HW 6, Physics 501
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Contents

1 Problem 1

Consider the equation $xy'' + (c - x)y' - ay = 0$. Identify a regular singular point and find two series solutions around this point. Test the solutions for convergence.

Solution

Writing the ODE as

$$y'' + A(x)y' + B(x)y = 0$$

Where

$$A(x) = \frac{(c-x)}{x}$$
$$B(x) = \frac{-a}{x}$$

The above shows that $x_0 = 0$ is a singularity point for both $A(x)$ and $B(x)$. Examining $A(x)$ and $B(x)$ to determine what type of singular point it is

$$\lim_{x\to x_0}(x - x_0)A(x) = \lim_{x\to 0} x\frac{(c-x)}{x} = \lim_{x\to 0}(c - x) = c$$

Because the limit exists, then $x_0 = 0$ is regular singular point for $A(x)$.

$$\lim_{x\to x_0}(x - x_0)^2 B(x) = \lim_{x\to 0} x^2\left(\frac{-a}{x}\right) = \lim_{x\to 0}(-ax) = 0$$

Because the limit exists, then $x_0 = 0$ is also regular singular point for $B(x)$.

Therefore $x_0 = 0$ is a regular singular point for the ODE.

Assuming the solution is Frobenius series gives

$$\begin{aligned} y(x) &= x^r \sum_{n=0}^{\infty} C_n (x - x_0)^n \qquad C_0 \neq 0 \\ &= x^r \sum_{n=0}^{\infty} C_n x^n \\ &= \sum_{n=0}^{\infty} C_n x^{n+r} \end{aligned}$$

Therefore

$$\begin{aligned} y' &= \sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-2} \end{aligned}$$

Substituting the above in the original ODE $xy'' + (c - x)y' - ay = 0$ gives

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-2} + (c-x)\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} - a\sum_{n=0}^{\infty}C_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + c\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} - x\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} - \sum_{n=0}^{\infty}aC_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-1} + \sum_{n=0}^{\infty}c(n+r)C_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)C_n x^{n+r} - \sum_{n=0}^{\infty}aC_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}((n+r)(n+r-1) + c(n+r))C_n x^{n+r-1} - \sum_{n=0}^{\infty}((n+r)+a)C_n x^{n+r} = 0$$

Since all powers of x have to be the same, adjusting indices and exponents gives (where in the second sum above, the outside index n is increased by 1 and n inside the sum is decreased by 1)

$$\sum_{n=0}^{\infty}((n+r)(n+r-1) + c(n+r))C_n x^{n+r-1} - \sum_{n=1}^{\infty}((n-1+r)+a)C_{n-1}x^{n+r-1} = 0 \tag{1}$$

Setting $n = 0$ gives the indicial equation, which only comes from the first sum above as the second sum starts from $n = 1$.

$$((r)(r-1) + cr) C_0 = 0$$

Since $C_0 \neq 0$ then

$$\begin{aligned}(r)(r-1) + cr &= 0\\ r^2 - r + cr &= 0\\ r(r + c - 1) &= 0\end{aligned}$$

The roots are

$$\begin{aligned}r_1 &= 1 - c\\ r_2 &= 0\end{aligned}$$

Assuming that $r_2 - r_1$ is not an integer, in other words, assuming $1-c$ is not an integer (problem did not say), then In this case, two linearly independent solutions can be constructed directly. The first is associated with $r_1 = 1 - c$ and the second is associated with $r_2 = 0$. These solutions are

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} C_n x^{n+1-c} \qquad C_0 \neq 0\\ y_2(x) &= \sum_{n=0}^{\infty} D_n x^n \qquad D_0 \neq 0\end{aligned}$$

The coefficients are not the same in each solution. For the first one C_n is used and for the second D_n is used.

