

HW 3, Physics 501  
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# 1 Problem 1

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## 1.1 Part (a)

Use Cauchy-Riemann equations to determine if  $|z|$  analytic function of the complex variable  $z$ .

Solution

$$f(z) = |z|$$

Let  $z = x + iy$ , then

$$\begin{aligned} f(z) &= (x^2 + y^2)^{\frac{1}{2}} \\ &= u + iv \end{aligned}$$

Hence

$$\begin{aligned} u &= \sqrt{x^2 + y^2} \\ v &= 0 \end{aligned}$$

Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

First equation above gives  $\frac{\partial v}{\partial y} = 0$  and  $\frac{\partial u}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2}}$ , which shows that  $\frac{\partial v}{\partial y} \neq \frac{\partial u}{\partial x}$ . Therefore  $|z|$  is not analytic.

## 1.2 Part (b)

Use Cauchy-Riemann equations to determine if  $\operatorname{Re}(z)$  analytic function of the complex variable  $z$ .

Solution

$$f(z) = \operatorname{Re}(z)$$

Let  $z = x + iy$ , then

$$\begin{aligned} f(z) &= x \\ &= u + iv \end{aligned}$$

Hence

$$\begin{aligned} u &= x \\ v &= 0 \end{aligned}$$

Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

First equation above gives  $\frac{\partial v}{\partial y} = 0$  and  $\frac{\partial u}{\partial x} = 1$ , which shows that  $\frac{\partial v}{\partial y} \neq \frac{\partial u}{\partial x}$ . Therefore  $\operatorname{Re}(z)$  is not analytic.

### 1.3 Part (c)

Use Cauchy-Riemann equations to determine if  $e^{\sin z}$  analytic function of the complex variable  $z$ .

#### Solution

$f(z) = e^{\sin z}$  is analytic since we can show that  $\exp(z)$  is analytic by applying Cauchy-Riemann (C-R), and also show that  $\sin(z)$  is analytic using C-R. Theory of analytic functions it says that the composition of analytic functions is also an analytic function, which means  $e^{\sin z}$  is analytic.

But this problems seems to ask to use C-R equations directly to show this. Therefore we need to first determine the real and complex parts ( $u, v$ ) of the function  $e^{\sin z}$ . Since

$$\sin z = \frac{z - z^{-1}}{2i}$$

Then

$$\begin{aligned} f(z) &= e^{\sin z} \\ &= \exp\left(\frac{z - z^{-1}}{2i}\right) \\ &= \exp\left(\frac{z}{2i}\right) \exp\left(\frac{-1}{2iz}\right) \end{aligned}$$

But  $z = x + iy$  and the above expands to

$$\begin{aligned} \exp(\sin z) &= \exp\left(\frac{1}{2i}(x + iy)\right) \exp\left(\frac{-1}{2i(x + iy)}\right) \\ &= \exp\left(\frac{-i}{2}x + \frac{1}{2}y\right) \exp\left(\frac{i}{2} \frac{1}{(x + iy)}\right) \\ &= \exp\left(\frac{-i}{2}x + \frac{1}{2}y\right) \exp\left(\frac{i}{2} \frac{x - iy}{(x + iy)(x - iy)}\right) \\ &= \exp\left(\frac{-i}{2}x + \frac{1}{2}y\right) \exp\left(\frac{i}{2} \frac{x - iy}{x^2 + y^2}\right) \\ &= \exp\left(\frac{-i}{2}x + \frac{1}{2}y\right) \exp\left(\frac{i}{2} \left(\frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}\right)\right) \\ &= \exp\left(\frac{-i}{2}x + \frac{1}{2}y\right) \exp\left(\frac{i}{2} \frac{x}{x^2 + y^2} + \frac{1}{2} \frac{y}{x^2 + y^2}\right) \\ &= \exp\left(\frac{-i}{2}x\right) \exp\left(\frac{1}{2}y\right) \exp\left(\frac{i}{2} \frac{x}{x^2 + y^2}\right) \exp\left(\frac{1}{2} \frac{y}{x^2 + y^2}\right) \end{aligned}$$

