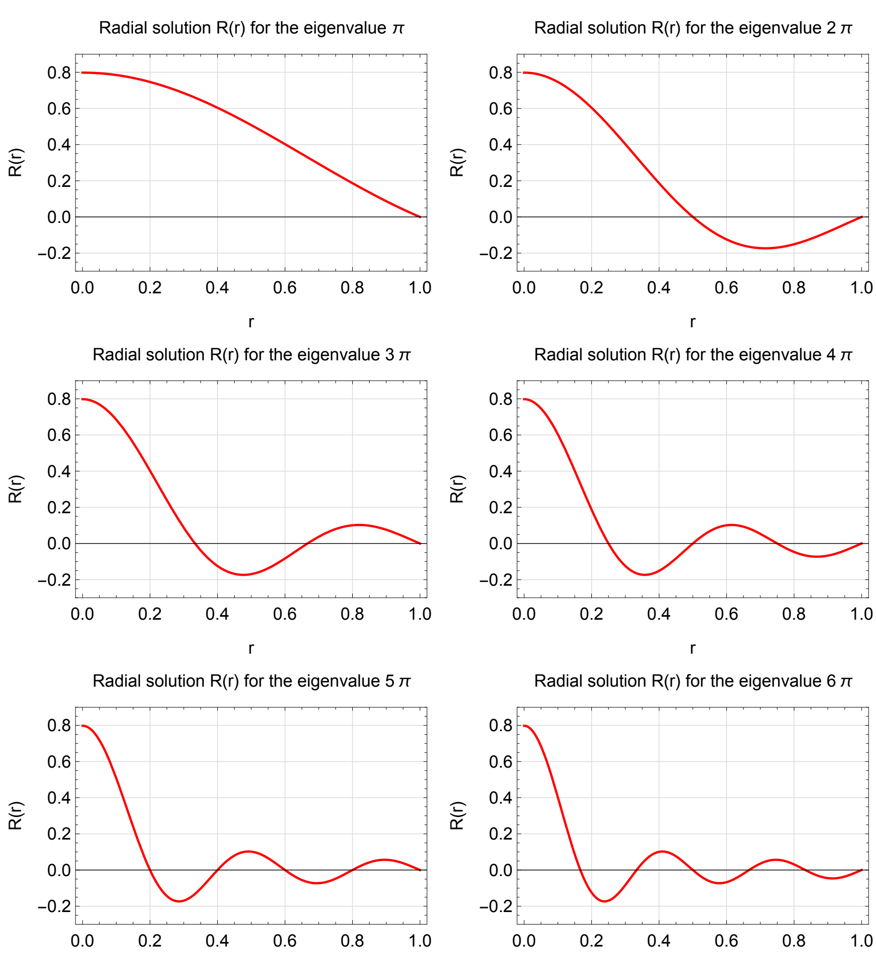
Using the above, the eigenfunctions (6) can also be written as

$$R_n(r) = \sqrt{\frac{2}{\pi^3}} \frac{\sin(n\pi r)}{nr} \qquad n = 1, 2, 3, \cdots$$

Note that

$$\begin{aligned} \lim_{r\to 0} \sqrt{\frac{2}{\pi^3}} \frac{\sin(n\pi r)}{nr} &= \sqrt{\frac{2}{\pi}} \\ &= 0.797885 \end{aligned}$$

For all n. Below is a plot of the first 6 eigenfunctions



5.1 References

In working on this exam, I have used a number of references such as Wikepidia, Wolfram Mathworld and the NIST Digital Library of Mathematical Functions.