The solution $y_1(x)$ associated with $r_1 = 1 - c$ is now found. From (1), and replacing r by $1 - c$ gives

$$\begin{aligned}\sum_{n=0}^{\infty} ((n+1-c)(n+1-c-1) + c(n+1-c)) C_n x^{n+1-c-1} - \sum_{n=1}^{\infty} ((n-1+1-c) + a) C_{n-1} x^{n+1-c-1} &= 0\\ \sum_{n=0}^{\infty} ((n+1-c)(n-c) + c(n+1-c)) C_n x^{n-c} - \sum_{n=1}^{\infty} ((n-c) + a) C_{n-1} x^{n-c} &= 0\\ \sum_{n=0}^{\infty} n(n-c+1) C_n x^{n-c} - \sum_{n=1}^{\infty} ((n-c) + a) C_{n-1} x^{n-c} &= 0\end{aligned}$$

For $n > 0$ the above gives the recursive relation ($n = 0$ is not used, since it was used to find r). For $n > 0$ the last equation above gives

$$\begin{aligned}n(n-c+1) C_n - ((n-c) + a) C_{n-1} &= 0\\ C_n &= \frac{((n-c) + a)}{n(n-c+1)} C_{n-1}\end{aligned}$$

Few terms are generated to see the pattern. For $n = 1$

$$C_1 = \frac{(1-c+a)}{1(1-c+1)} C_0 = \frac{(1-c+a)}{(2-c)} C_0$$

For $n = 2$

$$\begin{aligned}C_2 &= \frac{(2-c+a)}{2(2-c+1)} C_1\\ &= \frac{(2-c+a)}{2(3-c)} \frac{(1-c+a)}{(2-c)} C_0\end{aligned}$$

For $n = 3$

$$\begin{aligned}C_3 &= \frac{(3-c+a)}{3(3-c+1)} C_2\\ &= \frac{(3-c+a)}{3(4-c)} \frac{(2-c+a)}{2(3-c)} \frac{(1-c+a)}{(2-c)} C_0\end{aligned}$$

And so on. The pattern for general term is

$$\begin{aligned}C_n &= \frac{((n-c)+a)}{n(n-c+1)}\cdots\frac{(3-c+a)}{3(3-c+1)}\frac{(2-c+a)}{2(2-c+1)}\frac{(1-c+a)}{1(1-c+1)}C_0\\ &= \prod_{m=1}^{n}\frac{((m-c)+a)}{m(n-c+1)}\end{aligned}$$

Therefore the solution associated with $r_1 = 1-c$ is

$$\begin{aligned}y_1(x) &= \sum_{n=0}^{\infty} C_n x^{n+r}\\ &= \sum_{n=0}^{\infty} C_n x^{n+1-c}\\ &= C_0 x^{1-c} + C_1 x^{2-c} + C_2 x^{3-c} + \cdots\end{aligned}$$

Using results found above, and looking at few terms gives the <u>first solution</u> as

$$y_1(x) = C_0 x^{1-c}\left(1 + \frac{(1-c+a)}{(2-c)}x + \frac{1}{2}\frac{(2-c+a)}{(3-c)}\frac{(1-c+a)}{(2-c)}x^2 + \frac{1}{6}\frac{(3-c+a)}{(4-c)}\frac{(2-c+a)}{(3-c)}\frac{(1-c+a)}{(2-c)}x^3 + \cdots\right)$$

The second solution associated with $r_2 = 0$ is now found. As above, using (1) but with D_n instead of C_n for coefficients and replacing r by zero gives

$$\sum_{n=0}^{\infty}(n(n-1)+cn)D_n x^{n-1} - \sum_{n=1}^{\infty}((n-1)+a)D_{n-1}x^{n-1} = 0$$

For $n > 0$ the above gives the recursive relation for the second solution

$$\begin{aligned}(n(n-1)+cn)D_n - ((n-1)+a)D_{n-1} &= 0\\ D_n &= \frac{n-1+a}{n(n-1)+cn}D_{n-1}\\ &= \frac{n-1+a}{cn-n+n^2}D_{n-1}\end{aligned}$$

Few terms are now generated to see the pattern. For $n = 1$

$$D_1 = \frac{a}{c}D_0$$

For $n = 2$

$$\begin{aligned}D_2 &= \frac{1+a}{2c-2+4}D_1\\ &= \frac{1+a}{2(c+1)}\frac{a}{c}D_0\end{aligned}$$

For $n = 3$

$$\begin{aligned}D_3 &= \frac{3-1+a}{3c-3+9}D_2\\ &= \frac{2+a}{3(c+2)}\frac{1+a}{2(c+1)}\frac{a}{c}D_0\end{aligned}$$

