Exam 1, Physics 501
University Of Wisconsin, Milwaukee

Nasser M. Abbasi

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Contents

1 Problem 1

Using a well known sum, find a closed for expression for the following series

$$f(z) = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots$$

Using the ratio test, find for what values of z this series converges.

Solution

Method 1

Assume that the closed form is

$$(1-z)^a = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots$$

For some unknown a. Now a will be solved for. Using Binomial series definition $(1-z)^a = 1 - az + \frac{(a)(a-1)}{2!}z^2 - \frac{a(a-1)(a-2)}{3!}z^3 + \cdots$. in the LHS above gives

$$1 - az + \frac{a(a-1)}{2!}z^2 - \frac{a(a-1)(a-2)}{3!}z^3 + \cdots = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots$$

By comparing coefficients of z in the left side and on the right side shows that $a = -2$ from the coefficient of z term. Verifying this on the coefficient of z^2 shows it is correct since it gives $\frac{(-2)(-3)}{2} = 3$. Therefore

$$a = -2$$

The closed form is therefore

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots$$

Method 2

Starting with Binomial series expansion given by

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \cdots$$

Taking derivative w.r.t. z on both sides of the above results in

$$\begin{aligned}
\frac{d}{dz}\left(\frac{1}{1-z}\right) &= \frac{d}{dz}\left(1 + z + z^2 + z^3 + z^4 + \cdots\right) \\
-(1-z)^{-2}(-1) &= 0 + 1 + 2z + 3z^2 + 4z^3 + \cdots \\
\frac{1}{(1-z)^2} &= 1 + 2z + 3z^2 + 4z^3 + \cdots
\end{aligned}$$

Therefore the closed form expression is

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

Which is the same as method 1.

The series general term of the series is

$$1 + 2z + 3z^2 + 4z^3 + \cdots = \sum_{n=0}^{\infty}(n+1)z^n$$

Applying the ratio test

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+2)z^{n+1}}{(n+1)z^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+2)z}{n+1}\right| \\
&= z\lim_{n\to\infty}\left|\frac{n+2}{n+1}\right| \\
&= z\lim_{n\to\infty}\left|\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right|
\end{aligned}$$

But $\lim_{n\to\infty}\left|\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right| = 1$ and the above limit becomes

$$L = z$$

By the ratio test, the series converges when $|L| < 1$. Therefore $1 + 2z + 3z^2 + 4z^3 + \cdots$ converges absolutely when $|z| < 1$. An absolutely convergent series is also a convergent series. Hence the series converges for $|z| < 1$.

2 Problem 2

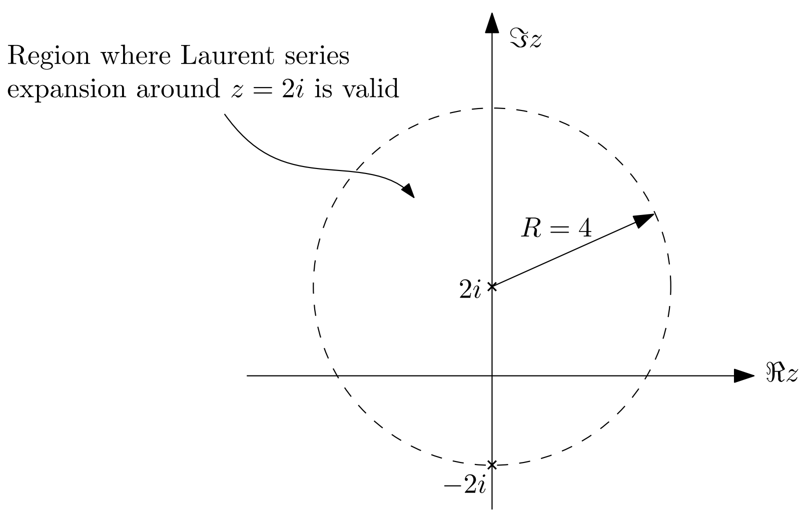
Find the Laurent series for the function

$$f(z) = \frac{1}{(z^2+4)^3}$$

About the isolated singular pole $z = 2i$. What is the order of this pole? What is the residue at this pole?

