HW 7, Physics 501
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Contents

1 Problem 1

Exercise 1: Consider Hermite's differential equation valid for $(-\infty < x < \infty)$:

$$y'' - 2xy' + 2ny = 0 \tag{1}$$

i) Assume the existence of a generating function $g(x,t) = \sum_{n=0}^{\infty} H_n(x)t^n/n!$. Differentiate $g(x,t)$ with respect to x and use the recurrence relation $H_n'(x) = 2nH_{n-1}(x)$to develop a first order differential equation for $g(x,t)$.
ii) Integrate this equation with respect to x holding t fixed.
iii) Use the relationships $H_{2n}(0) = (-1)^n(2n)!/n!$ and $H_{2n+1}(0) = 0$ to evaluate $g(0,t)$ and show $g(x,t) = \exp(-t^2 + 2tx)$.
iv) Use the generating function to find the recurrence relation $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.
v) By integrating the product $e^{-x^2}g(x,s)g(x,t)$ over all x, show

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = 2^n n! \sqrt{\pi}\delta_{nm}. \tag{2}$$

Figure 1: Problem statement

Solution

$$y'' - 2xy' + 2ny = 0 \qquad -\infty < x < \infty$$

1.1 Part 1

$$g(x,t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

Differentiating w.r.t x, and assuming term by term differentiation is allowed, gives

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

Using $H_n'(x) = 2nH_{n-1}(x)$ in the above results in

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=0}^{\infty} 2nH_{n-1}(x) \frac{t^n}{n!}$$

But for $n = 0$, the first term is zero, so the sum can start from 1 and give the same result

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=1}^{\infty} 2nH_{n-1}(x) \frac{t^n}{n!}$$

Now, decreasing the summation index by 1 and increasing the n inside the sum by 1 gives

$$\begin{aligned}
\frac{\partial g(x,t)}{\partial x} &= \sum_{n=0}^{\infty} 2(n+1)H_n(x) \frac{t^{n+1}}{(n+1)!} \\
&= \sum_{n=0}^{\infty} 2(n+1)H_n(x) \frac{t^{n+1}}{(n+1)\,n!} \\
&= \sum_{n=0}^{\infty} 2H_n(x) \frac{t^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} 2t \left(H_n(x) \frac{t^n}{n!}\right) \\
&= 2t \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}
\end{aligned}$$

But $\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = g(x,t)$ and the above reduces to

$$\frac{\partial g(x,t)}{\partial x} = 2tg(x,t)$$

The problem says it is supposed to be a first order differential equation and not a first order partial differential equation. Therefore, by assuming x to be a fixed parameter instead of another independent variable, the above can now be written as

$$\frac{d}{dx}g(x,t) - 2tg(x,t) = 0$$

1.2 Part 2

From the solution found in part (1)

$$\begin{aligned}\frac{\frac{d}{dx}g(x,t)}{g(x,t)} &= 2t \\ \frac{dg(x,t)}{g(x,t)} &= 2tdx\end{aligned}$$

Integrating both sides gives

$$\begin{aligned}\int \frac{dg(x,t)}{g(x,t)} &= \int 2tdx \\ \ln|g(x,t)| &= 2tx + C \\ g(x,t) &= e^{2tx+C} \\ g(x,t) &= C_1 e^{2tx}\end{aligned}$$

Where $C_1 = e^C$ a new constant. Let $g(0,t) = g_0$ then the above shows that $C_1 = g_0$ and the above can now be written as

$$g(x,t) = g(0,t)e^{2tx}$$

1.3 Part 3

Using the given definition of $g(x,t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$ and when $x = 0$ then

$$\begin{aligned}g(0,t) &= \sum_{n=0}^{\infty} H_n(0) \frac{t^n}{n!} \\ &= H_0(0) + H_1(0) + \sum_{n=2}^{\infty} H_n(0) \frac{t^n}{n!}\end{aligned}$$