And so on. Hence the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= \sum_{n=0}^{\infty} D_n x^n\\ &= D_0 + D_1 x + D_2 x^2 + \cdots\end{aligned}$$

Using result found above gives the <u>second solution</u> as

$$y_2(x) = D_0\left(1 + \frac{a}{c}x + \frac{1}{2}\frac{(1+a)a}{c(c+1)}x^2 + \frac{1}{6}\frac{a(1+a)(2+a)}{c(c+2)(c+1)}x^3 + \cdots\right)$$

The final solution is therefore the sum of the two solutions

$$\begin{aligned}y(x) &= C_0 x^{1-c}\left(1 + \frac{(1-c+a)}{(2-c)}x + \frac{1}{2}\frac{(2-c+a)}{(3-c)}\frac{(1-c+a)}{(2-c)}x^2 + \frac{1}{6}\frac{(3-c+a)}{(4-c)}\frac{(2-c+a)}{(3-c)}\frac{(1-c+a)}{(2-c)}x^3 + \cdots\right)\\ &\quad + D_0\left(1 + \frac{a}{c}x + \frac{1}{2}\frac{(1+a)a}{c(c+1)}x^2 + \frac{1}{6}\frac{a(1+a)(2+a)}{c(c+2)(c+1)}x^3 + \cdots\right)\end{aligned} \tag{2}$$

Where C_0, D_0 are the two constant of integration.

Testing for convergence. For $y_1(x)$ solution, the general term from above was

$$C_n x^n = \frac{((n-c)+a)}{n(n-c+1)} C_{n-1} x^n$$

Hence by ratio test

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{C_n x^n}{C_{n-1} x^{n-1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{((n-c)+a)}{n(n-c+1)} C_{n-1} x^n}{C_{n-1} x^{n-1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{((n-c)+a)\,x}{(n(n-c+1))} \right| \\
&= |x| \lim_{n\to\infty} \left| \frac{n-c+a}{n^2-nc+n} \right| \\
&= |x| \lim_{n\to\infty} \left| \frac{\frac{1}{n} - \frac{c}{n^2} + \frac{a}{n^2}}{1 - \frac{c}{n} + \frac{1}{n}} \right| \\
&= |x| \left| \frac{0}{1} \right| \\
&= 0
\end{aligned}$$

Therefore the series $y_1(x)$ converges for all x.

Testing for convergence. For $y_2(x)$ solution, the general term is

$$D_n x^n = \frac{n-1+a}{cn-n+n^2} D_{n-1} x^n$$

Hence by ratio test

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{D_n x^n}{D_{n-1} x^{n-1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{n-1+a}{cn-n+n^2} D_{n-1} x^n}{D_{n-1} x^{n-1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{n-1+a}{cn-n+n^2} x \right| \\
&= |x| \lim_{n\to\infty} \left| \frac{n-1+a}{cn-n+n^2} \right| \\
&= |x| \lim_{n\to\infty} \left| \frac{\frac{1}{n} - \frac{1}{n^2} + \frac{a}{n^2}}{\frac{c}{n} - \frac{1}{n} + 1} \right| \\
&= |x| \left| \frac{0}{1} \right| \\
&= 0
\end{aligned}$$

Therefore the series $y_2(x)$ also converges for all x. This means the solution $y(x) = y_1(x) + y_2(x)$ found in (2) above also converges for all x.

2 Problem 2

The Sturm Liouville equation can be expressed as

$$L\left[u\left(x\right)\right]=\lambda\rho\left(x\right)u\left(x\right)$$

Where L is given as in class. Show L is Hermitian on the domain $a\leq x\leq b$ with boundary conditions $u\left(a\right)=u\left(b\right)=0$. Find the orthogonality condition.