Solution

The poles are at $z^2 = 4$ or $z = \pm 2i$. The expansion of $f(z)$ is around the isolated pole at $z = 2i$. This pole has order 3. The region where this expansion is valid is inside a disk centered at $2i$ (but not including the point $z = 2i$ itself) and up to the nearest pole which is located at $-2i$. Therefore the disk will have radius 4.



Let

$$u = z - 2i$$
$$z = u + 2i$$

Substituting this expression for z back in $f(z)$ gives

$$\begin{aligned}
f(z) &= \frac{1}{\left((u+2i)^2+4\right)^3} \\
&= \frac{1}{(u^2-4+4ui+4)^3} \\
&= \frac{1}{(u^2+4ui)^3} \\
&= \frac{1}{(u(u+4i))^3} \\
&= \frac{1}{u^3}\frac{1}{(u+4i)^3} \\
&= \frac{1}{u^3}\frac{1}{\left[4i\left(\frac{u}{4i}+1\right)\right]^3} \\
&= \frac{1}{u^3}\frac{1}{(4i)^3\left(\frac{u}{4i}+1\right)^3} \\
&= \frac{1}{-i64u^3}\frac{1}{\left(\frac{u}{4i}+1\right)^3} \\
&= \left(\frac{i}{64u^3}\right)\frac{1}{\left(\frac{u}{4i}+1\right)^3} \qquad (1)
\end{aligned}$$

Expanding the term $\frac{1}{\left(1+\frac{u}{4i}\right)^3}$ using Binomial series, which is valid for $\left|\frac{u}{4i}\right| < 1$ or $|u| < 4$ gives

$$
\begin{aligned}
\frac{1}{\left(1+\frac{u}{4i}\right)^3} &= 1 + (-3)\frac{u}{4i} + \frac{(-3)(-4)}{2!}\left(\frac{u}{4i}\right)^2 + \frac{(-3)(-4)(-5)}{3!}\left(\frac{u}{4i}\right)^3 + \frac{(-3)(-4)(-5)(-6)}{4!}\left(\frac{u}{4i}\right)^4 + \cdots \\
&= 1 - 3\frac{u}{4i} + \frac{3\cdot 4}{2!}\frac{u^2}{16i^2} - \frac{3\cdot 4\cdot 5}{3!}\frac{u^3}{64i^3} + \frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u^4}{256i^4} + \cdots \\
&= 1 + 3i\frac{u}{4} - \frac{3\cdot 4}{2!}\frac{u^2}{16} - \frac{3\cdot 4\cdot 5}{3!}\frac{u^3}{64(-i)} + \frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u^4}{256} + \cdots \\
&= 1 + 3i\frac{u}{4} - \frac{3\cdot 4}{2!}\frac{u^2}{16} - i\frac{3\cdot 4\cdot 5}{3!}\frac{u^3}{64} + \frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u^4}{256} + \cdots \qquad (2)
\end{aligned}
$$

Substituting (2) into (1) and simplifying gives

$$
\begin{aligned}
f(z) &= \left(\frac{i}{64u^3}\right)\left(1 + 3i\frac{u}{4} - \frac{3\cdot 4}{2!}\frac{u^2}{16} - i\frac{3\cdot 4\cdot 5}{3!}\frac{u^3}{64} + \frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u^4}{256} + \cdots\right) \\
&= \frac{i}{64u^3} + \frac{i}{64u^3}\left(3i\frac{u}{4}\right) - \frac{i}{64u^3}\left(\frac{3\cdot 4}{2!}\frac{u^2}{16}\right) - \frac{i}{64u^3}\left(i\frac{3\cdot 4\cdot 5}{3!}\frac{u^3}{64}\right) + \frac{i}{64u^3}\left(\frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u^4}{256}\right) + \cdots \\
&= \frac{i}{64u^3} - \frac{1}{64u^2}\frac{3}{4} - \frac{i}{64u}\left(\frac{3\cdot 4}{2!}\frac{1}{16}\right) + \frac{1}{64}\left(\frac{3\cdot 4\cdot 5}{3!}\frac{1}{64}\right) + \frac{i}{64}\left(\frac{3\cdot 4\cdot 5\cdot 6}{4!}\frac{u}{256}\right) + \cdots \\
&= \frac{i}{64u^3} - \frac{3}{256u^2} - i\frac{3}{512}\frac{1}{u} + \frac{5}{2048} + i\frac{15}{16\,384}u + \cdots
\end{aligned}
$$

