HW 3, Physics 501
Fall 2018
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November 22, 2018 Compiled on November 22, 2018 at 2:03am

Contents

1 Problem 1

1.1 Part (a)

Use Cauchy-Riemann equations to determine if $|z|$ analytic function of the complex variable z.

Solution

$$f(z) = |z|$$

Let $z = x + iy$, then

$$\begin{aligned} f(z) &= \left(x^2 + y^2\right)^{\frac{1}{2}} \\ &= u + iv \end{aligned}$$

Hence

$$\begin{aligned} u &= \sqrt{x^2 + y^2} \\ v &= 0 \end{aligned}$$

Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

First equation above gives $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}$, which shows that $\frac{\partial v}{\partial y} \neq \frac{\partial u}{\partial x}$. Therefore $|z|$ is not analytic.

1.2 Part (b)

Use Cauchy-Riemann equations to determine if $\operatorname{Re}(z)$ analytic function of the complex variable z.

Solution

$$f(z) = \operatorname{Re}(z)$$

Let $z = x + iy$, then

$$\begin{aligned} f(z) &= x \\ &= u + iv \end{aligned}$$

Hence

$$\begin{aligned} u &= x \\ v &= 0 \end{aligned}$$

Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

First equation above gives $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial x} = 1$, which shows that $\frac{\partial v}{\partial y} \neq \frac{\partial u}{\partial x}$. Therefore $\operatorname{Re}(z)$ is not analytic.

1.3 Part (c)

Use Cauchy-Riemann equations to determine if $e^{\sin z}$ analytic function of the complex variable z.

Solution

$f(z) = e^{\sin z}$ is analytic since we can show that $\exp(z)$ is analytic by applying Cauchy-Riemann (C-R), and also show that $\sin(z)$ is analytic using C-R. Theory of analytic functions it says that the composition of analytic functions is also an analytic function, which means $e^{\sin z}$ is analytic.

But this problems seems to ask to use C-R equations directly to show this. Therefore we need to first determine the real and complex parts (u, v) of the function $e^{\sin z}$. Since

$$\sin z = \frac{z - z^{-1}}{2i}$$

Then

$$\begin{aligned} f(z) &= e^{\sin z} \\ &= \exp\left(\frac{z - z^{-1}}{2i}\right) \\ &= \exp\left(\frac{z}{2i}\right) \exp\left(\frac{-1}{2iz}\right) \end{aligned}$$

But $z=x+iy$ and the above expands to

$$\begin{aligned}
\exp(\sin z) &= \exp\left(\frac{1}{2i}(x+iy)\right)\exp\left(\frac{-1}{2i(x+iy)}\right)\\
&= \exp\left(\frac{-i}{2}x+\frac{1}{2}y\right)\exp\left(\frac{i}{2}\frac{1}{(x+iy)}\right)\\
&= \exp\left(\frac{-i}{2}x+\frac{1}{2}y\right)\exp\left(\frac{i}{2}\frac{x-iy}{(x+iy)(x-iy)}\right)\\
&= \exp\left(\frac{-i}{2}x+\frac{1}{2}y\right)\exp\left(\frac{i}{2}\frac{x-iy}{x^2+y^2}\right)\\
&= \exp\left(\frac{-i}{2}x+\frac{1}{2}y\right)\exp\left(\frac{i}{2}\left(\frac{x}{x^2+y^2}-\frac{iy}{x^2+y^2}\right)\right)\\
&= \exp\left(\frac{-i}{2}x+\frac{1}{2}y\right)\exp\left(\frac{i}{2}\frac{x}{x^2+y^2}+\frac{1}{2}\frac{y}{x^2+y^2}\right)\\
&= \exp\left(\frac{-i}{2}x\right)\exp\left(\frac{1}{2}y\right)\exp\left(\frac{i}{2}\frac{x}{x^2+y^2}\right)\exp\left(\frac{1}{2}\frac{y}{x^2+y^2}\right)
\end{aligned}$$