But $H_0(x) = 1$, hence $H_0(0) = 1$ and $H_1(x) = 2x$, hence $H_1(0) = 0$ and the above becomes

$$g(0,t) = 1 + \sum_{n=2}^{\infty} H_n(0) \frac{t^n}{n!}$$

For the remaining series, it can be written as sum of even and odd terms

$$g(0,t) = 1 + \sum_{n=2,4,6,\cdots}^{\infty} H_n(0) \frac{t^n}{n!} + \sum_{n=3,5,7,\cdots}^{\infty} H_n(0) \frac{t^n}{n!}$$

Or, equivalently

$$g(0,t) = 1 + \sum_{n=1,2,3,\cdots}^{\infty} H_{2n}(0) \frac{t^{2n}}{(2n)!} + \sum_{n=1,2,3,\cdots}^{\infty} H_{2n+1}(0) \frac{t^{2n+1}}{(2n+1)!}$$

But using the hint given that $H_{2n+1}(0) = 0$ and $H_{2n}(0) = \frac{(-1)^n (2n)!}{n!}$ the above simplifies to

$$\begin{aligned}g(0,t) &= 1 + \sum_{n=1,2,3,\cdots}^{\infty} \frac{(-1)^n (2n)!}{n!} \frac{t^{2n}}{(2n)!} \\ &= 1 + \sum_{n=1,2,3,\cdots}^{\infty} (-1)^n \frac{t^{2n}}{n!}\end{aligned}$$

But since $(-1)^n \frac{t^{2n}}{n!} = 1$ when $n = 0$, then the above sum can be made to start as zero and it simplifies to

$$g(0,t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$$

Therefore the solution $g(x,t) = g(0,t)\, e^{tx}$ found in part (2) becomes

$$g(x,t) = \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}\right) e^{2tx} \tag{1}$$

Now the sum $\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots$ and comparing this sum to standard series of $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$, then this shows that when $z = -t^2$ and series for e^{-t^2} becomes

$$\begin{aligned} e^{-t^2} &= 1 + \left(-t^2\right) + \frac{\left(-t^2\right)^2}{2!} + \frac{\left(-t^2\right)^3}{3!} + \frac{\left(-t^2\right)^4}{4!} \cdots \\ &= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} \cdots \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} = e^{-t^2}$$

Substituting this into (1) gives

$$\begin{aligned} g(x,t) &= e^{-t^2} e^{2tx} \\ &= e^{2tx - t^2} \end{aligned}$$

1.4 Part 4

Since $g(x,t) = e^{2tx-t^2}$ from part (3), then

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= (2x - 2t)\, e^{2tx-t^2} \\ &= (2x - 2t)\, g(x,t) \end{aligned}$$

But $g(x,t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$, therefore the above can be written as

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= (2x - 2t) \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \\ &= 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2t \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \\ &= 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2 \sum_{n=0}^{\infty} H_n(x) \frac{t^{n+1}}{n!} \\ &= 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2 \sum_{n=1}^{\infty} H_{n-1}(x) \frac{t^n}{(n-1)!} \\ &= 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2 \sum_{n=1}^{\infty} nH_{n-1}(x) \frac{t^n}{n(n-1)!} \\ &= 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2 \sum_{n=1}^{\infty} nH_{n-1}(x) \frac{t^n}{n!} \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= \frac{\partial}{\partial t} \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} nH_n(x) \frac{t^{n-1}}{n!} \end{aligned}$$

Since at $n = 0$ the sum is zero, then it can be started from $n = 1$ without changing the result

$$\begin{aligned}\frac{\partial}{\partial t}g(x,t) &= \sum_{n=1}^{\infty} nH_n(x)\frac{t^{n-1}}{n!}\\ &= \sum_{n=0}^{\infty}(n+1)H_{n+1}(x)\frac{t^n}{(n+1)!}\\ &= \sum_{n=0}^{\infty}(n+1)H_{n+1}(x)\frac{t^n}{(n+1)n!}\\ &= \sum_{n=0}^{\infty}H_{n+1}(x)\frac{t^n}{n!} \qquad (2)\end{aligned}$$