Collecting terms gives

$$\begin{aligned} \exp(\sin z) &= \exp\left(\frac{1}{2}y + \frac{1}{2} \frac{y}{x^2 + y^2}\right) \exp\left(\frac{i}{2} \frac{x}{x^2 + y^2} - \frac{i}{2}x\right) \\ &= \exp\left(\frac{1}{2} \frac{y(1 + (x^2 + y^2))}{x^2 + y^2}\right) \exp\left(\frac{i}{2} \frac{x}{x^2 + y^2} - \frac{i}{2(x^2 + y^2)}x(x^2 + y^2)\right) \\ &= \exp\left(\frac{1}{2} \frac{y(1 + (x^2 + y^2))}{x^2 + y^2}\right) \exp\left(i \frac{1}{2} \frac{x(1 - (x^2 + y^2))}{x^2 + y^2}\right) \\ &= \exp\left(\frac{1}{2} \frac{y(1 + (x^2 + y^2))}{x^2 + y^2}\right) \left[ \cos\left(\frac{1}{2} \frac{x(1 - (x^2 + y^2))}{x^2 + y^2}\right) + i \sin\left(\frac{1}{2} \frac{x(1 - (x^2 + y^2))}{x^2 + y^2}\right) \right] \\ &= \exp\left(\frac{y + y(x^2 + y^2)}{2(x^2 + y^2)}\right) \cos\left(\frac{x - x(x^2 + y^2)}{2(x^2 + y^2)}\right) + i \exp\left(\frac{y + y(x^2 + y^2)}{2(x^2 + y^2)}\right) \sin\left(\frac{1}{2} \frac{x - x(x^2 + y^2)}{(x^2 + y^2)}\right) \end{aligned}$$

Therefore, since  $\exp(\sin z) = u + iv$ , then we see from above that

$$\begin{aligned} u &= \exp\left(\frac{1}{2} \frac{y + y(x^2 + y^2)}{x^2 + y^2}\right) \cos\left(\frac{1}{2} \frac{x - x(x^2 + y^2)}{x^2 + y^2}\right) \\ v &= \exp\left(\frac{1}{2} \frac{y + y(x^2 + y^2)}{x^2 + y^2}\right) \sin\left(\frac{1}{2} \frac{x - x(x^2 + y^2)}{x^2 + y^2}\right) \end{aligned}$$

Now we need to check the Cauchy-Riemann equations on the above  $u, v$  functions we found.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (1)$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad (2)$$

Evaluating each partial derivative gives

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{d}{dx} \left( \frac{1}{2} \frac{y + y(x^2 + y^2)}{x^2 + y^2} \right) \exp \left( \frac{1}{2} \frac{y + y(x^2 + y^2)}{x^2 + y^2} \right) \cos \left( \frac{1}{2} \frac{x - x(x^2 + y^2)}{x^2 + y^2} \right) \\ &+ \exp \left( \frac{1}{2} \frac{y + y(x^2 + y^2)}{x^2 + y^2} \right) \frac{d}{dx} \cos \left( \frac{1}{2} \frac{x - x(x^2 + y^2)}{x^2 + y^2} \right) \\ &= \frac{1}{2} \frac{2yx(x^2 + y^2) - (y + y(x^2 + y^2))2x}{(x^2 + y^2)^2} \exp \left( \frac{1}{2} \frac{y((x^2 + y^2) + 1)}{x^2 + y^2} \right) \cos \left( \frac{1}{2} \frac{x(1 - (x^2 + y^2))}{x^2 + y^2} \right) \\ &- \exp \left( \frac{1}{2} \frac{y + y(x^2 + y^2)}{x^2 + y^2} \right) \sin \left( \frac{1}{2} \frac{x - x(x^2 + y^2)}{x^2 + y^2} \right) \frac{d}{dx} \left( \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right) \\ &= \frac{-xy}{(x^2 + y^2)^2} \exp \left( \frac{1}{2} \frac{y((x^2 + y^2) + 1)}{x^2 + y^2} \right) \cos \left( \frac{x(1 - (x^2 + y^2))}{2(x^2 + y^2)} \right) \\ &- \exp \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \sin \left( \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right) \left( \frac{(1 - 3x^2 - y^2)(x^2 + y^2) - (x - x(x^2 + y^2))2x}{2(x^2 + y^2)^2} \right) \\ &= \frac{-xy}{(x^2 + y^2)^2} \exp \left( \frac{1}{2} \frac{y(x^2 + y^2 + 1)}{x^2 + y^2} \right) \cos \left( \frac{1}{2} \frac{x(1 - (x^2 + y^2))}{x^2 + y^2} \right) \\ &- \exp \left( \frac{1}{2} \frac{y(x^2 + y^2 + 1)}{x^2 + y^2} \right) \sin \left( \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right) \left( \frac{-x^4 - 2x^2y^2 - x^2 - y^4 + y^2}{2(x^2 + y^2)^2} \right) \end{aligned}$$

The above can be simplified more to become

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{-1}{2(x^2 + y^2)^2} \exp \left( \frac{y(x^2 + y^2 + 1)}{2(x^2 + y^2)} \right) \\ &\left[ 2xy \cos \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} + (-x^4 - 2x^2y^2 - x^2 - y^4 + y^2) \sin \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right] \quad (3) \end{aligned}$$