Collecting terms gives

$$\begin{aligned}
\exp(\sin z) &= \exp\left(\frac{1}{2}y+\frac{1}{2}\frac{y}{x^2+y^2}\right)\exp\left(\frac{i}{2}\frac{x}{x^2+y^2}-\frac{i}{2}x\right)\\
&= \exp\left(\frac{1}{2}\frac{y\left(1+\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\exp\left(\frac{i}{2}\frac{x}{x^2+y^2}-\frac{i}{2\left(x^2+y^2\right)}x\left(x^2+y^2\right)\right)\\
&= \exp\left(\frac{1}{2}\frac{y\left(1+\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\exp\left(i\frac{1}{2}\frac{x\left(1-\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\\
&= \exp\left(\frac{1}{2}\frac{y\left(1+\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\left[\cos\left(\frac{1}{2}\frac{x\left(1-\left(x^2+y^2\right)\right)}{x^2+y^2}\right)+i\sin\left(\frac{1}{2}\frac{x\left(1-\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\right]\\
&= \exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)+i\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)
\end{aligned}$$

Therefore, since $\exp(\sin z)=u+iv$, then we see from above that

$$\begin{aligned}
u &= \exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
v &= \exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)
\end{aligned}$$

Now we need to check the Cauchy-Riemann equations on the above u,v functions we found.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \tag{2}$$

Evaluating each partial derivative gives

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{d}{dx}\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&+\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{d}{dx}\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&=\frac{1}{2}\frac{2yx\left(x^2+y^2\right)-\left(y+y\left(x^2+y^2\right)\right)2x}{\left(x^2+y^2\right)^2}\exp\left(\frac{1}{2}\frac{y\left(\left(x^2+y^2\right)+1\right)}{x^2+y^2}\right)\cos\left(\frac{1}{2}\frac{x\left(1-\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\\
&-\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{d}{dx}\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&=\frac{-xy}{\left(x^2+y^2\right)^2}\exp\left(\frac{1}{2}\frac{y\left(\left(x^2+y^2\right)+1\right)}{x^2+y^2}\right)\cos\left(\frac{x\left(1-\left(x^2+y^2\right)\right)}{2\left(x^2+y^2\right)}\right)\\
&-\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{\left(1-3x^2-y^2\right)\left(x^2+y^2\right)-\left(x-x\left(x^2+y^2\right)\right)2x}{2\left(x^2+y^2\right)^2}\right)\\
&=\frac{-xy}{\left(x^2+y^2\right)^2}\exp\left(\frac{1}{2}\frac{y\left(x^2+y^2+1\right)}{x^2+y^2}\right)\cos\left(\frac{1}{2}\frac{x\left(1-\left(x^2+y^2\right)\right)}{x^2+y^2}\right)\\
&-\exp\left(\frac{1}{2}\frac{y\left(x^2+y^2+1\right)}{x^2+y^2}\right)\sin\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{\left(-x^4-2x^2y^2-x^2-y^4+y^2\right)}{2\left(x^2+y^2\right)^2}\right)\end{aligned}$$

The above can be simplified more to become

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{-1}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y\left(x^2+y^2+1\right)}{2\left(x^2+y^2\right)}\right)\\
&\left[2xy\cos\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}+\left(-x^4-2x^2y^2-x^2-y^4+y^2\right)\sin\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right] \qquad (3)\end{aligned}$$

Now we evaluate $\frac{\partial v}{\partial y}$ to see if it the same as above. Since $v=\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)$

then

$$\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{d}{dy}\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\frac{d}{dy}\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&=\left(\frac{1}{2}\frac{\left(1+x^2+3y^2\right)\left(x^2+y^2\right)-\left(y+y\left(x^2+y^2\right)\right)2y}{\left(x^2+y^2\right)^2}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{1}{2}\frac{(-2xy)\left(x^2+y^2\right)-\left(x-x\left(x^2+y^2\right)\right)(2y)}{\left(x^2+y^2\right)^2}\right)\\
&=\left(\frac{1}{2}\frac{x^4+2x^2y^2+x^2+y^4-y^2}{\left(x^2+y^2\right)^2}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{1}{2}\frac{-2xy}{\left(x^2+y^2\right)^2}\right)\\
&=\left(\frac{1}{2}\frac{x^4+2x^2y^2+x^2+y^4-y^2}{\left(x^2+y^2\right)^2}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&\quad-\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{xy}{\left(x^2+y^2\right)^2}\right)
\end{aligned}$$