Equating (1) and (2) gives

$$\sum_{n=0}^{\infty}H_{n+1}(x)\frac{t^n}{n!} = 2x\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} - 2\sum_{n=1}^{\infty}nH_{n-1}(x)\frac{t^n}{n!}$$

But $\sum_{n=1}^{\infty} nH_{n-1}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} nH_{n-1}(x)\frac{t^n}{n!}$ because at $n = 0$ it is zero, so it does not affect the result to start the sum from zero, and now the above can be written as

$$\sum_{n=0}^{\infty}H_{n+1}(x)\frac{t^n}{n!} = 2x\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty}nH_{n-1}(x)\frac{t^n}{n!}$$

Now since all the sums start from $n = 0$ then the above means the same as

$$H_{n+1}(x)\frac{t^n}{n!} = 2xH_n(x)\frac{t^n}{n!} - 2nH_{n-1}(x)\frac{t^n}{n!}$$

Canceling $\frac{t^n}{n!}$ from each term gives

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Which is the result required to show.

1.5 Part 5

The problem is asking to show that

$$\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)\,dx = \begin{cases} 0 & n \neq m \\ 2^n n!\sqrt{\pi} & n = m \end{cases}$$

The first part below will show the case for $n \neq m$ and the second part part will show the case for $n = m$

case $n \neq m$ This is shown by using the differential equation directly. I found this method easier and more direct. Before starting, the ODE $y'' - 2xy' + 2ny = 0$ is rewritten as

$$e^{x^2}\frac{d}{dx}\left(e^{-x^2}y'\right) + 2ny = 0 \qquad (1)$$

The above form is exactly the same as the original ODE as can be seen by expanding it. Now, Let $H_n(x)$ be one solution to (1) and let $H_m(x)$ be another solution to (1) which results in the following two ODE's

$$e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right) + 2nH_n = 0 \qquad (1A)$$

$$e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right) + 2mH_m = 0 \qquad (2A)$$

Multiplying (1A) by H_m and (2A) by H_n and subtracting gives

$$\begin{aligned} H_m\left(e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right) + 2nH_n\right) - H_n\left(e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right) + 2mH_m\right) &= 0\\ \left(H_me^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right) + 2nH_nH_m\right) - \left(H_ne^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right) + 2mH_nH_m\right) &= 0\\ H_me^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right) - H_ne^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right) + 2(n-m)H_nH_m &= 0\\ H_m\frac{d}{dx}\left(e^{-x^2}H_n'\right) - H_n\frac{d}{dx}\left(e^{-x^2}H_m'\right) + 2(n-m)H_nH_me^{-x^2} &= 0 \qquad (3)\end{aligned}$$

But

$$H_m \frac{d}{dx}\left(e^{-x^2} H_n'\right) = \frac{d}{dx}\left(e^{-x^2} H_n' H_m\right) - e^{-x^2} H_n' H_m'$$

And

$$H_n \frac{d}{dx}\left(e^{-x^2} H_m'\right) = \frac{d}{dx}\left(e^{-x^2} H_m' H_n\right) - e^{-x^2} H_m' H_n'$$

Therefore

$$\begin{aligned} H_m \frac{d}{dx}\left(e^{-x^2} H_n'\right) - H_n \frac{d}{dx}\left(e^{-x^2} H_m'\right) &= \left(\frac{d}{dx}\left(e^{-x^2} H_n' H_m\right) - e^{-x^2} H_n' H_m'\right) - \left(\frac{d}{dx}\left(e^{-x^2} H_m' H_n\right) - e^{-x^2} H_m' H_n'\right) \\ &= \frac{d}{dx}\left(e^{-x^2} H_n' H_m\right) - \frac{d}{dx}\left(e^{-x^2} H_m' H_n\right) \\ &= \frac{d}{dx}\left(e^{-x^2}\left(H_n' H_m - H_m' H_n\right)\right) \end{aligned}$$