Now we evaluate  $\frac{\partial v}{\partial y}$  to see if it the same as above. Since  $v = \exp \left( \frac{1}{2} \frac{y + y(x^2 + y^2)}{x^2 + y^2} \right) \sin \left( \frac{1}{2} \frac{x - x(x^2 + y^2)}{x^2 + y^2} \right)$  then

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{d}{dy} \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \exp \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \sin \left( \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right) \\ &+ \exp \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \cos \left( \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right) \frac{d}{dy} \left( \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right) \\ &= \left( \frac{1}{2} \frac{(1 + x^2 + 3y^2)(x^2 + y^2) - (y + y(x^2 + y^2))2y}{(x^2 + y^2)^2} \right) \exp \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \sin \left( \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right) \\ &+ \exp \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \cos \left( \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right) \left( \frac{1}{2} \frac{(-2xy)(x^2 + y^2) - (x - x(x^2 + y^2))(2y)}{(x^2 + y^2)^2} \right) \\ &= \left( \frac{1}{2} \frac{x^4 + 2x^2y^2 + x^2 + y^4 - y^2}{(x^2 + y^2)^2} \right) \exp \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \sin \left( \frac{1}{2} \frac{x - x(x^2 + y^2)}{x^2 + y^2} \right) \\ &+ \exp \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \cos \left( \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right) \left( \frac{1}{2} \frac{-2xy}{(x^2 + y^2)^2} \right) \\ &= \left( \frac{1}{2} \frac{x^4 + 2x^2y^2 + x^2 + y^4 - y^2}{(x^2 + y^2)^2} \right) \exp \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \sin \left( \frac{1}{2} \frac{x - x(x^2 + y^2)}{x^2 + y^2} \right) \\ &- \exp \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \cos \left( \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right) \left( \frac{xy}{(x^2 + y^2)^2} \right) \end{aligned}$$

Simplifying the above more gives

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{-1}{2(x^2 + y^2)^2} \exp \left( \frac{y + y(x^2 + y^2)}{2(x^2 + y^2)} \right) \\ &\left[ 2xy \cos \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} + (-x^4 - 2x^2y^2 - x^2 - y^4 + y^2) \sin \frac{x - x(x^2 + y^2)}{2(x^2 + y^2)} \right] \quad (4) \end{aligned}$$

Comparing (3) and (4) shows they are the same expressions. Therefore the first equation is verified.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Now we verify the second equation  $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ . Since  $u = \exp\left(\frac{1}{2} \frac{y+y(x^2+y^2)}{(x^2+y^2)}\right) \cos\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{(x^2+y^2)}\right)$  then

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{d}{dy} \left( \frac{1}{2} \frac{y+y(x^2+y^2)}{(x^2+y^2)} \right) \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \cos\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2}\right) \\ &\quad - \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \sin\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2}\right) \frac{d}{dy} \left( \frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2} \right) \\ &= \frac{(1+x^2+3y^2)(x^2+y^2) - (y+y(x^2+y^2))2y}{2(x^2+y^2)^2} \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \cos\left(\frac{x-x(x^2+y^2)}{2(x^2+y^2)}\right) \\ &\quad - \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \sin\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2}\right) \frac{(-2y)(x^2+y^2) - (x-x(x^2+y^2))2y}{2(x^2+y^2)^2} \\ &= \frac{(x^4+2x^2y^2+x^2+y^4-y^2)}{2(x^2+y^2)^2} \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \cos\left(\frac{x-x(x^2+y^2)}{2(x^2+y^2)}\right) \\ &\quad - \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \sin\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2}\right) \frac{(-2y)(x^2+y^2) - 2yx + 2yx(x^2+y^2)}{2(x^2+y^2)^2} \\ &= \frac{(x^4+2x^2y^2+x^2+y^4-y^2)}{2(x^2+y^2)^2} \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \cos\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2}\right) \\ &\quad + \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \sin\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2}\right) \frac{yx}{(x^2+y^2)^2} \end{aligned}$$

The above can simplified more to give

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{1}{2(x^2+y^2)^2} \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \\ &\quad \left[ (x^4+2x^2y^2+x^2+y^4-y^2) \cos\frac{x-x(x^2+y^2)}{2(x^2+y^2)} + 2xy \sin\frac{x-x(x^2+y^2)}{2(x^2+y^2)} \right] \end{aligned}$$

Hence

$$\begin{aligned} -\frac{\partial u}{\partial y} &= \frac{1}{2(x^2+y^2)^2} \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \\ &\quad \left[ -(x^4+2x^2y^2+x^2+y^4-y^2) \cos\frac{x-x(x^2+y^2)}{2(x^2+y^2)} - 2xy \sin\frac{x-x(x^2+y^2)}{2(x^2+y^2)} \right] \quad (5) \end{aligned}$$

