HW 4, Physics 501
Fall 2018
University Of Wisconsin, Milwaukee

Nasser M. Abbasi

November 22, 2018 Compiled on November 22, 2018 at 2:04am

Contents

1 Problem 1

Using series expansion evaluate the integral $I=\int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x}$

Solution

We first need to find the Taylor series for $\ln\left(\frac{1+x}{1-x}\right)$ expanded around $x=0$. Since

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \ln\left((1+x)\left(\frac{1}{1-x}\right)\right)\\ &= \ln(1+x)+\ln\left(\frac{1}{1-x}\right)\\ &= \ln(1+x)-\ln(1-x) \qquad (1)\end{aligned}$$

Looking at $\ln(1+x)$, where now $f(x)=\ln(1+x)$, then we see that $f'(x)=\frac{1}{1+x}, f''(x)=\frac{-1}{(1+x)^2}, f'''(x)=\frac{2}{(1+x)^3}, f^{(4)}(x)=-\frac{2\cdot 3}{(1+x)^4},\cdots$, therefore

$$\begin{aligned}\ln(1+x) &= f(0)+xf'(0)+\frac{x^2}{2}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(4)}(0)+\cdots\\ &= 0+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \qquad (2)\end{aligned}$$

Similarly for $\ln(1-x)$, where now $f'(x)=\frac{-1}{1-x}, f''(x)=\frac{-1}{(1-x)^2}, f'''(x)=\frac{-2}{(1-x)^3}, f^{(4)}(x)=-\frac{2\cdot 3}{(1-x)^4},\cdots$, therefore

$$\begin{aligned}\ln(1-x) &= f(0)+xf'(0)+\frac{x^2}{2}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(4)}(0)+\cdots\\ &= 0-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\cdots \qquad (3)\end{aligned}$$

Using (2,3) in (1) gives the series expansion for $\ln\left(\frac{1+x}{1-x}\right)$ as

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)\\ &= 2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\frac{2}{7}x^7+\cdots \qquad (4)\end{aligned}$$

Using (4) in the integral given results in

$$\begin{aligned}I &= \int_0^1\left(2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\frac{2}{7}x^7+\cdots\right)\frac{dx}{x}\\ &= \int_0^1\left(2+\frac{2}{3}x^2+\frac{2}{5}x^4+\frac{2}{7}x^6+\cdots\right)dx\\ &= \left[2x+\frac{2}{3}\frac{x^3}{3}+\frac{2}{5}\frac{x^5}{5}+\frac{2}{7}\frac{x^7}{7}+\cdots\right]_0^1\end{aligned}$$

Which simplifies to

$$\begin{aligned}I &= 2+\frac{2}{3}\frac{1}{3}+\frac{2}{5}\frac{1}{5}+\frac{2}{7}\frac{1}{7}+\cdots\\ &= 2+\frac{2}{3^2}+\frac{2}{5^2}+\frac{2}{7^2}+\frac{2}{9^2}+\cdots\\ &= 2\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots\right)\\ &= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \qquad (5)\end{aligned}$$

The following are two methods to obtain closed form sum for $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. The first method is based on writing

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n)^2}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \qquad (6)$$

Where the sum on the left is broken into odd and even terms on the right, as in

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\left(\frac{1}{2^2}+\frac{1}{4^2}+\cdots\right)+\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

But, from lecture Sept. 12, 2018, we showed in class that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)=\frac{\pi^2}{6} \tag{7}$$

(This is called the Basel problem, and the above closed form sum was first given by Euler in 1734). Now using (7) into (6) results in

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}&=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\\&=\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}\\&=\frac{3}{4}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)\\&=\frac{3}{4}\left(\frac{\pi^2}{6}\right)\\&=\frac{\pi^2}{8}\end{aligned}$$

Another way to obtained closed form sum for $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$ is to use Fourier series. Considering the Fourier series for the following periodic function

$$f(x)=\begin{cases}-x & -\pi<x<0\\0 & 0\leq x\leq\pi\end{cases}$$

Using

$$f(x)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos(nx)+\sum_{n=1}^{\infty}B_n\sin(nx)$$

Therefore

$$A_0=\frac{1}{\pi}\int_{-\pi}^{0}-xdx=\frac{-1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi}^{0}=\frac{-1}{2\pi}\left(x^2\right)_{-\pi}^{0}=\frac{-1}{2\pi}\left(-\pi^2\right)=\frac{1}{2}\pi$$