Substituting the above relation back into (3) gives

$$\frac{d}{dx}\left(e^{-x^2}\left(H_n' H_m - H_m' H_n\right)\right) + 2(n-m) H_n H_m e^{-x^2} = 0$$

Integrating gives

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d}{dx}\left(e^{-x^2}\left(H_n' H_m - H_m' H_n\right)\right) dx + \int_{-\infty}^{\infty} 2(n-m) H_n H_m e^{-x^2} dx &= 0 \\ \int_{-\infty}^{\infty} d\left(e^{-x^2}\left(H_n' H_m - H_m' H_n\right)\right) + 2(n-m) \int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx &= 0 \\ \left[e^{-x^2}\left(H_n' H_m - H_m' H_n\right)\right]_{-\infty}^{\infty} + 2(n-m) \int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx &= 0 \end{aligned}$$

But $\lim_{x\to\pm\infty} e^{-x^2} \longrightarrow 0$ so the first term above vanishes and the above becomes

$$2(n-m) \int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx = 0$$

Since this is the case where $n \neq m$ then the above shows that

$$\int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx = 0 \qquad n \neq m$$

Now the <u>case $n = m$</u> is proofed. When $H_n = H_m$ then the integral becomes $\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx$. Using the known Rodrigues formula for Hermite polynomials, given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Then applying the above the above to one of the $H_n(x)$ in the integral $\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx$, gives

$$\begin{aligned} \int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx &= \int_{-\infty}^{\infty} \left((-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}\right) H_n e^{-x^2} dx \\ &= (-1)^n \int_{-\infty}^{\infty} \left(\frac{d^n}{dx^n} e^{-x^2}\right) H_n dx \end{aligned}$$

Now integration by parts is carried out. $\int u dv = uv - \int v du$. Let $u = H_n$ and let $dv = \frac{d^n}{dx^n} e^{-x^2}$, therefore $du = H_n'(x) = 2n H_{n-1}(x)$ and $v = \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}$, therefore

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx = (-1)^n \left(\left[H_n(x) \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) 2n H_{n-1}(x) dx\right)$$

But $\left[H_n(x) \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right]_{-\infty}^{\infty} \longrightarrow 0$ as $x \longrightarrow \pm\infty$ because each derivative of $\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}$ produces a term with e^{-x^2} which vanishes at both ends of the real line. Hence the above integral now becomes

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx = (-1)^n \left(-2n \int_{-\infty}^{\infty} \left(\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) H_{n-1}(x) dx\right)$$

Now the process is repeated, doing one more integration by parts. This results in

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx = (-1)^n \left(-2n \left(-2(n-1) \int_{-\infty}^{\infty} \left(\frac{d^{n-2}}{dx^{n-2}} e^{-x^2} \right) H_{n-2}(x)\, dx \right) \right)$$

And again

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx = (-1)^n \left(-2n \left(-2(n-1) \left(-2(n-2) \int_{-\infty}^{\infty} \left(\frac{d^{n-3}}{dx^{n-3}} e^{-x^2} \right) H_{n-3}(x)\, dx \right) \right) \right)$$

This process continues n times. After n integrations by parts, the above becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx &= (-1)^n \left(-2n \left(-2(n-1) \left(-2(n-2) \left(\cdots \left(\int_{-\infty}^{\infty} e^{-x^2} H_0(x)\, dx \right) \right) \right) \right) \right) \\
&= (-1)^n (-2)^n n! \int_{-\infty}^{\infty} e^{-x^2} H_0(x)\, dx \\
&= 2^n n! \int_{-\infty}^{\infty} e^{-x^2} H_0(x)\, dx
\end{aligned}$$

But $H_0(x) = 1$, therefore the above becomes

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx = 2^n n! \int_{-\infty}^{\infty} e^{-x^2} dx$$

But

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2} &= 2 \int_{0}^{\infty} e^{-x^2} \\
&= 2 \frac{\sqrt{\pi}}{2} \\
&= \sqrt{\pi}
\end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx = 2^n n! \sqrt{\pi}$$

This completes the case for $n = m$. Hence

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \begin{cases} 0 & n \neq m \\ 2^n n! \sqrt{\pi} & n = m \end{cases}$$

Which is what the problem asked to show.