And since  $v = \exp\left(\frac{1}{2} \frac{y+y(x^2+y^2)}{(x^2+y^2)}\right) \sin\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{(x^2+y^2)}\right)$  then

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{d}{dx} \left( \frac{1}{2} \frac{y+y(x^2+y^2)}{x^2+y^2} \right) \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \sin\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2}\right) \\ &\quad + \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \cos\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2}\right) \frac{d}{dx} \left( \frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2} \right) \\ &= \frac{1}{2} \left( \frac{2xy(x^2+y^2) - (y+y(x^2+y^2))2x}{(x^2+y^2)^2} \right) \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \sin\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2}\right) \\ &\quad + \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \cos\left(\frac{1}{2} \frac{x-x(x^2+y^2)}{x^2+y^2}\right) \left( \frac{(1-3x^2-y^2)(x^2+y^2) - (x-x(x^2+y^2))2x}{2(x^2+y^2)^2} \right) \\ &= \frac{-xy}{(x^2+y^2)^2} \exp\left(\frac{1}{2} \frac{y+y(x^2+y^2)}{x^2+y^2}\right) \sin\left(\frac{x-x(x^2+y^2)}{2(x^2+y^2)}\right) \\ &\quad + \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \cos\left(\frac{x-x(x^2+y^2)}{2(x^2+y^2)}\right) \left( \frac{-x^4-2x^2y^2-x^2-y^4+y^2}{2(x^2+y^2)^2} \right) \end{aligned}$$

The above can simplified more to give

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{1}{2(x^2+y^2)^2} \exp\left(\frac{y+y(x^2+y^2)}{2(x^2+y^2)}\right) \\ &\quad \left[ -(x^4+2x^2y^2+x^2+y^4-y^2) \cos\frac{x-x(x^2+y^2)}{2(x^2+y^2)} - 2xy \sin\frac{x-x(x^2+y^2)}{2(x^2+y^2)} \right] \quad (6) \end{aligned}$$

Comparing (5,6) shows they are the same, i.e.

$$\frac{-\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

C-R equations are satisfied, and because it is clear that all partial derivatives  $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$  are continuous functions in  $x, y$  as they are made up of exponential and trigonometric functions which are continuous, then we conclude that  $f(z) = e^{\sin z}$  is analytic function everywhere.

## 2 Problem 2

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### 2.1 Part (a)

Represent  $\frac{z+3}{z-3}$  by its Maclaurin series and give the region of validity for the representation. Next expand this in powers of  $\frac{1}{z}$  to find a Laurent series. What is the range of validity of the Laurent series?

#### Solution

Maclaurin series is expansion of  $f(z)$  around  $z = 0$ . Since  $f(z)$  has simple pole at  $z = 3$ , then the region of validity will be a disk centered at  $z = 0$  up to the nearest pole, which is at  $z = 3$ . Hence  $|z| < 3$  is the region.

$$\begin{aligned} f(z) &= \frac{z+3}{z-3} \\ &= \frac{z+3}{-3\left(1-\frac{z}{3}\right)} \\ &= \frac{z+3}{-3} \left( \frac{1}{1-\frac{z}{3}} \right) \end{aligned}$$

Now we can expand using Binomial to obtain

$$\begin{aligned} f(z) &= \frac{3+z}{-3} \left( 1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \dots \right) \\ &= \left(-1 - \frac{z}{3}\right) \left( 1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \dots \right) \\ &= \left(-1 - \frac{z}{3}\right) + \left(-1 - \frac{z}{3}\right) \left(\frac{z}{3}\right) + \left(-1 - \frac{z}{3}\right) \left(\frac{z}{3}\right)^2 + \left(-1 - \frac{z}{3}\right) \left(\frac{z}{3}\right)^3 + \dots \\ &= -1 - \frac{z}{3} - \frac{z}{3} - \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 - \left(\frac{z}{3}\right)^3 - \left(\frac{z}{3}\right)^4 + \dots \\ &= -1 - \frac{2}{3}z - \frac{2}{9}z^2 - \frac{2}{27}z^3 - \frac{2}{81}z^4 - \dots \end{aligned}$$

Or

$$f(z) = -1 - \sum_{n=1}^{\infty} \frac{2}{3^n} z^n$$

To expand in negative powers of  $z$ , or in  $\frac{1}{z}$ , then

$$\begin{aligned} f(z) &= \frac{z+3}{z\left(1-\frac{3}{z}\right)} \\ &= \frac{z+3}{z} \left( \frac{1}{1-\frac{3}{z}} \right) \end{aligned}$$