And

$$\begin{aligned}A_n&=\frac{-1}{\pi}\int_{-\pi}^{0}x\cos(nx)\,dx=\frac{1+(-1)^{n+1}}{n^2}\\B_n&=\frac{-1}{\pi}\int_{-\pi}^{0}x\sin(nx)\,dx=\frac{(-1)^{n+1}}{n}\pi\end{aligned}$$

Hence the Fourier series for $f(x)$ is

$$\begin{aligned}f(x)&=\frac{\pi}{4}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2}\cos(nx)-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\pi(\sin nx)\\&=\frac{\pi}{4}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2}\cos(nx)-\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)\end{aligned}$$

Evaluating the above at $x=0$ then all the sin terms vanish and we obtain

$$\begin{aligned}0&=\frac{\pi}{4}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2}\\&=\frac{\pi}{4}-\frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)\\&=\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\end{aligned}$$

Therefore

$$\begin{aligned}\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} &= \frac{\pi}{4}\\ \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} &= \frac{\pi^2}{8}\end{aligned}$$

Now that we found closed form sum for $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$, we can find the value of the integral. Since $I = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$, then

$$\begin{aligned}\int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x} &= 2\left(\frac{\pi^2}{8}\right)\\ &= \frac{\pi^2}{4}\end{aligned}$$

2 Problem 2

Let $I(x) = \int_0^\infty e^{xf(t)} dt$ with $f(t) = t - \frac{e^t}{x}$, find a large x approximation for this integral.

Solution

$$\begin{aligned} I &= \int_0^\infty \exp(xf(t))\, dt \\ &= \int_0^\infty \exp\left(x\left(t - \frac{e^t}{x}\right)\right) dt \\ &= \int_0^\infty \exp\left(xt - e^t\right) dt \\ &= \int_0^\infty \exp(F(t))\, dt \end{aligned} \tag{1}$$

Where $F(t) = xt - e^t$. We need to find saddle point where $F(t)$ is maximum. Hence

$$\begin{aligned} \frac{d}{dt}F(t) &= 0 \\ x - e^t &= 0 \\ e^t &= x \\ t_0 &= \ln(x) \end{aligned}$$

Where t_0 is location of t where $F(t)$ is maximum. We called this in class t_{peak}. We now expand $F(t)$ around t_0 using Taylor series

$$F(t) = F(t_0) + F'(t_0)(t - t_0) + \frac{1}{2}F''(t_0)(t - t_0)^2 + \cdots \tag{2}$$

But

$$\begin{aligned} F(t_0) &= x\ln(x) - e^{\ln x} \\ &= x\ln x - x \end{aligned}$$

And $F'(t) = x - e^t$, hence as expected $F'(t_0) = 0$. And $F''(t) = -e^t$, therefore $F''(t_0) = -e^{\ln x} = -x$. We see also that $F''(t_0) < 0$, which means the saddle point was a maximum and not a minimum (since x is positive). Using these in (2) gives

$$\begin{aligned} F(t) &\approx (x\ln x - x) + \frac{1}{2}(-x)(t - \ln x)^2 \\ &= x\ln x - x - \frac{1}{2}x(t - \ln x)^2 \end{aligned}$$

Substituting the above into (1) gives

$$\begin{aligned} I &= \int_0^\infty \exp\left(x\ln x - x - \frac{1}{2}x(t - \ln x)^2\right) dt \\ &= \int_0^\infty \exp(x\ln x)\exp(-x)\exp\left(-\frac{1}{2}x(t - \ln x)^2\right) dt \\ &= \exp(x\ln x)\exp(-x)\int_0^\infty \exp\left(-\frac{1}{2}x(t - \ln x)^2\right) dt \\ &= x^x e^{-x}\int_0^\infty e^{-\frac{1}{2}x(t-\ln x)^2} dt \end{aligned} \tag{3}$$