2 Problem 2

Exercise 2: a) Consider the differential equation for $0 < r < \infty$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)y(r) = 0 \tag{3}$$

where $n = 0, 1, 2, 3, ...$ Find two independent solutions, one which vanishes as $r \to 0$ and the other that vanishes as $r \to \infty$. Hint let $x = \ln r$.
b) Given the result of part a), find the solution to the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)y(r) = \frac{1}{r}\delta(r - r') \tag{4}$$

with the boundary condition that the solution vanishes as $r \to 0$ and $r \to \infty$.

Figure 2: Problem statement

Solution

2.1 Part (a)

$$y''(r) + \frac{1}{r}y'(r) - \frac{n^2}{r^2}y(r) = 0 \qquad 0 < r < \infty$$

Or

$$r^2 y''(r) + r y'(r) - n^2 y(r) = 0$$

case $n = 0$

The ode becomes $r^2 y''(r) + r y'(r) = 0$. Let $z = y'$ and it becomes $r^2 z'(r) + r z(r) = 0$ or $z'(r) + \frac{1}{r} z(r) = 0$. This is linear in $z(r)$. Integrating factor is $I = e^{\int \frac{1}{r} dr} = r$. Multiplying the ode by I it becomes exact differential $\frac{d}{dr}(zr) = 0$ or $d(zr) = 0$, hence $z = \frac{c_1}{r}$ where c_1 is constant of integration. Therefore

$$y'(r) = \frac{c_1}{r}$$

Integrating again gives

$$y(r) = \frac{c_1}{\ln r} + c_2$$

Since $\lim_{r\to 0}$ the solution is bounded, then c_1 must be zero. Therefore $0 = c_2$ and this implies $c_2 = 0$ also. Therefore when $n = 0$ the solution is

$$y(r) = 0$$

Case $n \neq 0$

Since powers of r is the same as order of derivative in each term, this is an Euler ODE. It is solved by assuming $y = r^\alpha$. Hence $y' = \alpha r^{\alpha-1}, y'' = \alpha(\alpha-1) r^{\alpha-2}$. Substituting these into the above ODE gives

$$\begin{aligned} r^2 \alpha(\alpha-1) r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2 r^\alpha &= 0 \\ \alpha(\alpha-1) r^\alpha + \alpha r^\alpha - n^2 r^\alpha &= 0 \\ r^\alpha\left(\alpha(\alpha-1) + \alpha - n^2\right) &= 0 \end{aligned}$$

Assuming non-trivial solution $r^\alpha \neq 0$, then the indicial equation is

$$\begin{aligned} \alpha(\alpha-1) + \alpha - n^2 &= 0 \\ \alpha^2 &= n^2 \\ \alpha &= \pm n \end{aligned}$$

Hence one solution is

$$y_1(r) = r^n$$

And second solution is

$$y_2(r) = r^{-n}$$

And the general solution is linear combination of these solutions

$$y(r) = c_1 r^n + c_2 r^{-n}$$

The above shows that $\lim_{r\to 0} y_1(r) = 0$ and $\lim_{r\to\infty} y_2(r) = 0$.

2.2 Part (b)

Short version of the solution

To simplify the notations, r_0 is used instead of r' in all the following.

$$y''(r) + \frac{1}{r}y'(r) - \frac{n^2}{r^2}y(r) = \frac{1}{r}\delta(r - r_0) \qquad 0 < r < \infty$$