Replacing u back by $z - 2i$ in the above results in

$$
f(z) = \frac{i}{64}\frac{1}{(z-2i)^3} - \frac{3}{256}\frac{1}{(z-2i)^2} - \frac{3i}{512}\frac{1}{(z-2i)} + \frac{5}{2048} + \frac{15i}{16\,384}(z-2i) + \cdots \qquad (3)
$$

This expansion is valid for $|z - 2i| < 4$. The above shows that the residue is

$$\boxed{-\frac{3i}{512}}$$

Which is the coefficient of the $\frac{1}{(z-2i)}$ term in (3).

3 Problem 3

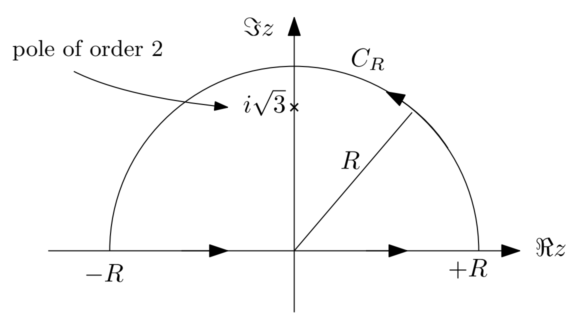
Use residues to evaluate the following integral

$$I = \int_0^{\infty} \frac{dx}{x^4 + 6x^2 + 9}$$

Solution

The integrand is an even function. Therefore the integral $\int_{-\infty}^{\infty} \frac{dx}{x^4+6x^2+9}$ is evaluated instead and then the required integral I will be half the value obtained. The poles of $\frac{1}{x^4+6x^2+9}$ are the zeros of the denominator. Factoring the denominator as $\left(x^2+3\right)\left(x^2+3\right) = 0$, shows the roots are $x = \pm i\sqrt{3}$ from the first factor and $x = \pm i\sqrt{3}$ from the second factor.

Since the upper half plane will be used, the pole located there is $+i\sqrt{3}$ and it is of order two. Now that pole locations are known, the following contour is used to evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^4+6x^2+9}$ as shown in the plot below



$$\begin{aligned}\oint_C f(z)\,dz &= \lim_{R\to\infty} \int_C f(z)\,dz + \lim_{R\to\infty} \int_{-R}^{+R} f(x)\,dx \\ &= \lim_{R\to\infty} \int_C \frac{dz}{z^4+6z^2+9} + \lim_{R\to\infty} \int_{-R}^{+R} \frac{dx}{x^4+6x^2+9} dx \end{aligned} \tag{2}$$

Where the integral $\int_{-R}^{+R}$ is Cauchy principal integral. Since the contour C is closed and because $f(z)$ is analytic on and inside C except for the isolated singularity inside at $z = i\sqrt{3}$, then by Cauchy integral formula $\oint_C f(z)\,dz = 2\pi i \sum \text{Residue}$. Where the sum of residues is over all poles inside C. Therefore (2) can becomes

$$\int_{-\infty}^{+\infty} \frac{dx}{x^4+6x^2+9} dx = 2\pi i \sum \text{Residue} - \lim_{R\to\infty} \int_C f(z)\,dz \tag{3}$$

But

$$\begin{aligned}\left|\int_C f(z)\,dz\right|_{\max} &\le ML \\ &= |f(z)|_{\max}\,\pi R\end{aligned} \tag{4}$$