Now, since the peak value where $F(t)$ occurs is on the positive real axis, because $t_0 = \ln(x)$, therefore $x > 1$ to have a maximum, and assuming a narrow peak, then all the contribution to the integral comes from x close to the peak location, so we can change $\int_0^\infty e^{-\frac{1}{2}x(t-\ln x)^2} dt$ to $\int_{-\infty}^\infty e^{-\frac{1}{2}x(t-\ln x)^2} dt$ without affecting the final result. Therefore (3) becomes

$$I = x^x e^{-x}\int_{-\infty}^\infty e^{-\frac{1}{2}x(t-\ln x)^2} dt \tag{4}$$

Now comparing $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x(t-\ln x)^2} dt$ to the Gaussian integral $\int_{-\infty}^{\infty} e^{-a(t-b)^2} dt = \sqrt{\frac{\pi}{a}}$, shows that $a = \frac{x}{2}$ for our case. Hence

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x(t-\ln x)^2} dt = \sqrt{\frac{2\pi}{x}}$$

Therefore (4) becomes

$$I \approx x^x e^{-x} \sqrt{\frac{2\pi}{x}}$$

For large x.

3 Problem 3

Evaluate the following integrals with aid of residue theorem $a \geq 0$. (a) $\int_0^\infty \frac{1}{x^4+1} dx$ (b) $\int_0^\infty \frac{\cos(ax)}{x^2+1} dx$

3.1 Part (a)

Since the integrand is even, then

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$$

Now we consider the following contour

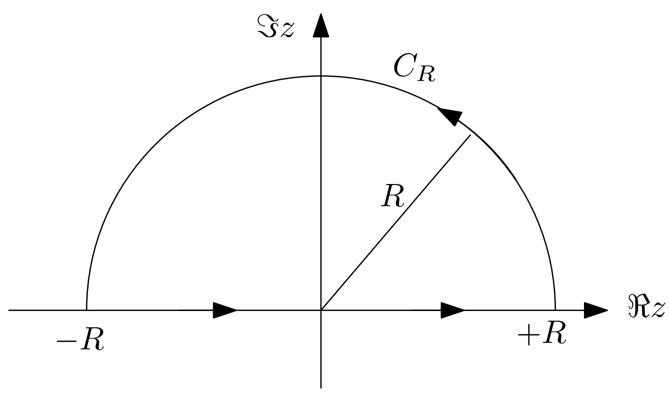

Figure 1: contour used for problem 3

Therefore

$$\oint_C f(z)\,dz = \left(\lim_{R\to\infty} \int_{-R}^{0} f(x)\,dx + \lim_{\tilde{R}\to\infty} \int_0^{\tilde{R}} f(x)\,dx \right) + \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

Using Cauchy principal value the integral above can be written as

$$\begin{aligned} \oint_C f(z)\,dz &= \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{C_R} f(z)\,d \\ &= 2\pi i \sum \text{Residue} \end{aligned}$$

Where $\sum$ Residue is sum of residues of $\frac{1}{z^4+1}$ for poles that are inside the contour C. Therefore the above becomes

$$\begin{aligned} \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx &= 2\pi i \sum \text{Residue} - \lim_{R\to\infty} \int_{C_R} f(z)\,dz \\ \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx &= 2\pi i \sum \text{Residue} - \lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz \end{aligned} \tag{1}$$

Now we will show that $\lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz = 0$. Since

$$\begin{aligned} \left| \int_{C_R} \frac{1}{z^4+1} dz \right| &\leq ML \\ &= |f(z)|_{\max} (\pi R) \end{aligned} \tag{2}$$

But

$$f(z) = \frac{1}{(z^2-i)(z^2+i)}$$

Hence, and since $z = R\,e^{i\theta}$ then

$$|f(z)|_{\max} \leq \frac{1}{|z^2-i|_{\min} |z^2+i|_{\min}}$$

But but inverse triangle inequality $\left|z^2-i\right| \geq |z|^2+1$ and $\left|z^2+i\right| \geq |z|^2-1$, and since $|z|=R$ then the above becomes

$$
\begin{aligned}
|f(z)|_{\max} &\leq \frac{1}{\left(R^2+1\right)\left(R^2-1\right)} \\
&= \frac{1}{R^4-1}
\end{aligned}
$$

Therefore (2) becomes

$$
\left|\int_{C_R} \frac{1}{z^4+1} dz\right| \leq \frac{\pi R}{R^4-1}
$$

Then it is clear that as $R \longrightarrow \infty$ the above goes to zero since $\lim_{R\to\infty} \frac{\pi R}{R^4-1} = \lim_{R\to\infty} \frac{\frac{\pi}{R^3}}{1-\frac{1}{R^4}} = \frac{0}{1} = 0$. Then (1) now simplifies to

$$
\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i \sum \text{Residue} \tag{2A}
$$