For  $\left|\frac{3}{z}\right| < 1$  or  $|z| < 3$  the above becomes

$$\begin{aligned} f(z) &= \frac{z+3}{z} \left( 1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \dots \right) \\ &= \left(1 + \frac{3}{z}\right) \left( 1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \dots \right) \\ &= \left(1 + \frac{3}{z}\right) + \left(1 + \frac{3}{z}\right) \frac{3}{z} + \left(1 + \frac{3}{z}\right) \left(\frac{3}{z}\right)^2 + \left(1 + \frac{3}{z}\right) \left(\frac{3}{z}\right)^3 + \dots \\ &= 1 + \frac{3}{z} + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \left(\frac{3}{z}\right)^3 + \left(\frac{3}{z}\right)^4 + \dots \\ &= 1 + \frac{6}{z} + \frac{18}{z^2} + \frac{54}{z^3} + \dots \end{aligned}$$

This is valid for  $|z| > 3$ . The residue is 6, which can be confirmed using

$$\begin{aligned} \text{Residue}(3) &= \lim_{z \rightarrow 3} (z-3) f(z) \\ &= \lim_{z \rightarrow 3} (z-3) \frac{z+3}{z-3} \\ &= \lim_{z \rightarrow 3} (z+3) \\ &= 6 \end{aligned}$$

### Summary

$$f(z) = \frac{z+3}{z-3} = \begin{cases} -1 - \frac{2}{3}z - \frac{2}{9}z^2 - \frac{2}{27}z^3 - \frac{2}{81}z^4 - \dots & |z| < 3 \\ 1 + \frac{6}{z} + \frac{18}{z^2} + \frac{54}{z^3} + \dots & |z| > 3 \end{cases}$$

## 2.2 Part (b)

Find Laurent series for  $\frac{z}{(z+1)(z-3)}$  in each of the following domains (i)  $|z| < 1$  (ii)  $1 < |z| < 3$  (iii)  $|z| > 3$

### Solution

The possible region are shown below. Since there is a pole at  $z = -1$  and pole at  $z = 3$ , then there are three different regions. They are named A, B, C in the following diagram

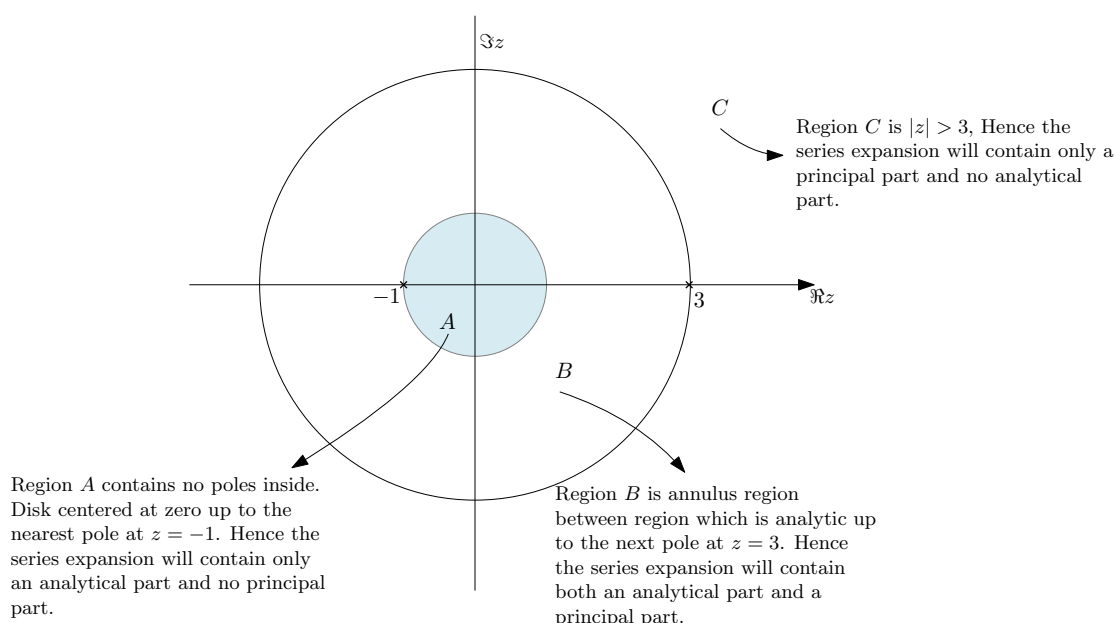


Figure 1: Laurent series regions

First the expression  $\frac{z}{(z+1)(z-3)}$  is expanded using partial fractions

$$\frac{z}{(z+1)(z-3)} = \frac{A}{z+1} + \frac{B}{z-3} \quad (1)$$

Hence

$$\begin{aligned} z &= A(z-3) + B(z+1) \\ &= z(A+B) - 3A + B \end{aligned}$$

The above gives two equations

$$\begin{aligned} A + B &= 1 \\ 0 &= -3A + B \end{aligned}$$

First equation gives  $A = 1 - B$ . Substituting in the second equation gives  $0 = -3(1 - B) + B$  or  $0 = -3 + 4B$ , hence  $B = \frac{3}{4}$ , which implies  $A = 1 - \frac{3}{4} = \frac{1}{4}$ , therefore (1) becomes

$$\frac{z}{(z+1)(z-3)} = \frac{1}{4} \frac{1}{(z+1)} + \frac{3}{4} \frac{1}{(z-3)}$$