Multiplying both sides by r the above becomes

$$ry''(r) + y'(r) - \frac{n^2}{r}y(r) = \delta(r - r_0) \tag{1}$$

But the two solutions[1] to the homogeneous ODE $ry''(r) + y'(r) - \frac{n^2}{r}y(r) = 0$ were found in part (a). These are

$$\begin{aligned} y_1(r) &= r^n \qquad (1A)\\ y_2(r) &= r^{-n} \end{aligned}$$

The Green function is the solution to

$$\begin{aligned} rG(r, r_0) + G(r, r_0) - \frac{n^2}{r}G(r, r_0) &= \delta(r - r_0) \qquad (1B)\\ \lim_{r\to 0} G(r, r_0) &= 0\\ \lim_{r\to\infty} G(r, r_0) &= 0 \end{aligned}$$

Which is given by (Using class notes, Lecture December 5, 2018) as

$$G(r, r_0) = \frac{1}{C}\begin{cases} y_1(r)\, y_2(r_0) & 0 < r < r_0\\ y_1(r_0)\, y_2(r) & r_0 < r < \infty \end{cases} \tag{2}$$

Note, I used $\frac{+1}{C}$ and not $\frac{-1}{C}$ as in class notes, since I am using $L = -\left((py')' - qy\right)$ as the operator and not $L = +\left((py')' + qy\right)$. Now C is given by

$$C = p(r_0)\left(y_1(r_0)\, y_2'(r_0) - y_1'(r_0)\, y_2(r_0)\right)$$

Where from (1A) we see that

$$\begin{aligned} y_1(r_0) &= r_0^n\\ y_2'(r_0) &= -nr_0^{-n-1}\\ y_1'(r_0) &= nr_0^{n-1}\\ y_2(r_0) &= r_0^{-n} \end{aligned}$$

Therefore C becomes

$$\begin{aligned} C &= p(r_0)\left(-nr_0^{-n-1}r_0^n - nr_0^{n-1}r_0^{-n}\right)\\ &= 2nr_0^{-1}p(r_0) \end{aligned}$$

[1] All the following is for $n \neq 0$, since for $n = 0$, only trivial solution exist

We just need now to find $p(r_0)$. This comes from Sturm Liouville form. We need to convert the ODE $r^2y''(r)+ry'(r)-n^2y(r)=0$ to Sturm Liouville. Writing this ODE as $ay''+by'+(c+\lambda)y=0$ where $a=r^2, b=r, c=0, \lambda=-n^2$, therefore

$$
\begin{aligned}
p &= e^{\int \frac{b}{a}dr} = e^{\int \frac{r}{r^2}dr} = r\\
q &= -p\frac{c}{a} = 0\\
\rho &= \frac{p}{a} = \frac{r}{r^2} = \frac{1}{r}
\end{aligned}
$$

Hence the SL form is $(py')' - qy + \lambda\rho y = 0$. Hence the SL form is $(py')' - qy + \lambda\rho y = 0$ or

$$
(ry')' - \frac{1}{r}n^2 y = 0 \tag{2A}
$$

Hence the operator is $L[y] = -\left(\frac{d}{dr}\left(r\frac{d}{dr}\right)\right)[y]$ and in standard form it becomes $L[y]+\frac{1}{r}n^2y=0$.

The above shows that $p(r_0)=r_0$. Therefore

$$
C = 2n
$$

Hence Green function is now found from (2) as, for $n\neq 0$

$$
G(r,r_0) = \frac{1}{2n}\begin{cases} r^n r_0^{-n} & 0<r<r_0\\ r_0^n r^{-n} & r_0<r<\infty\end{cases}
$$

Since $f(r)$ in the original ODE is zero, there is nothing to convolve with. i.e. $y(r)=\int_0^\infty G(r,r_0)f(r_0)\,dr_0$ here is not needed since there is no $f(r)$. Therefore the above is the final solution.