Solution

$$L=-\left(p\frac{d^2}{dx^2}+p'\frac{d}{dx}-q\right)$$

The operator L is Hermitian if

$$\int_a^b\bar{v}L\left[u\right]dx=\overline{\int_a^b\bar{u}L\left[v\right]dx}$$

Where in the above u,v are any two functions defined over the domain that satisfy the boundary conditions given. Starting from the left integral to show it will result in the right integral. Replacing $L\left[u\right]$ by $-\left(p\frac{d^2}{dx^2}+p'\frac{d}{dx}-q\right)u$ in the LHS of the above gives

$$\begin{aligned}-\int_a^b\bar{v}\left(p\frac{d^2}{dx^2}+p'\frac{d}{dx}-q\right)u\ dx&=-\int_a^b\bar{v}\left(p\frac{d^2u}{dx^2}+p'\frac{du}{dx}-qu\right)\ dx\\&=-\int_a^b\bar{v}p\frac{d^2u}{dx^2}+\bar{v}p'\frac{du}{dx}-q\bar{v}u\ dx\\&=\overbrace{-\int_a^b p\bar{v}\frac{d^2u}{dx^2}dx}^{I_1}-\int_a^b\bar{v}p'\frac{du}{dx}dx+\int_a^b q\bar{v}u\ dx\end{aligned}\tag{1}$$

Looking at the first integral above, which is $I_1=\int_a^b\left(p\bar{v}\right)\left(\frac{d^2u}{dx^2}\right)dx$. The idea is to integrate this twice to move the second derivative from u to $\bar{v}$. Applying $\int AdB=AB-\int BdA$, where

$$\begin{aligned}A&\equiv p\bar{v}\\dB&\equiv\frac{d^2u}{dx^2}\end{aligned}$$

Hence

$$\begin{aligned}dA&=p\frac{d\bar{v}}{dx}+p'\bar{v}\\B&=\frac{du}{dx}\end{aligned}$$

Therefore the integral I_1 in (1) becomes

$$\begin{aligned}I_1&=\int_a^b p\bar{v}\frac{d^2}{dx^2}u\\&=\left[p\bar{v}\frac{du}{dx}\right]_a^b-\int_a^b\frac{du}{dx}\left(p\frac{d\bar{v}}{dx}+p'\bar{v}\right)dx\end{aligned}$$

But $\bar{v}\left(a\right)=0$ and $\bar{v}\left(b\right)=0$, hence the boundary terms above vanish and simplifies to

$$\begin{aligned}I_1&=-\int_a^b p\frac{du}{dx}\frac{d\bar{v}}{dx}+p'\bar{v}\ \frac{du}{dx}dx\\&=-\int_a^b p\frac{du}{dx}\frac{d\bar{v}}{dx}dx-\int_a^b p'\bar{v}\ \frac{du}{dx}dx\end{aligned}\tag{2}$$

Before integrating by parts a second time, putting the result of I_1 back into (1) first simplifies the result. Substituting (2) into (1) gives

$$\begin{aligned}\int_a^b \bar{v} L[u]\,dx &= -I_1 - \int_a^b \bar{v} p' \frac{du}{dx} dx + \int_a^b q\bar{v}u\,dx \\ &= -\overbrace{\left(-\int_a^b p\frac{du}{dx}\frac{d\bar{v}}{dx}dx - \int_a^b p'\bar{v}\,\frac{du}{dx}dx\right)}^{I_1} - \int_a^b \bar{v}p'\frac{du}{dx}dx + \int_a^b q\bar{v}u\,dx \\ &= \int_a^b p\frac{du}{dx}\frac{d\bar{v}}{dx}dx + \int_a^b p'\bar{v}\,\frac{du}{dx}dx - \int_a^b \bar{v}p'\frac{du}{dx}dx + \int_a^b q\bar{v}u\,dx\end{aligned}$$

The second and third terms above cancel and the result becomes

$$\int_a^b \bar{v}L[u]\,dx = \overbrace{\int_a^b p\frac{du}{dx}\frac{d\bar{v}}{dx}dx}^{I_2} + \int_a^b q\bar{v}u\,dx \tag{3}$$

Now integration by parts is applied on the first integral above. Let $I_2 = \int_a^b \frac{du}{dx}\left(p\frac{d\bar{v}}{dx}\right)dx$. Applying $\int AdB = AB - \int BdA$, where

$$\begin{aligned}A &\equiv p\frac{d\bar{v}}{dx} \\ dB &\equiv \frac{du}{dx}\end{aligned}$$