We just now need to find the residues of $\frac{1}{z^4+1}$ located in upper half plane. The zeros of the denominator $z^4+1=0$ are at $z=-1^{\frac{1}{4}}=\left(e^{i\pi}\right)^{\frac{1}{4}}$, then the first zero is at $e^{i\frac{\pi}{4}}$, and the second zero at $e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)}=e^{i\left(\frac{3}{4}\pi\right)}$ and the third zero at $e^{i\left(\frac{3}{4}\pi+\frac{\pi}{2}\right)}=e^{i\left(\frac{5}{4}\pi\right)}$ and the fourth zero at $e^{i\left(\frac{5}{4}\pi+\frac{\pi}{2}\right)}=e^{i\frac{7}{4}\pi}$. Hence poles are at

$$
\begin{aligned}
z_1 &= e^{i\frac{\pi}{4}} \\
z_2 &= e^{i\frac{3}{4}\pi} \\
z_3 &= e^{i\frac{5}{4}\pi} \\
z_4 &= e^{i\frac{7}{4}\pi}
\end{aligned}
$$

Out of these only the first two are in upper half plane z_1 and z_1. Hence

$$
\begin{aligned}
\text{Residue}\left(z_1\right) &= \lim_{z\to z_1}\left(z-z_1\right) f(z) \\
&= \lim_{z\to z_1}\left(z-z_1\right)\frac{1}{z^4-1}
\end{aligned}
$$

Applying L'Hopitals

$$
\begin{aligned}
\text{Residue}\left(z_1\right) &= \lim_{z\to z_1}\frac{1}{4z^3} \\
&= \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3} \\
&= \frac{1}{4e^{i\frac{3\pi}{4}}}
\end{aligned}
$$

Similarly for the other residue

$$
\begin{aligned}
\text{Residue}\left(z_2\right) &= \lim_{z\to z_2}\left(z-z_2\right) f(z) \\
&= \lim_{z\to z_1}\left(z-z_2\right)\frac{1}{z^4-1}
\end{aligned}
$$

Applying L'Hopitals

$$
\begin{aligned}
\text{Residue}\left(z_2\right) &= \lim_{z\to z_2}\frac{1}{4z^3} \\
&= \frac{1}{4\left(e^{i\frac{3}{4}\pi}\right)^3} \\
&= \frac{1}{4e^{i\frac{9\pi}{4}}} \\
&= \frac{1}{4e^{i\frac{\pi}{4}}}
\end{aligned}
$$

Hence (2A) becomes

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx &= 2\pi i\left(\frac{1}{4e^{i\frac{3\pi}{4}}}+\frac{1}{4e^{i\frac{\pi}{4}}}\right)\\ &= 2\pi i\left(\frac{\sqrt{2}}{4i}\right)\\ &= \frac{1}{2}\sqrt{2}\pi\end{aligned}$$

But $\int_0^\infty \frac{1}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx$, therefore

$$\begin{aligned}\int_0^\infty \frac{1}{x^4+1}dx &= \frac{\sqrt{2}}{4}\pi\\ &= \frac{2}{4\sqrt{2}}\pi\\ &= \frac{\pi}{2\sqrt{2}}\end{aligned}$$

3.2 Part (b)

Since the integrand is even, then

$$I = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos(ax)}{x^2+1}dx$$

We will evaluate $\int_{-\infty}^{\infty}\frac{e^{iaz}}{x^2+1}dx$ and at the end take the real part of the answer. Considering the following contour

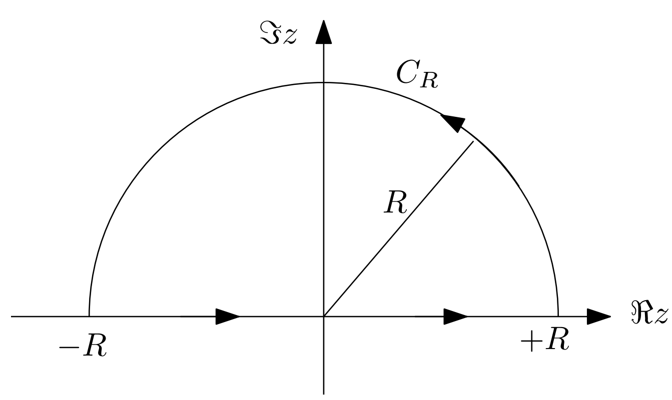

Figure 2: contour used for part b

Then

$$\oint_C f(z)\,dz = \left(\lim_{R\to\infty}\int_{-R}^{0} f(x)\,dx + \lim_{\tilde{R}\to\infty}\int_0^{\tilde{R}} f(x)\,dx\right) + \lim_{R\to\infty}\int_{C_R} f(z)\,dz$$