Considering each term in turn. For  $\frac{1}{4} \frac{1}{(z+1)}$ , we can expand this as

$$\frac{1}{4} \frac{1}{(z+1)} = \frac{1}{4} (1 - z + z^2 - z^3 + z^4 + \dots) \quad |z| < 1 \quad (2a)$$

$$\frac{1}{4} \frac{1}{(z+1)} = \frac{1}{4z} \frac{1}{(1 + \frac{1}{z})} = \frac{1}{4z} \left( 1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4 - \dots \right) \quad |z| > 1 \quad (2b)$$

And for the term  $\frac{3}{4} \frac{1}{(z-3)}$ , we can expand this as

$$\frac{3}{4} \frac{1}{(z-3)} = -\frac{1}{4} \frac{1}{(1 - \frac{z}{3})} = -\frac{1}{4} \left( 1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \left(\frac{z}{3}\right)^4 + \dots \right) \quad |z| < 3 \quad (3a)$$

$$\frac{3}{4} \frac{1}{(z-3)} = \frac{3}{4z} \frac{1}{(1 - \frac{3}{z})} = \frac{3}{4z} \left( 1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \dots \right) \quad |z| > 3 \quad (3b)$$

Now that we expanded all the terms in the two possible ways for each each, we now consider each region of interest, and look at the above 4 expansions, and simply pick for each region the expansion which is valid in for that region of interest.

For (i), region A: In this region, we want  $|z| < 1$ . From (2,3) we see that (2a) and (3a) are valid expansions in  $|z| < 1$ . Hence

$$\begin{aligned} \frac{z}{(z+1)(z-3)} &= \frac{1}{4} (1 - z + z^2 - z^3 + z^4 + \dots) - \frac{1}{4} \left( 1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \left(\frac{z}{3}\right)^4 + \dots \right) \\ &= \frac{1}{4} (1 - z + z^2 - z^3 + z^4 - \dots) - \frac{1}{4} \left( 1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \frac{z^4}{81} + \dots \right) \\ &= \left( \frac{1}{4} - \frac{1}{4}z + \frac{1}{4}z^2 - \frac{1}{4}z^3 + \frac{1}{4}z^4 - \dots \right) - \left( \frac{1}{4} + \frac{z}{12} + \frac{z^2}{36} + \frac{z^3}{108} + \frac{z^4}{324} + \dots \right) \\ &= -\frac{1}{4}z - \frac{z}{12} + \frac{1}{4}z^2 - \frac{z^2}{36} - \frac{1}{4}z^3 - \frac{z^3}{108} + \frac{1}{4}z^4 - \frac{z^4}{324} - \dots \\ &\quad - \frac{1}{3}z + \frac{2}{9}z^2 - \frac{7}{27}z^3 + \frac{20}{81}z^4 - \dots \end{aligned}$$

For (ii), region B: This is for  $1 < |z| < 3$ . From equations (2,3) we see that (2b) and (3a) are valid in this region. Hence

$$\begin{aligned} \frac{z}{(z+1)(z-3)} &= \frac{1}{4z} \left( 1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4 - \dots \right) - \frac{1}{4} \left( 1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \left(\frac{z}{3}\right)^4 + \dots \right) \\ &= \frac{1}{4z} \left( 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \dots \right) - \frac{1}{4} \left( 1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \frac{z^4}{81} + \dots \right) \\ &= \underbrace{\left( \frac{1}{4z} - \frac{1}{4z^2} + \frac{1}{4z^3} - \frac{1}{4z^4} + \frac{1}{4z^5} - \dots \right)}_{\text{principal part}} - \underbrace{\left( \frac{1}{4} + \frac{z}{12} + \frac{z^2}{36} + \frac{z^3}{108} + \frac{z^4}{324} + \dots \right)}_{\text{analytical part}} \\ &= \dots + \frac{1}{4z^5} - \frac{1}{4z^4} + \frac{1}{4z^3} - \frac{1}{4z^2} + \frac{1}{4z} - \frac{1}{4} - \frac{z}{12} - \frac{z^2}{36} - \frac{z^3}{108} - \frac{z^4}{324} - \dots \end{aligned}$$



The residue is  $\frac{1}{4}$  by looking at the above. The value for the residue can be verified as follows. Using