Simplifying the above more gives

$$\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{-1}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\quad\left[2xy\cos\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}+\left(-x^4-2x^2y^2-x^2-y^4+y^2\right)\sin\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right] \qquad (4)
\end{aligned}$$

Comparing (3) and (4) shows they are the same expressions. Therefore the first equation is verified.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

Now we verify the second equation $-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$. Since $u=\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)$

then

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{d}{dy}\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&\quad-\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{d}{dy}\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&=\frac{\left(1+x^2+3y^2\right)\left(x^2+y^2\right)-\left(y+y\left(x^2+y^2\right)\right)2y}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\quad-\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{(-2y)\left(x^2+y^2\right)-\left(x-x\left(x^2+y^2\right)\right)2y}{2\left(x^2+y^2\right)^2}\\
&=\frac{\left(x^4+2x^2y^2+x^2+y^4-y^2\right)}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\quad-\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{(-2y)\left(x^2+y^2\right)-2yx+2yx\left(x^2+y^2\right)}{2\left(x^2+y^2\right)^2}\\
&=\frac{\left(x^4+2x^2y^2+x^2+y^4-y^2\right)}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{yx}{\left(x^2+y^2\right)^2}
\end{aligned}$$

The above can simplified more to give

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{1}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\qquad\left[\left(x^4+2x^2y^2+x^2+y^4-y^2\right)\cos\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}+2xy\sin\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right]
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{-\partial u}{\partial y} &= \frac{1}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\qquad\left[-\left(x^4+2x^2y^2+x^2+y^4-y^2\right)\cos\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}-2xy\sin\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right] \qquad (5)
\end{aligned}$$

And since $v=\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{\left(x^2+y^2\right)}\right)$ then

$$
\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{d}{dx}\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\frac{d}{dx}\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&= \frac{1}{2}\left(\frac{2xy\left(x^2+y^2\right)-\left(y+y\left(x^2+y^2\right)\right)2x}{\left(x^2+y^2\right)^2}\right)\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\sin\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{1}{2}\frac{x-x\left(x^2+y^2\right)}{x^2+y^2}\right)\left(\frac{\left(1-3x^2-y^2\right)\left(x^2+y^2\right)-\left(x-x\left(x^2+y^2\right)\right)2x}{2\left(x^2+y^2\right)^2}\right)\\
&= \frac{-xy}{\left(x^2+y^2\right)^2}\exp\left(\frac{1}{2}\frac{y+y\left(x^2+y^2\right)}{x^2+y^2}\right)\sin\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\quad+\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\cos\left(\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\left(\frac{-x^4-2x^2y^2-x^2-y^4+y^2}{2\left(x^2+y^2\right)^2}\right)
\end{aligned}
$$

The above can simplified more to give

$$
\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{1}{2\left(x^2+y^2\right)^2}\exp\left(\frac{y+y\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right)\\
&\qquad\left[-\left(x^4+2x^2y^2+x^2+y^4-y^2\right)\cos\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}-2xy\sin\frac{x-x\left(x^2+y^2\right)}{2\left(x^2+y^2\right)}\right] \qquad (6)
\end{aligned}
$$

Comparing (5,6) shows they are the same, i.e.

$$\frac{-\partial u}{\partial y}=\frac{\partial v}{\partial x}$$

C-R equations are satisfied, and because it is clear that all partial derivatives $\frac{\partial v}{\partial x},\frac{\partial v}{\partial y},\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$ are continuous functions in x,y as they are made up of exponential and trigonometric functions which are continuous, then we conclude that $f(z)=e^{\sin z}$ is analytic function everywhere.