Using Cauchy principal value the integral above can be written as

$$\begin{aligned}\oint_C f(z)\,dz &= \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty}\int_{C_R} f(z)\,d\\ &= 2\pi i\sum \text{Residue}\end{aligned}$$

Where $\sum$ Residue is sum of residues of $\frac{e^{iaz}}{x^2+1}$ for poles that are inside the contour C. Therefore the above becomes

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx &= 2\pi i\sum\text{Residue} - \lim_{R\to\infty}\int_{C_R} f(z)\,dz\\ \int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+1}dx &= 2\pi i\sum\text{Residue} - \lim_{R\to\infty}\int_{C_R}\frac{e^{iaz}}{z^2+1}dz \qquad (1)\end{aligned}$$

Now we will show that $\lim_{R\to\infty}\int_{C_R}\frac{e^{iaz}}{z^2+1}dz=0$. Since

$$\left|\int_{C_R}\frac{e^{iaz}}{z^2+1}dz\right| \leq ML$$
$$= |f(z)|_{\max}(\pi R) \tag{2}$$

But

$$\begin{aligned} f(z) &= \frac{e^{iaz}}{(z-i)(z+i)} \\ &= \frac{e^{ia(x+iy)}}{(z-i)(z+i)} \\ &= \frac{e^{iax-ay}}{(z-i)(z+i)} \\ &= \frac{e^{iax}e^{-ay}}{(z-i)(z+i)} \end{aligned}$$

Hence

$$\begin{aligned} |f(z)|_{\max} &= \frac{\left|e^{iaz}\right|_{\max}\left|e^{-ay}\right|_{\max}}{|z-i|_{\min}|z+i|_{\min}} \\ &= \frac{\left|e^{-ay}\right|_{\max}}{(R+1)(R-1)} \\ &= \frac{\left|e^{-ay}\right|_{\max}}{R^2-1} \end{aligned}$$

Since $a>0$ and since in upper half $y>0$ then $\left|e^{-ay}\right|_{\max}=\left|e^{-aR}\right|_{\max}=1$. Jordan inequality was not needed here, since there is no extra x in the numerator of the integrand in this problem. The above now reduces to

$$|f(z)|_{\max}=\frac{1}{R^2-1}$$

Equation (2) becomes

$$\left|\int_{C_R}\frac{e^{iaz}}{z^2+1}dz\right| \leq \frac{\pi R}{R^2-1}$$

$R\to\infty$ the above goes to zero since $\lim_{R\to\infty}\frac{\pi R}{R^2-1}=\lim_{R\to\infty}\frac{\frac{\pi}{R^2}}{1-\frac{1}{R^2}}=\frac{0}{1}=0$. Equation (1) now simplifies to

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{x^4+1}dx=2\pi i\sum \text{Residue}$$

We just now need to find the residues of $\frac{1}{z^2+1}$ that are located in upper half plane. The zeros of the denominator $z^2+1=0$ are at $z=\pm i$, hence poles are at

$$\begin{aligned} z_1 &= i \\ z_2 &= -i \end{aligned}$$

Only z_1 is in upper half plane. Therefore

$$\begin{aligned} \text{Residue}(z_1) &= \lim_{z\to z_1}(z-z_1)f(z) \\ &= \lim_{z\to z_1}(z-z_1)\frac{e^{iaz}}{(z-z_1)(z-z_2)} \\ &= \lim_{z\to z_1}\frac{e^{iaz}}{(z-z_2)} \\ &= \frac{e^{ia(i)}}{(i+i)} \\ &= \frac{e^{-a}}{2i} \end{aligned}$$

Since $\int_{-\infty}^{\infty}\frac{e^{ax}}{x^4+1}dx=2\pi i\sum\text{Residue}$ then

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{e^{iax}}{x^4+1}dx &= 2\pi i\left(\frac{e^{-a}}{2i}\right) \\ &= \pi e^{-a} \end{aligned}$$

Therefore

$$\begin{aligned}\int_0^{\infty} \frac{e^{iax}}{x^4+1} dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{ax}}{x^4+1} dx \\ &= \frac{\pi}{2} e^{-a}\end{aligned}$$

But real part of the above is

$$\int_0^{\infty} \frac{\cos(ax)}{x^4+1} dx = \frac{\pi}{2} e^{-a}$$

4 Problem 4

Using residues evaluate(a) $\int_0^{2\pi} \frac{1}{1+a\cos\theta} d\theta$ for $|a| < 1$ (b) $\int_0^{\pi} (\cos(\theta))^{2n} d\theta$ for n integer.