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

Where in the above  $z_0$  is the location of the pole and  $n$  is the coefficient of the  $\frac{1}{z^n}$  is the principal part. Since we want the residue, then  $n = 1$  and the above becomes

$$b_1 = \frac{1}{2\pi i} \oint_C f(z) dz$$

In the above, the contour  $C$  is circle somewhere inside the annulus  $1 < |z| < 3$ . It does not matter that the radius is, as long as it is located in this range. For example, choosing radius 2 will work. The above then becomes

$$b_1 = \frac{1}{2\pi i} \oint_C \frac{z}{(z+1)(z-3)} dz \quad (5)$$

However, since  $f(z)$  is analytic in this region, then  $\oint_C f(z) dz = 2\pi i \Sigma$  (residues inside). There is only one pole now inside  $C$ , which is at  $z = -1$ . So all what we have to do is find the residue at  $z = -1$ .

$$\begin{aligned} \text{Residue}(-1) &= \lim_{z \rightarrow -1} (z+1) f(z) \\ &= \lim_{z \rightarrow -1} (z+1) \frac{z}{(z+1)(z-3)} \\ &= \lim_{z \rightarrow -1} \frac{z}{(z-3)} \\ &= \frac{-1}{(-1-3)} \\ &= \frac{1}{4} \end{aligned}$$

Using this in (5) gives

$$\begin{aligned} b_1 &= \frac{1}{2\pi i} \left( 2\pi i \frac{1}{4} \right) \\ &= \frac{1}{4} \end{aligned}$$

Which agrees with what we found in (4) above.

For (iii), region  $C$ : This is for  $|z| > 3$ . From (2,3) we see that (2b) and (3b) are valid expansions in  $z > 3$ , Hence

$$\begin{aligned} \frac{z}{(z+1)(z-3)} &= \frac{1}{4z} \left( 1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4 - \dots \right) + \frac{3}{4z} \left( 1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \dots \right) \\ &= \left( \frac{1}{4z} - \frac{1}{4z^2} + \frac{1}{4z^3} - \frac{1}{4z^4} + \frac{1}{4z^5} - \dots \right) + \frac{3}{4z} \left( 1 + \frac{3}{z} + \frac{9}{z^2} + \frac{27}{z^3} + \dots \right) \\ &= \left( \frac{1}{4z} - \frac{1}{4z^2} + \frac{1}{4z^3} - \frac{1}{4z^4} + \frac{1}{4z^5} - \dots \right) + \left( \frac{3}{4z} + \frac{9}{4z^2} + \frac{27}{4z^3} + \frac{81}{4z^4} + \dots \right) \\ &= \dots + \frac{20}{z^4} + \frac{7}{z^3} + \frac{2}{z^2} + \frac{1}{z} \end{aligned}$$

This is as expected contains only a principal part and no analytical part. The residue is 1. This above value for the residue can be verified as follows. Using

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

Where in the above  $z_0$  is the location of the pole and  $n$  is the coefficient of the  $\frac{1}{z^n}$  is the principal part. Since we want the residue, then  $n = 1$  and the above becomes

$$b_1 = \frac{1}{2\pi i} \oint_C f(z) dz$$

In the above, the contour  $C$  is circle somewhere in  $|z| > 3$ . It does not matter that the radius is. The above integral then becomes

$$b_1 = \frac{1}{2\pi i} \oint_C \frac{z}{(z+1)(z-3)} dz \quad (7)$$

However, since  $f(z)$  is analytic in  $|z| > 3$ , then  $\oint_C f(z) dz = 2\pi i \sum$  (residues inside). There are now two poles inside  $C$ , one at  $z = -1$  and one at  $z = 3$ . So all what we have to do is find the residues at each. We found earlier that  $\text{Residue}(-1) = \frac{1}{4}$ . Now

$$\begin{aligned} \text{Residue}(3) &= \lim_{z \rightarrow 3} (z-3) f(z) \\ &= \lim_{z \rightarrow 3} (z-3) \frac{z}{(z+1)(z-3)} \\ &= \lim_{z \rightarrow 3} \frac{z}{z+1} \\ &= \frac{3}{4} \end{aligned}$$

Therefore the sum of residues is 1. Using this result in (7) gives

$$\begin{aligned} b_1 &= \frac{1}{2\pi i} \left( 2\pi i \left( \frac{1}{4} + \frac{3}{4} \right) \right) \\ &= 1 \end{aligned}$$

Which agrees with what result from (6) above.