Hence

$$\begin{aligned}dA &= p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx} \\ B &= u\end{aligned}$$

Therefore the integral I_2 becomes

$$I_2 = \left[p\frac{d\bar{v}}{dx}u\right]_a^b - \int_a^b u\left(p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx}\right)dx$$

But $u(a) = 0, u(b) = 0$, hence the boundary term vanishes and the above simplifies to

$$I_2 = -\int_a^b u\left(p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx}\right)dx$$

Substituting the above back into (3) gives

$$\begin{aligned}\int_a^b \bar{v}L[u]\,dx &= -\int_a^b u\left(p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx}\right)dx + \int_a^b q\bar{v}u\,dx \\ &= -\int_a^b u\left(p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx} - q\bar{v}\right)dx\end{aligned}$$

But $-\left(p\frac{d^2\bar{v}}{dx^2} + p'\frac{d\bar{v}}{dx} - q\bar{v}\right) = L[\bar{v}]$ by definition, and the above becomes

$$\int_a^b \bar{v}L[u]\,dx = \int_a^b uL[\bar{v}]\,dx$$

But $\int_a^b uL[\bar{v}]\,dx = \overline{\int_a^b \bar{u}L[v]\,dx}$, and the above becomes

$$\int_a^b \bar{v}L[u]\,dx = \overline{\int_a^b \bar{u}\left(L[v]\right)dx}$$

Therefore L is Hermitian.

3 Problem 3

1. For the equation $y'' + \frac{1-\alpha^2}{4x^2} y = 0$ show that two solutions are $y_1(x) = a_0 x^{\frac{1+\alpha}{2}}$ and $y_2(x) = a_0 x^{\frac{1-\alpha}{2}}$

2. For $\alpha = 0$, the two solutions are not independent. Find a second solution y_{20} by solving $W' = 0$ (W is the Wronskian).

3. Show that the second solution found in (2) is a limiting case of the two solutions from part (1). That is

$$y_{20} = \lim_{\alpha \to 0} \frac{y_1 - y_2}{\alpha}$$

Solution

3.1 Part 1

The point $x_0 = 0$ is a regular singular point. This is shown as follows.

$$\begin{aligned}
\lim_{x \to x_0} (x - x_0)^2 \frac{1-\alpha^2}{4x^2} &= \lim_{x \to 0} x^2 \frac{1-\alpha^2}{4x^2} \\
&= \lim_{x \to 0} \frac{1-\alpha^2}{4} \\
&= \frac{1-\alpha^2}{4}
\end{aligned}$$

Since the limit exist, then $x_0 = 0$ is a regular singular point. Assuming the solution is a Frobenius series given by

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad c_0 \neq 0$$

Therefore

$$\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \\
y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}
\end{aligned}$$

Substituting the above 2 expressions back into the original ODE gives

$$\begin{aligned}
4x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \right) + \left(1 - \alpha^2\right) \left(\sum_{n=0}^{\infty} c_n x^{n+r} \right) &= 0 \\
\sum_{n=0}^{\infty} 4(n+r)(n+r-1) c_n x^{n+r} + \left(1 - \alpha^2\right) \left(\sum_{n=0}^{\infty} c_n x^{n+r} \right) &= 0 \qquad (1)
\end{aligned}$$

Looking at $n = 0$ first, in order to obtain the indicial equation gives

$$\begin{aligned}
4(r)(r-1) c_0 + \left(1 - \alpha^2\right) c_0 &= 0 \\
c_0 \left(4r^2 - 4r + \left(1 - \alpha^2\right)\right) &= 0
\end{aligned}$$

But $c_0 \neq 0$, therefore

$$r^2 - r + \frac{\left(1 - \alpha^2\right)}{4} = 0$$

The roots are $r = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$, but $a = 1, b = -1, c = \frac{(1-\alpha^2)}{4}$, hence the roots are

$$\begin{aligned}
r &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - (1 - \alpha^2)} \\
&= \frac{1}{2} \pm \frac{1}{2}\sqrt{\alpha^2} \\
&= \frac{1}{2} \pm \frac{1}{2}\alpha
\end{aligned}$$