Using

$$|f(z)|_{\max} \le \frac{1}{|z^2+3|_{\min}\,|z^2+3|_{\min}}$$

By inverse triangle inequality $\left|z^2+3\right| \ge |z|^2 - 3$. But $|z| = R$ on C, therefore $\left|z^2+3\right| \ge R^2 - 3$ and the above can now be written as

$$|f(z)|_{\max} \le \frac{1}{(R^2-3)(R^2-3)}$$

Using the above in (4) gives

$$\begin{aligned}\left|\int_C f(z)\,dz\right|_{\max} &\leq \frac{\pi R}{(R^2-3)(R^2-3)}\\ &= \frac{\pi R}{R^4-6R^2+9}\\ &= \frac{\frac{\pi}{R}}{R^2-6+\frac{9}{R^2}}\end{aligned}$$

In the limit as $R \longrightarrow \infty$ then $\left|\int_C f(z)\,dz\right|_{\max} \longrightarrow 0$. Using this result in (3) it simplifies to

$$\int_{-\infty}^{+\infty} \frac{dx}{x^4+6x^2+9} dx = 2\pi i \sum \text{Residue} \tag{5}$$

What is left now is to determine the residue at pole $z_0 = i\sqrt{3}$ which is of order 2. This is done using

$$\text{Residue}(z_0) = \lim_{z\longrightarrow z_0} \frac{d}{dz}\left((z-z_0)^2 f(z)\right)$$

But $z_0 = i\sqrt{3}$ and the above becomes

$$\begin{aligned}\text{Residue}\left(i\sqrt{3}\right) &= \lim_{z\longrightarrow i\sqrt{3}} \frac{d}{dz}\left(\left(z-i\sqrt{3}\right)^2 \frac{1}{\left(z-i\sqrt{3}\right)^2\left(z+i\sqrt{3}\right)^2}\right)\\ &= \lim_{z\longrightarrow i\sqrt{3}} \frac{d}{dz}\frac{1}{\left(z+i\sqrt{3}\right)^2}\\ &= \lim_{z\longrightarrow i\sqrt{3}} \frac{-2}{\left(z+i\sqrt{3}\right)^3}\\ &= \frac{-2}{\left(i\sqrt{3}+i\sqrt{3}\right)^3}\\ &= \frac{-2}{\left(2i\sqrt{3}\right)^3}\\ &= \frac{-2}{-(8)(3)\,i\sqrt{3}}\\ &= \frac{1}{12i\sqrt{3}}\end{aligned}$$

Using the above value of the residue in (5) gives

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{dx}{x^4+6x^2+9} dx &= 2\pi i\left(\frac{1}{12i\sqrt{3}}\right)\\ &= \frac{\pi}{6\sqrt{3}}\end{aligned}$$

Therefore the integral $\int_0^\infty \frac{dx}{x^4+6x^3+9}$ is half of the above result which is

$$\int_0^\infty \frac{dx}{x^4+6x^2+9} = \frac{\pi}{12\sqrt{3}}$$

4 Problem 4

Find two approximations for the integral $x > 0$

$$I(x) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x\cos^2\theta} d\theta$$

One for small x (keeping up to linear order in x) and one for large values of x (keeping only the leading order term).

Solution

The integrand has the form e^z. This has a known Taylor series expansion around zero given by

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots$$

Replacing z by $x\cos^2\theta$ in the above gives

$$e^{x\cos^2\theta} = 1 + x\cos^2\theta + \frac{\left(x\cos^2\theta\right)^2}{2} + \cdots$$

The problem is asking to keep linear terms in x. Therefore

$$e^{x\cos^2\theta} \approx 1 + x\cos^2\theta$$

Replacing the integrand in the original integral by the above approximation gives

$$\begin{aligned}
I(x) &\approx \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + x\cos^2\theta\right) d\theta \\
&\approx \frac{1}{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta + x\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta\right) \\
&\approx \frac{1}{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta + x\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos 2\theta d\theta\right) \\
&\approx \frac{1}{2\pi}\left(\pi + x\left(\frac{1}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) \\
&\approx \frac{1}{2\pi}\left(\pi + \frac{x}{4}\left(2\theta + \sin 2\theta\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) \\
&\approx \frac{1}{2\pi}\left(\pi + \frac{x}{4}\left(2\pi + 0\right)\right) \\
&\approx \frac{1}{2\pi}\left(\pi + \frac{x}{2}\pi\right) \\
&\approx \frac{1}{2}\left(1 + \frac{x}{2}\right)
\end{aligned}$$