#### Summary of results

$f(z) = \frac{z}{(z+1)(z-3)} = \begin{cases} -\frac{1}{3}z + \frac{2}{9}z^2 - \frac{7}{27}z^3 + \frac{20}{81}z^4 - \dots &  z  < 1 \\ \dots + \frac{1}{4z^5} - \frac{1}{4z^4} + \frac{1}{4z^3} - \frac{1}{4z^2} + \frac{1}{4z} - \frac{1}{4} - \frac{z}{12} - \frac{z^2}{36} - \frac{z^3}{108} - \frac{z^4}{324} - \dots & 1 <  z  < 3 \\ \dots + \frac{20}{z^4} + \frac{7}{z^3} + \frac{2}{z^2} + \frac{1}{z} &  z  > 3 \end{cases}$
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### 3 Problem 3

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#### 3.1 Part (a)

Use residue theorem to evaluate  $\oint_C \frac{e^{-2z}}{z^2} dz$  on contour  $C$  which is circle  $|z| = 1$  in positive sense.

#### Solution

For  $f(z)$  which is analytic on and inside  $C$ , the Cauchy integral formula says

$$\oint_C f(z) dz = 2\pi i \sum_j \text{Residue}(z = z_j) \quad (1)$$

Where the sum is over all residues located inside  $C$ . for  $f(z) = \frac{e^{-2z}}{z^2}$  there is a simple pole at  $z = 0$  of order 2. To find the residue, we use the formula for pole of order  $m$  given by

$$\text{Residue}(z_0) = \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} \frac{(z - z_0)^m}{(m-1)!} f(z)$$

Hence for  $m = 2$  and  $z_0 = 0$  the above becomes

$$\begin{aligned} \text{Residue}(0) &= \lim_{z \rightarrow 0} \frac{d}{dz} z^2 f(z) \\ &= \lim_{z \rightarrow 0} \frac{d}{dz} z^2 \frac{e^{-2z}}{z^2} \\ &= \lim_{z \rightarrow 0} \frac{d}{dz} e^{-2z} \\ &= \lim_{z \rightarrow 0} (-2e^{-2z}) \\ &= -2 \end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}\oint_C \frac{e^{-2z}}{z^2} dz &= 2\pi i(-2) \\ &= -4\pi i\end{aligned}$$

### 3.2 Part (b)

Use residue theorem to evaluate  $\oint_C ze^{\frac{1}{z}} dz$  on contour  $C$  which is circle  $|z| = 1$  in positive sense.

Solution

The singularity is at  $z = 0$ , but we can not use the simple pole residue finding method here, since this is an essential singularity now due to the  $e^{\frac{1}{z}}$  term. To find the residue, we expand  $f(z)$  around  $z = 0$  in Laurent series and look for the coefficient of  $\frac{1}{z}$  term.

$$\begin{aligned}f(z) &= ze^{\frac{1}{z}} \\ &= z \left( 1 + \frac{1}{z} + \frac{1}{2} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + \dots \right) \\ &= z + 1 + \frac{1}{2} \frac{1}{z} + \frac{1}{3!} \frac{1}{z^2} + \dots\end{aligned}$$

Hence residue is  $\frac{1}{2}$ . Therefore

$$\begin{aligned}\oint_C ze^{\frac{1}{z}} dz &= 2\pi i \left( \frac{1}{2} \right) \\ &= \pi i\end{aligned}$$

### 3.3 Part (c)

Use residue theorem to evaluate  $\oint_C \frac{z+2}{z^2 - \frac{z}{2}} dz$  on contour  $C$  which is circle  $|z| = 1$  in positive sense.

Solution

$$\begin{aligned}f(z) &= \frac{z+2}{z^2 - \frac{z}{2}} \\ &= \frac{z+2}{z(z - \frac{1}{2})}\end{aligned}$$

Hence there is a simple pole at  $z = 0$  and simple pole at  $z = \frac{1}{2}$

$$\begin{aligned}\text{Residue}(0) &= \lim_{z \rightarrow 0} (z) f(z) \\ &= \lim_{z \rightarrow 0} z \frac{z+2}{z(z - \frac{1}{2})} \\ &= \lim_{z \rightarrow 0} \frac{z+2}{(z - \frac{1}{2})} \\ &= \frac{2}{-\frac{1}{2}} \\ &= -4\end{aligned}$$

And

$$\begin{aligned}\text{Residue}\left(\frac{1}{2}\right) &= \lim_{z \rightarrow \frac{1}{2}} \left(z - \frac{1}{2}\right) f(z) \\ &= \lim_{z \rightarrow \frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{z+2}{z\left(z - \frac{1}{2}\right)} \\ &= \lim_{z \rightarrow \frac{1}{2}} \frac{z+2}{z} \\ &= \frac{\frac{1}{2} + 2}{\frac{1}{2}} \\ &= 5\end{aligned}$$

Therefore

$$\begin{aligned}\oint_C \frac{z+2}{z^2 - \frac{z}{2}} dz &= 2\pi i(5 - 4) \\ &= 2\pi i\end{aligned}$$