4.1 Part (a)

Using contour which is anti-clockwise over the unit circle

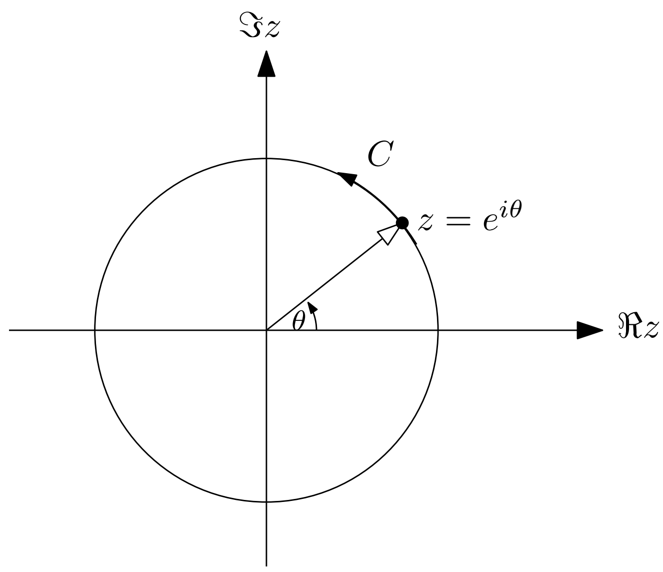

Figure 3: contour used for problem 4

Let $z = e^{i\theta}$, hence $dz = d\theta i e^{i\theta} = d\theta iz$. Using $\cos\theta = \frac{z+z^{-1}}{2}$ then the integral can be written in complex domain as

$$\begin{aligned}
\oint_C \frac{\frac{1}{iz}dz}{1 + a\frac{z+z^{-1}}{2}} &= \frac{2}{i}\oint_C \frac{\frac{1}{z}dz}{2 + a\left(z + \frac{1}{z}\right)} \\
&= \frac{2}{i}\oint_C \frac{dz}{2z + az^2 + a} \\
&= \frac{2}{ai}\oint_C \frac{dz}{z^2 + \frac{2}{a}z + 1} \\
&= \frac{2}{ai}\oint_C \frac{dz}{(z - z_1)(z - z_2)}
\end{aligned}$$

Where z_1, z_2 are roots of $z^2 + \frac{2}{a}z + 1 = 0$ which are found to be (using the quadratic formula) as

$$\begin{aligned}
z_1 &= \frac{-1 - \sqrt{1 - a^2}}{a} \\
z_2 &= \frac{-1 + \sqrt{1 - a^2}}{a}
\end{aligned}$$

Since $|a| < 1$ then only z_2 will be inside the unit disk for all a values. Therefore

$$\begin{aligned}
\frac{2}{ai}\oint_C \frac{dz}{(z - z_1)(z - z_2)} &= \left(\frac{2}{ai}\right) 2\pi i \operatorname{Residue}(z_2) \\
&= \frac{4}{a}\pi \operatorname{Residue}(z_2) \qquad (1)
\end{aligned}$$

Now we will find the Residue (z_2) where in this case $f(z) = \frac{1}{(z-z_1)(z-z_2)}$. Hence

$$\begin{aligned}
\text{Residue}(z_2) &= \lim_{z\to z_2}(z-z_2)\,f(z)\\
&= \lim_{z\to z_2}(z-z_2)\frac{1}{(z-z_1)(z-z_2)}\\
&= \lim_{z\to z_2}\frac{1}{(z-z_1)}\\
&= \frac{1}{\left(\frac{-1+\sqrt{1-a^2}}{a}\right)-\left(\frac{-1-\sqrt{1-a^2}}{a}\right)}\\
&= \frac{a}{2\sqrt{1-a^2}}
\end{aligned}$$

Using the above result in (1) gives

$$\begin{aligned}
\int_0^{2\pi}\frac{1}{1+a\cos\theta}d\theta &= \left(\frac{4}{a}\pi\right)\frac{a}{2\sqrt{1-a^2}}\\
&= \frac{2\pi}{\sqrt{1-a^2}} \qquad a\neq 1
\end{aligned}$$

Using Maple, verified that the above result is correct.