For large value of x, The integrand is written as $e^{f(\theta)}$ where $f(\theta) = x\cos^2\theta$. The value of θ where $f(\theta)$ is maximum is first found. Then solving for θ in

$$\begin{aligned}
\frac{d}{d\theta}x\cos^2\theta &= 0 \\
-2x\cos\theta\sin\theta &= 0
\end{aligned}$$

Hence solving for θ in

$$\cos\theta\sin\theta = 0$$

There are two solutions to this. Either $\theta = \frac{\pi}{2}$ or $\theta = 0$. To find which is the correct choice, the sign of $\frac{d^2}{d\theta^2}f(\theta)$ is checked for each choice.

$$\begin{aligned}
\frac{d^2}{d\theta^2}x\cos^2\theta &= \frac{d}{d\theta}\left(-2x\cos\theta\sin\theta\right) \\
&= -2x\frac{d}{d\theta}\left(\cos\theta\sin\theta\right) \\
&= -2x\left(-\sin\theta\sin\theta + \cos\theta\cos\theta\right) \\
&= -2x\left(-\sin^2\theta + \cos^2\theta\right) \qquad (1)
\end{aligned}$$

Substituting $\theta = \frac{\pi}{2}$ in (1) and using $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$ gives

$$\begin{aligned}\left.\frac{d^2}{d\theta^2} x\cos^2\theta\right|_{\theta=\frac{\pi}{2}} &= -2x(-1)\\ &= 2x\end{aligned}$$

Since the problem says that $x > 0$ then $\left.\frac{d^2}{d\theta^2} x\cos^2\theta\right|_{\theta=\frac{\pi}{2}} > 0$. Therefore this is a minimum. Using the second choice $\theta = 0$, then (1) becomes (after using $\cos(0) = 1$ and $\sin(0) = 0$)

$$\left.\frac{d^2}{d\theta^2} x\cos^2\theta\right|_{\theta=0} = -2x$$

And because $x > 0$ then $\left.\frac{d^2}{d\theta^2} x\cos^2\theta\right|_{\theta=0} < 0$. Therefore the integrand is maximum at

$$\theta_{peak} = 0$$

Now that peak θ is found, then $f(\theta)$ is expanded in Taylor series around $\theta_{peak} = 0$. Since $f(\theta) = x\cos^2\theta$, then

$$f\left(\theta_{peak}\right) = x$$

And $f'(\theta) = -2x\cos\theta\sin\theta$. At θ_{peak} this becomes $f'\left(\theta_{peak}\right) = 0$. The next term is the quadratic one, given by

$$\begin{aligned}f''(\theta) &= -2x\frac{d}{d\theta}(\cos\theta\sin\theta)\\ &= -2x\left(-\sin^2\theta + \cos^2\theta\right)\end{aligned}$$

Evaluating the above at $\theta_{peak} = 0$ gives

$$f''\left(\theta_{peak}\right) = -2x$$

The problem says to keep leading term, so no need for more terms. Therefore the Taylor series expansion of $f(\theta) = x\cos^2\theta$ around $\theta = \theta_{peak}$ is

$$\begin{aligned}x\cos^2\theta &\approx f\left(\theta_{peak}\right) + f'\left(\theta_{peak}\right)\theta + \frac{1}{2!}f''\left(\theta_{peak}\right)\theta^2\\ &= x + 0 - \frac{2x}{2!}\theta^2\\ &= x - x\theta^2\\ &= x\left(1-\theta^2\right)\end{aligned}$$

