Problem Set 7 Solutions

i) $g(x,t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} 2n\, H_{n-1}(x) \frac{t^n}{n!}$$

$$= 2t \underbrace{\sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^{n-1}}_{\text{let } \tilde{n}=n-1} = 2t \sum_{\tilde{n}=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

or $\boxed{\frac{\partial g}{\partial x} = 2tg}$

ii) $\frac{dg}{g} = 2t\,dx \implies \ln g = 2tx + f(t)$

$$\implies \boxed{g = e^{2tx} e^{f(t)}}$$

iii) $g(0,t) = \sum_{n=0}^{\infty} H_n(0) \frac{t^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} \frac{t^{2k}}{(2k)!}$ — let $n=2k$

$$= \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} = e^{-t^2}$$

$$\implies e^{f(t)} = e^{-t^2}$$

$$\implies g(x,t) = e^{-t^2+2tx}$$

iv) $e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$

take $\frac{d}{dx}$ of the above equation

$$\Rightarrow (-2t+2x)\, e^{-t^2+2tx} = \sum_{n=0}^{\infty} \frac{n H_n(x)}{n!} t^{n-1}$$

$$\Rightarrow (-2t+2x) \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} \frac{n H_n(x)\, t^{n-1}}{n!} \quad \sim \tilde{n} = n-1$$

$$\Rightarrow -2 \sum_{n=0}^{\infty} H_n(x) \frac{t^{\overset{\tilde{n}=n+1}{n+1}}}{n!} + 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{\tilde{n}=0}^{\infty} \frac{H_{\tilde{n}+1}(x)\, t^{\tilde{n}}}{\tilde{n}!}$$

or $-2 \sum_{\tilde{n}=1}^{\infty} \frac{\tilde{n} H_{\tilde{n}-1}\, t^{\tilde{n}}}{\tilde{n}!} + 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{\tilde{n}=0}^{\infty} \frac{H_{\tilde{n}+1}(x)\, t^{\tilde{n}}}{\tilde{n}!}$

$$\Rightarrow \boxed{-2n H_{n-1} + 2x H_n = H_{n+1}} \quad \text{as required}$$

v) $e^{-x^2} g(x,s)\, g(x,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} H_m(x) H_n(x) \frac{s^m}{m!} \frac{t^n}{n!} e^{-x^2}$

consider $\int_{-\infty}^{\infty} e^{-x^2} g(x,s)\, g(x,t) = \int_{-\infty}^{\infty} e^{-x^2} e^{-s^2+2xs} e^{-t^2+2tx} dx$

$$= \int_{-\infty}^{\infty} e^{-x^2} e^{2x(s+t)} e^{-s^2} e^{-t^2} dx$$

$$= e^{-s^2} e^{-t^2} e^{(s+t)^2} \int_{-\infty}^{\infty} e^{-(x-s-t)^2} dx$$

or $\int_{-\infty}^{\infty} e^{-x^2} g(x,s)\, g(x,t)\, dx = \sqrt{\pi}\, e^{2st}$

$$\Rightarrow \sqrt{\pi} \sum_{p=0}^{\infty} \frac{(st)^p 2^p}{p!} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^n s^m}{m!\,n!} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx$$

L equating power by power in s and t

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = f_m \delta_{m,n}$$

$$\Rightarrow \sqrt{\pi} \sum_{m=0}^{\infty} \frac{(st)^m 2^m}{m!} = \sum_{m=0}^{\infty} \frac{(st)^m f_m}{(m!)^2}$$

$$\Rightarrow$$

$f_m = \sqrt{\pi}\, (m!)\, 2^m$ as required.

Exercise 2

a) $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) y(r) = 0 \qquad 0 < r < \infty$

let $\quad x = \ln r \qquad \frac{d}{dx} = \frac{dr}{dx}\frac{d}{dr} = r\frac{d}{dr}$

note $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) y(r) = 0$ can be rewritten

as $\quad \underbrace{r\frac{d}{dr}}_{\frac{d}{dx}}\left[\underbrace{r\frac{d}{dr}}_{\frac{d}{dx}} y(r)\right] - n^2 y(r) = 0$

$\longrightarrow \quad \frac{d^2}{dx^2} y(x) - n^2 y(x) = 0$

$\Rightarrow \quad y = e^{\pm n x} = \begin{cases} r^n & \text{vanishes as } r \to 0 \\ r^{-n} & \text{vanishes as } r \to \infty \end{cases}$

b) Let $\quad G(r,r') = \begin{cases} A r^n & 0 < r < r' \\ B r^{-n} & r' < r < \infty \end{cases}$

continuity at $r = r'$ $\longrightarrow$ $A(r')^n = B(r')^{-n} = C$

$\Rightarrow \quad A = C(r')^{-n}$ and $B = C(r')^n$

$\therefore \quad G(r,r') = C \begin{cases} (r/r')^n & 0 < r < r' \\ (r'/r)^n & r' < r < \infty \end{cases}$

Also $\quad \frac{d}{dr}\left[r\frac{d}{dr}G(r,r')\right] - \frac{n^2}{r}G(r,r') = \delta(r - r')$

integrate from $r = r' - \varepsilon$ to $r = r' + \varepsilon$ and let $\varepsilon \to 0$

$\lim_{\varepsilon \to 0}\left[r\frac{d}{dr}G(r,r')\right]_{r'-\varepsilon}^{r'+\varepsilon} - \lim_{\varepsilon \to 0}\left[\frac{n^2}{r}G(r,r')\right]_{r'-\varepsilon}^{r'+\varepsilon} = 1$

$$\Rightarrow \quad c(-n)\left(\frac{r'}{r}\right)^n \Big|_{r'} - cn\left(\frac{r}{r'}\right)^n \Big|_{r'} = 1$$

$$\Rightarrow \quad c = \frac{-1}{2n}$$

$$\therefore \quad \boxed{G(r,r') = -\frac{1}{2n}\begin{cases} (r/r')^n & 0 < r < r' \\ (r'/r)^n & r' < r < \infty \end{cases}}$$