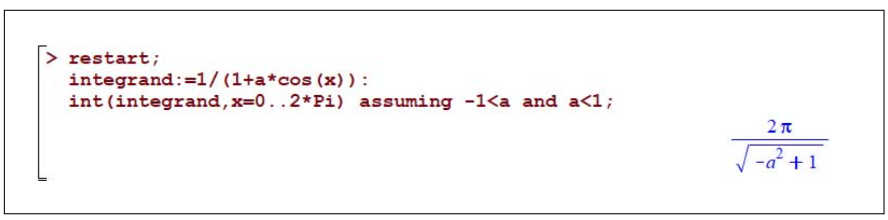

Figure 4: Verification using Maple

4.2 Part (b)

Since integrand is even, then $\int_0^{\pi}(\cos(\theta))^{2n}\,d\theta = \frac{1}{2}\int_0^{2\pi}(\cos(\theta))^{2n}\,d\theta$. Using same contour as in part (a), and letting $z = e^{i\theta}$, hence $dz = d\theta i e^{i\theta} = d\theta i z$ and using $\cos\theta = \frac{z+z^{-1}}{2}$ then the integral can be written in complex domain as

$$\begin{aligned}
\int_0^{2\pi}(\cos(\theta))^{2n}\,d\theta &= \oint_C\left(\frac{z+\frac{1}{z}}{2}\right)^{2n}\frac{dz}{iz}\\
&= \frac{1}{i}\oint_C\frac{\left(z+\frac{1}{z}\right)^{2n}}{2^{2n}}\frac{dz}{z}\\
&= \frac{1}{4^n i}\oint_C\left(z+\frac{1}{z}\right)^{2n}\frac{dz}{z}\\
&= \frac{1}{4^n i}\oint_C\left(\frac{z^2+1}{z}\right)^{2n}\frac{dz}{z}\\
&= \frac{1}{4^n i}\oint_C\frac{\left(z^2+1\right)^{2n}}{z^{2n}}\frac{dz}{z}\\
&= \frac{1}{4^n i}\oint_C\frac{\left(z^2+1\right)^{2n}}{z^{2n+1}}dz
\end{aligned}$$

Considering $f(z) = \frac{\left(z^2+1\right)^{2n}}{z^{2n+1}}$, this has a pole at $z = 0$ of order $m = 2n+1$. Therefore

$$\frac{1}{4^n i}\oint_C\frac{\left(z^2+1\right)^{2n}}{z^{2n+1}}dz = \left(\frac{1}{4^n i}\right)2\pi i\,\text{Residue}(z=0) \tag{1}$$

So we now need to find residue of $f(z)$ at $z=0$ but for pole of order $m=2n+1$. Using the formula for finding residue for pole of order m gives

$$\text{Residue}\left(z_0=0\right)=\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\frac{\left(z-z_0\right)^m f(z)}{(m-1)!}$$

But $m=2n+1$, and $z_0=0$, hence the above becomes

$$\begin{aligned}\text{Residue}(0)&=\lim_{z\to 0}\frac{d^{2n}}{dz^{2n}}\frac{z^{2n+1}}{(2n)!}\frac{\left(z^2+1\right)^{2n}}{z^{2n+1}}\\&=\frac{1}{(2n)!}\lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)\end{aligned}$$

Equation (1) becomes

$$\int_0^{2\pi}(\cos(\theta))^{2n}\,d\theta=\left(\frac{1}{4^n}\right)2\pi\left(\frac{1}{(2n)!}\lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)\right)$$

Therefore

$$\begin{aligned}\int_0^{\pi}(\cos(\theta))^{2n}\,d\theta&=\frac{1}{2}\left(\frac{1}{4^n}\right)2\pi\left(\frac{1}{(2n)!}\lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)\right)\\&=\frac{1}{4^n}\frac{\pi}{(2n)!}\lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)\end{aligned}$$

Will now try to obtained closed form solution. Trying for different n values in order to see the pattern. From few lectures ago, we learned also that