Hence $r_1 = \frac{1}{2}(1+\alpha)$ and $r_2 = \frac{1}{2}(1-\alpha)$. Each one of these roots gives a solution. The difference is

$$\begin{aligned} r_2 - r_1 &= \frac{1}{2}(1+\alpha) - \frac{1}{2}(1-\alpha) \\ &= \alpha \end{aligned}$$

Therefore, to use the same solution form $y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$ and $y_2(x) = \sum_{n=0}^{\infty} d_n x^{n+r_2}$ for each, it is assumed that α is not an integer. In this case, the recursive relation for $y_1(x)$ is found from (1) by using $r = \frac{1}{2}(1+\alpha)$ which results in

$$\sum_{n=0}^{\infty} 4\left(n + \frac{1}{2}(1+\alpha)\right)\left(n + \frac{1}{2}(1+\alpha) - 1\right) c_n x^{n+\frac{1}{2}(1+\alpha)} + \left(1-\alpha^2\right)\left(\sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}(1+\alpha)}\right) = 0$$

For $n > 0$ the above becomes

$$\begin{aligned} 4\left(n + \frac{1}{2}(1+\alpha)\right)\left(n + \frac{1}{2}(1+\alpha) - 1\right) c_n + \left(1-\alpha^2\right) c_n &= 0 \\ \left(4\left(n + \frac{1}{2}(1+\alpha)\right)\left(n + \frac{1}{2}(1+\alpha) - 1\right) + \left(1-\alpha^2\right)\right) c_n &= 0 \\ 4n(n+\alpha) c_n &= 0 \end{aligned}$$

The above can be true for all $n > 0$ only when $c_n = 0$ for $n > 0$. Therefore the solution is only the term with c_0

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} = c_0 x^{r_1} = c_0 x^{\frac{1}{2}(1+\alpha)}$$

To find the second solution $y_2(x)$, the above is repeated but with

$$y_2(x) = \sum_{n=0}^{\infty} d_n x^{n+r_2}$$

Where the constants are not the same and by replacing r in (1) by $r_2 = \frac{1}{2}(1-\alpha)$. This results in

$$\sum_{n=0}^{\infty} 4\left(n + \frac{1}{2}(1-\alpha)\right)\left(n + \frac{1}{2}(1-\alpha) - 1\right) d_n x^{n+\frac{1}{2}(1-\alpha)} + \left(1-\alpha^2\right)\left(\sum_{n=0}^{\infty} d_n x^{n+\frac{1}{2}(1-\alpha)}\right) = 0$$

For $n > 0$

$$\begin{aligned} \left(4\left(n + \frac{1}{2}(1-\alpha)\right)\left(n + \frac{1}{2}(1-\alpha) - 1\right) + \left(1-\alpha^2\right)\right) d_n &= 0 \\ 4n(n-\alpha) d_n &= 0 \end{aligned}$$

The above is true for all $n > 0$ only when $c_n = 0$ for $n > 0$. Therefore the solution is just the term with d_0

$$y_2(x) = \sum_{n=0}^{\infty} d_n x^{n+r_2} = d_0 x^{r_2} = d_0 x^{\frac{1}{2}(1-\alpha)}$$

Therefore the two solutions are

$$\begin{aligned} y_1(x) &= c_0 x^{\frac{1}{2}(1+\alpha)} \\ y_2(x) &= d_0 x^{\frac{1}{2}(1-\alpha)} \end{aligned}$$

3.2 Part 2

When $\alpha = 0$ then the ODE becomes

$$4x^2 y'' + y = 0$$

And the two solutions found in part (1) simplify to

$$\begin{aligned} y_1(x) &= c_0 \sqrt{x} \\ y_2(x) &= d_0 \sqrt{x} \end{aligned}$$

Therefore the two solutions are not linearly independent. Let $y_{20}(x)$ be the second solution. The Wronskian is

$$W(x) = \begin{vmatrix} y_1 & y_{20} \\ y_1' & y_{20}' \end{vmatrix} = y_1 y_{20}' - y_{20} y_1' \tag{1}$$