Extended solution

This solution shows derivation of (2) above. It can be considered as an appendix. The Green function is the solution to

$$
\begin{aligned}
rG(r,r_0) + G(r,r_0) - \frac{n^2}{r}G(r,r_0) &= \delta(r-r_0) \qquad (1B)\\
\lim_{r\to 0} G(r,r_0) &= 0\\
\lim_{r\to\infty} G(r,r_0) &= 0
\end{aligned}
$$

In (1B), r_0 is the location of the impulse and r is the location of the observed response due to this impulse. The solution to the above ODE is now broken to two regions

$$
G(r,r_0) = \begin{cases} A_1y_1(r) + A_2y_2(r) & 0<r<r_0\\ B_1y_1(r)+B_1y_2(r) & r_0<r<\infty\end{cases} \tag{2}
$$

Where $y_1(r), y_2(r)$ are the solution to $ry''(r)+y'(r)-\frac{n^2}{r}y(r)=0$ and these were found in part (a) to be $y_1(r)=r^n, y_2(r)=r^{-n}$ and A_1,A_2,B_1,B_2 needs to be determined. Hence (2) becomes

$$
G(r,r_0) = \begin{cases} A_1r^n + A_2r^{-n} & 0<r<r_0\\ B_1r^n+B_2r^{-n} & r_0<r<\infty\end{cases} \tag{3}
$$

The left boundary condition $\lim_{r\to 0} G(r,r_0)=0$ implies $A_2=0$ and the right boundary condition $\lim_{r\to\infty} G(r,r_0)=0$ implies $B_1=0$. This is needed to keep the solution bounded. Hence (3) simplifies to

$$
G(r,r_0) = \begin{cases} A_1r^n & 0<r<r_0\\ B_2r^{-n} & r_0<r<\infty\end{cases} \tag{4}
$$

To determine the remaining two constants A_1, B_2, two additional conditions are needed. The first is that $G(r,r_0)$ is continuous at $r=r_0$ which implies

$$
A_1r_0^n = B_2r_0^{-n} \tag{5}
$$

The second condition is the jump in the derivative of $G(r, r_0)$ given by

$$\left.\frac{d}{dr}G(r, r_0)\right|_{r>r_0} - \left.\frac{d}{dr}G(r, r_0)\right|_{r<r_0} = \frac{-1}{p(r_0)}$$

Where $p(r_0)$ comes from the Sturm Liouville form of the homogeneous ODE. This was found above as $p(r_0) = r_0$. Hence the above condition becomes

$$\left.\frac{d}{dr}G(r, r_0)\right|_{r>r_0} - \left.\frac{d}{dr}G(r, r_0)\right|_{r<r_0} = \frac{-1}{r_0}$$

Equation (4) shows that $\left.\frac{d}{dr}G(r, r_0)\right|_{r>r_0} = -nB_2 r_0^{-n-1}$ and that $\left.\frac{d}{dr}G(r, r_0)\right|_{r<r_0} = nA_1 r_0^{n-1}$. Using these in the above gives the second equation needed

$$-nB_2 r_0^{-n-1} - nA_1 r_0^{n-1} = \frac{-1}{r_0} \tag{6}$$

Solving (5,6) for A_1, B_1: From (5) $A_1 = B_2 r_0^{-2n}$. Substituting this in (6) gives

$$\begin{aligned}
-nB_2 r_0^{-n-1} - n\left(B_2 r_0^{-2n}\right) r^{n-1} &= \frac{-1}{r_0} \\
-nB_2 r^{-n-1} - nB_2 r^{-n-1} &= \frac{-1}{r_0} \\
-2nB_2 r_0^{-n-1} &= -r_0^{-1} \\
B_2 &= \frac{-r_0^{-1}}{-2n r_0^{-n-1}} \\
&= \frac{1}{2n} r_0^n
\end{aligned}$$

But since $A_1 = B_2 r_0^{-2n}$, then

$$\begin{aligned}
A_1 &= \frac{1}{2n} r_0^n r_0^{-2n} \\
&= \frac{1}{2n} r_0^{-n}
\end{aligned}$$

Therefore the solution (4), which is the Green function, becomes, for $n \neq 0$

$$G(r, r_0) = \begin{cases} \frac{1}{2n} r_0^{-n} r^n & 0 < r < r_0 \\ \frac{1}{2n} r_0^n r^{-n} & r_0 < r < \infty \end{cases} \tag{7}$$