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n}\sqrt{\pi}$$

Now will generate a table to see the pattern

n	$\frac{1}{4^n}\frac{\pi}{(2n)!}\lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)$	result of integral	$\Gamma\left(n+\frac{1}{2}\right)$
1	$\frac{1}{4}\frac{\pi}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(z^2+1\right)^2$	$\frac{\pi}{2}$	$\Gamma\left(1+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$
2	$\frac{1}{4^2}\frac{\pi}{4!}\lim_{z\to 0}\frac{d^4}{dz^4}\left(z^2+1\right)^4$	$\frac{3\pi}{8}$	$\Gamma\left(2+\frac{1}{2}\right)=\frac{3\sqrt{\pi}}{4}$
3	$\frac{1}{4^3}\frac{\pi}{6!}\lim_{z\to 0}\frac{d^6}{dz^6}\left(z^2+1\right)^6$	$\frac{5\pi}{16}$	$\Gamma\left(3+\frac{1}{2}\right)=\frac{15\sqrt{\pi}}{8}$
4	$\frac{1}{4^4}\frac{\pi}{8!}\lim_{z\to 0}\frac{d^8}{dz^8}\left(z^2+1\right)^8$	$\frac{35\pi}{128}$	$\Gamma\left(4+\frac{1}{2}\right)=\frac{105\sqrt{\pi}}{16}$
5	$\frac{1}{4^5}\frac{\pi}{10!}\lim_{z\to 0}\frac{d^{10}}{dz^{10}}\left(z^2+1\right)^{10}$	$\frac{63\pi}{256}$	$\Gamma\left(5+\frac{1}{2}\right)=\frac{945\sqrt{\pi}}{32}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Based on the above, we see that $I=\frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{n!}$, which is verified as follows

n	result of integral	$\Gamma\left(n+\frac{1}{2}\right)$	$\frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{n!}$
1	$\frac{\pi}{2}$	$\Gamma\left(1+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$	$\frac{\sqrt{\pi}\left(\frac{\sqrt{\pi}}{2}\right)}{1}=\frac{1}{2}\pi$
2	$\frac{3\pi}{8}$	$\Gamma\left(2+\frac{1}{2}\right)=\frac{3\sqrt{\pi}}{4}$	$\frac{\sqrt{\pi}\left(\frac{3\sqrt{\pi}}{4}\right)}{2!}=\frac{3}{8}\pi$
3	$\frac{5\pi}{16}$	$\Gamma\left(3+\frac{1}{2}\right)=\frac{15\sqrt{\pi}}{8}$	$\frac{\sqrt{\pi}\left(\frac{15\sqrt{\pi}}{8}\right)}{3!}=\frac{15\pi}{(6)(8)}=\frac{15\pi}{48}=\frac{3}{16}\pi$
4	$\frac{35\pi}{128}$	$\Gamma\left(4+\frac{1}{2}\right)=\frac{105\sqrt{\pi}}{16}$	$\frac{\sqrt{\pi}\left(\frac{105\sqrt{\pi}}{16}\right)}{4!}=\frac{\sqrt{\pi}\left(105\sqrt{\pi}\right)}{(24)(16)}=\frac{105\pi}{384}=\frac{35}{128}\pi$
5	$\frac{63\pi}{256}$	$\Gamma\left(5+\frac{1}{2}\right)=\frac{945\sqrt{\pi}}{32}$	$\frac{\sqrt{\pi}\left(\frac{945\sqrt{\pi}}{32}\right)}{5!}=\frac{945\pi}{(120)(32)}=\frac{945\pi}{3840}=\frac{63}{256}\pi$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Therefore

$$\int_0^{\pi}(\cos(\theta))^{2n}\,d\theta=\frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{n!}$$

Tried to do pole/zero cancellation on the integrand of $\oint\limits_{C} \frac{\left(z^2+1\right)^{2n}}{z^{2n+1}} dz$ in order to find a simpler method than the above but was not able to. The above result was verified using the computer

```
In[ ]:= Assuming[Element[n, Integers] && n > 0, Integrate[Cos[x]^(2 n), {x, 0, π}]];
        TraditionalForm[%]
Out[ ]//TraditionalForm=
```

$$\frac{\sqrt{\pi}\ \Gamma\left(n+\frac{1}{2}\right)}{n!}$$

Figure 5: Verification using Mathematica