Using Abel's theorem which says that for ODE of form $y'' + p(x)y' + q(x)y = 0$, the Wronskian is $W(x) = Ce^{-\int p(x)dx}$. Applying this to the given ODE above and since $p(x) = 0$ then the above becomes

$$W(x) = C$$

Where C is constant. For y_{20} to be linearly independent from y_1 $W(x) \neq 0$. Using $W(x) = C$ in (1) results in the following equation (here it is also assumed that $y_1 \neq 0$, or $x \neq 0$, because the equation is divided by y_1)

$$\begin{aligned} y_1 y_{20}' - y_{20} y_1' &= C \\ y_{20}' - y_{20}\frac{y_1'}{y_1} &= \frac{C}{y_1} \end{aligned}$$

Since $y_1 = \sqrt{x}$ and $y_1' = \frac{1}{2}\frac{1}{\sqrt{x}}$ the above simplifies to

$$\begin{aligned} y_{20}' - y_{20}\frac{\frac{1}{2}\frac{1}{\sqrt{x}}}{\sqrt{x}} &= \frac{C}{\sqrt{x}} \\ y_{20}' - y_{20}\frac{1}{2x} &= \frac{C}{\sqrt{x}} \end{aligned} \tag{2}$$

But the above is linear first order ODE of the form $Y' + pY = q$, therefore the standard integrating factor to use is $I = e^{\int p(x)dx}$ which results in

$$\begin{aligned} I &= e^{\int \frac{-1}{2x}dx} \\ &= e^{-\frac{1}{2}\int \frac{1}{x}dx} \\ &= e^{-\frac{1}{2}\ln x} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

Multiplying both sides of (2) by this integrating factor, makes the left side of (2) an exact differential

$$\frac{d}{dx}\left(y_{20}\frac{1}{\sqrt{x}}\right) = \frac{C}{x}$$

Integrating both sides gives

$$\begin{aligned} y_{20}\frac{1}{\sqrt{x}} &= C\int \frac{1}{x}dx + C_1 \\ y_{20}\frac{1}{\sqrt{x}} &= 2C\ln x + C_1 \\ y_{20} &= 2C\ln x\sqrt{x} + C_1\sqrt{x} \end{aligned}$$

Or

$$y_{20} = C_1 \ln x\sqrt{x} + C_2\sqrt{x} \tag{3}$$

The above is the second solution. Therefore the final solution is

$$y(x) = C_0 y_1(x) + C_3 y_{20}(x)$$

Substituting $y_1 = \sqrt{x}$ and y_{20} found above and combining the common term $\sqrt{x}$ and renaming constants gives

$$y(x) = C_1\sqrt{x} + C_2 \ln x\sqrt{x}$$

Another method to find the second solution

This method is called the reduction of order method. It does not require finding $W(x)$ first. Let the second solution be

$$y_{20} = Y = v(x)\, y_1(x) \tag{4}$$

Where $v(x)$ is unknown function to be determined, and $y_1(x) = \sqrt{x}$ which is the first solution that is already known. Therefore

$$\begin{aligned} Y' &= v'y_1 + vy_1' \\ Y'' &= v''y_1 + v'y_1' + v'y_1' + vy_1'' \\ &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

Since Y is a solution to the ODE $4x^2y'' + y = 0$, then substituting the above equations back into the ODE $4x^2y'' + y = 0$ gives

$$\begin{aligned} 4x^2\left(v''y_1 + 2v'y_1' + vy_1''\right) + vy_1 &= 0 \\ v''\left(4x^2y_1\right) + v'\left(8x^2y_1'\right) + v\left(\overbrace{4x^2y_1'' + y_1}^{0}\right) &= 0 \end{aligned}$$

But $4x^2y_1'' + y_1 = 0$ because y_1 is a solution. The above simplifies to

$$v''\left(4x^2y_1\right) + v'\left(8x^2y_1'\right) = 0$$

But $y_1 = x^{\frac{1}{2}}$, hence $y_1' = \frac{1}{2}x^{\frac{-1}{2}}$ and the above simplifies to

$$\begin{aligned} v''\left(4x^2x^{\frac{1}{2}}\right) + v'\left(4x^2x^{\frac{-1}{2}}\right) &= 0 \\ x^{\frac{5}{2}}v'' + v'x^{\frac{3}{2}} &= 0 \\ xv'' + v' &= 0 \\ v'' + \frac{1}{x}v' &= 0 \end{aligned}$$