The integral now becomes

$$\begin{aligned}I(x) &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x\left(1-\theta^2\right)}\,d\theta\\ &\approx \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x}e^{-x\theta^2}\,d\theta\\ &= \frac{1}{2\pi}\left(e^{x}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x\theta^2}\,d\theta\right)\end{aligned}$$

Comparing $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x\theta^2}d\theta$ to the Gaussian integral $\int_{-\infty}^{\infty} e^{-a\theta^2}d\theta = \sqrt{\frac{\pi}{x}}$, then the above can be approximated as

$$I(x) = \frac{e^x}{2\pi}\sqrt{\frac{\pi}{x}}$$

Summary of result

Small x approximation	$\frac{1}{2}\left(1+\frac{x}{2}\right)$
Large x approximation	$\frac{e^x}{2\pi}\sqrt{\frac{\pi}{x}}$

Note that using the computer, the exact solution is

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{x\cos^2\theta}\,d\theta = \frac{1}{2}e^{\frac{x}{2}}\,\text{BesselI}\left(0,\frac{x}{2}\right)$$

5 Problem 5

Use the Cauchy-Riemann equations to determine where the function

$$f(z) = z + \overline{z^2}$$

Is analytic. Evaluate $\oint_C f(z)\,dz$ where contour C is on the unit circle $|z| = 1$ in a counterclockwise sense.

Solution

Using $z = x + iy$, the function $f(z)$ becomes

$$\begin{aligned} f(z) &= x + iy + \overline{(x+iy)^2} \\ &= x + iy + \overline{(x^2 - y^2 + 2ixy)} \\ &= x + iy + \left(x^2 - y^2 - 2ixy\right) \\ &= \left(x + x^2 - y^2\right) + i\,(y - 2xy) \end{aligned}$$

Writing $f(z) = u + iv$, and comparing this to the above result shows that

$$\begin{aligned} u &= x + x^2 - y^2 \\ v &= y - 2xy \end{aligned} \tag{1}$$

Cauchy-Riemann are given by

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{-\partial u}{\partial y} &= \frac{\partial v}{\partial x} \end{aligned}$$

Using result in (1), Cauchy-Riemann are checked to see if they are satisfied or not. The first equation above results in

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1 + 2x \\ \frac{\partial v}{\partial y} &= 1 - 2x \end{aligned}$$

Therefore $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$. This shows that $f(z)$ is not analytic for all x, y.

Since $f(z)$ is not analytic, Cauchy integral formula can not be used. Instead this can be integrated using parameterization. Let $z = e^{i\theta}$ (No need to use $re^{i\theta}$ since $r = 1$ in this case because it is the unit circle). The function $f(z)$ becomes

$$\begin{aligned} f(z) &= e^{i\theta} + \overline{\left(e^{i\theta}\right)^2} \\ &= e^{i\theta} + \overline{e^{2i\theta}} \\ &= e^{i\theta} + e^{-2i\theta} \end{aligned}$$

And because $z = e^{i\theta}$ then $dz = d\theta e^{i\theta}$. The integral now becomes

$$\begin{aligned} \oint_C f(z)\,dz &= \int_0^{2\pi} \left(e^{i\theta} + e^{-2i\theta}\right) e^{i\theta} d\theta \\ &= \int_0^{2\pi} \left(e^{2i\theta} + e^{-i\theta}\right) d\theta \\ &= \left[\frac{e^{2i\theta}}{2i}\right]_0^{2\pi} + \left[\frac{e^{-i\theta}}{-i}\right]_0^{2\pi} \\ &= \frac{1}{2i}\left[\cos 2\theta + i\sin 2\theta\right]_0^{2\pi} + i\left[\cos\theta - i\sin\theta\right]_0^{2\pi} \\ &= \frac{1}{2i}\left[(\cos 4\pi + i\sin 4\pi) - (\cos 0 + i\sin 0)\right] + i\left[(\cos 2\pi - i\sin 2\pi) - (\cos 0 - i\sin 0)\right] \\ &= \frac{1}{2i}\left[1 - 1\right] + i\left[1 - 1\right] \end{aligned}$$

Hence

$$\oint_C f(z)\, dz = 0$$