This ODE is now easy to solve because the $v(x)$ term is missing. Let $w = v'$ and the above first order ODE $w' + \frac{1}{x}w = 0$. This is linear in w. Hence using integrating factor $I = e^{\int \frac{1}{x}dz} = x$, this ODE becomes

$$\begin{aligned} \frac{d}{x}(wx) &= 0 \\ wx &= C \\ w &= \frac{C}{x} \end{aligned}$$

Where C is constant of integration. Since $v' = w$, then $v' = \frac{C_1}{x}$. Now $v(x)$ is found by integrating both sides

$$v = C_1 \ln x + C_2$$

Therefore the second solution from (4) becomes

$$\begin{aligned} y_{20} &= C_1 \ln x y_1 + C_2 y_1 \\ &= C_1\sqrt{x}\ln x + C_2\sqrt{x} \end{aligned} \tag{5}$$

Comparing the above to (3), shows it is the same solution. Both methods can be used, but reduction of order method is a more common method and it does not require finding the Wronskian first, although it is not hard to find by using Abel's theorem.

3.3 Part 3

The solutions we found in part (1) are

$$\begin{aligned} y_1(x) &= C_1 x^{\frac{1}{2}(1+\alpha)} \\ y_2(x) &= C_2 x^{\frac{1}{2}(1-\alpha)} \end{aligned}$$

Therefore

$$\lim_{\alpha\to 0}\frac{y_1 - y_2}{\alpha} = \lim_{\alpha\to 0}\frac{C_1 x^{\frac{1}{2}(1+\alpha)} - C_2 x^{\frac{1}{2}(1-\alpha)}}{\alpha}$$

Applying L'Hopital's

$$\lim_{\alpha\to 0}\frac{y_1-y_2}{\alpha}=\lim_{\alpha\to 0}\frac{C_1\frac{d}{d\alpha}\left(x^{\frac{1}{2}(1+\alpha)}\right)-C_2\frac{d}{d\alpha}\left(x^{\frac{1}{2}(1-\alpha)}\right)}{1} \tag{1}$$

But

$$\begin{aligned}\frac{d}{d\alpha}\left(x^{\frac{1}{2}(1+\alpha)}\right)&=\frac{d}{d\alpha}e^{\frac{1}{2}(1+\alpha)\ln x}\\&=\frac{d}{d\alpha}e^{\left(\frac{1}{2}\ln x+\alpha\ln x\right)}\\&=\ln xe^{\left(\frac{1}{2}\ln x+\alpha\ln x\right)}\end{aligned}$$

And

$$\begin{aligned}\frac{d}{d\alpha}\left(x^{\frac{1}{2}(1-\alpha)}\right)&=\frac{d}{d\alpha}e^{\frac{1}{2}(1-\alpha)\ln x}\\&=\frac{d}{d\alpha}e^{\left(\frac{1}{2}\ln x-\alpha\ln x\right)}\\&=-\ln xe^{\left(\frac{1}{2}\ln x-\alpha\ln x\right)}\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}\lim_{\alpha\to 0}\frac{y_1-y_2}{\alpha}&=\lim_{\alpha\to 0}C_1\ln xe^{\left(\frac{1}{2}\ln x+\alpha\ln x\right)}+C_2\ln xe^{\left(\frac{1}{2}\ln x-\alpha\ln x\right)}\\&=\ln x\left(\lim_{\alpha\to 0}C_1e^{\left(\frac{1}{2}\ln x+\alpha\ln x\right)}+C_2e^{\left(\frac{1}{2}\ln x-\alpha\ln x\right)}\right)\\&=\ln x\left(C_1e^{\frac{1}{2}\ln x}+C_2e^{\frac{1}{2}\ln x}\right)\\&=\ln x\left(C_1\sqrt{x}+C_2\sqrt{x}\right)\\&=C\sqrt{x}\ln x\end{aligned}$$

The above is the same as (3) found in part (2). Hence

$$y_{20}(x)=\lim_{\alpha\to 0}\frac{y_1-y_2}{\alpha}$$

Which is what the problem asked to show.