2 Problem 2

2.1 Part (a)

Represent $\frac{z+3}{z-3}$ by its Maclaurin series and give the region of validity for the representation. Next expand this in powers of $\frac{1}{z}$ to find a Laurent series. What is the range of validity of the Laurent series?

Solution

Maclaurin series is expansion of $f(z)$ around $z=0$. Since $f(z)$ has simple pole at $z=3$, then the region of validity will be a disk centered at $z=0$ up to the nearest pole, which is at $z=3$.

Hence $|z| < 3$ is the region.

$$\begin{aligned} f(z) &= \frac{z+3}{z-3} \\ &= \frac{z+3}{-3\left(1-\frac{z}{3}\right)} \\ &= \frac{z+3}{-3}\left(\frac{1}{1-\frac{z}{3}}\right) \end{aligned}$$

Now we can expand using Binomial to obtain

$$\begin{aligned} f(z) &= \frac{3+z}{-3}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right) \\ &= \left(-1-\frac{z}{3}\right)\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right) \\ &= \left(-1-\frac{z}{3}\right)+\left(-1-\frac{z}{3}\right)\left(\frac{z}{3}\right)+\left(-1-\frac{z}{3}\right)\left(\frac{z}{3}\right)^2+\left(-1-\frac{z}{3}\right)\left(\frac{z}{3}\right)^3+\cdots \\ &= -1-\frac{z}{3}-\frac{z}{3}-\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3-\left(\frac{z}{3}\right)^3-\left(\frac{z}{3}\right)^4+\cdots \\ &= -1-\frac{2}{3}z-\frac{2}{9}z^2-\frac{2}{27}z^3-\frac{2}{81}z^4-\cdots \end{aligned}$$

Or

$$f(z) = -1-\sum_{n=1}^{\infty}\frac{2}{3^n}z^n$$

To expand in negative powers of z, or in $\frac{1}{z}$, then

$$\begin{aligned} f(z) &= \frac{z+3}{z\left(1-\frac{3}{z}\right)} \\ &= \frac{z+3}{z}\left(\frac{1}{1-\frac{3}{z}}\right) \end{aligned}$$

For $\left|\frac{3}{z}\right| < 1$ or $|z| < 3$ the above becomes

$$\begin{aligned} f(z) &= \frac{z+3}{z}\left(1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right) \\ &= \left(1+\frac{3}{z}\right)\left(1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right) \\ &= \left(1+\frac{3}{z}\right)+\left(1+\frac{3}{z}\right)\frac{3}{z}+\left(1+\frac{3}{z}\right)\left(\frac{3}{z}\right)^2+\left(1+\frac{3}{z}\right)\left(\frac{3}{z}\right)^3+\cdots \\ &= 1+\frac{3}{z}+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\left(\frac{3}{z}\right)^3+\left(\frac{3}{z}\right)^4+\cdots \\ &= 1+\frac{6}{z}+\frac{18}{z^2}+\frac{54}{z^3}+\cdots \end{aligned}$$

This is valid for $|z| > 3$. The residue is 6, which can be confirmed using

$$\begin{aligned} \text{Residue}(3) &= \lim_{z\to 3}(z-3)\, f(z) \\ &= \lim_{z\to 3}(z-3)\frac{z+3}{z-3} \\ &= \lim_{z\to 3}(z+3) \\ &= 6 \end{aligned}$$

Summary

$$f(z) = \frac{z+3}{z-3} = \begin{cases} -1 - \frac{2}{3}z - \frac{2}{9}z^2 - \frac{2}{27}z^3 - \frac{2}{81}z^4 - \cdots & |z| < 3 \\ 1 + \frac{6}{z} + \frac{18}{z^2} + \frac{54}{z^3} + \cdots & |z| > 3 \end{cases}$$

2.2 Part (b)

Find Laurent series for $\frac{z}{(z+1)(z-3)}$ in each of the following domains (i) $|z| < 1$ (ii) $1 < |z| < 3$ (iii) $|z| > 3$

Solution

The possible region are shown below. Since there is a pole at $z = -1$ and pole at $z = 3$, then there are three different regions. They are named A, B, C in the following diagram

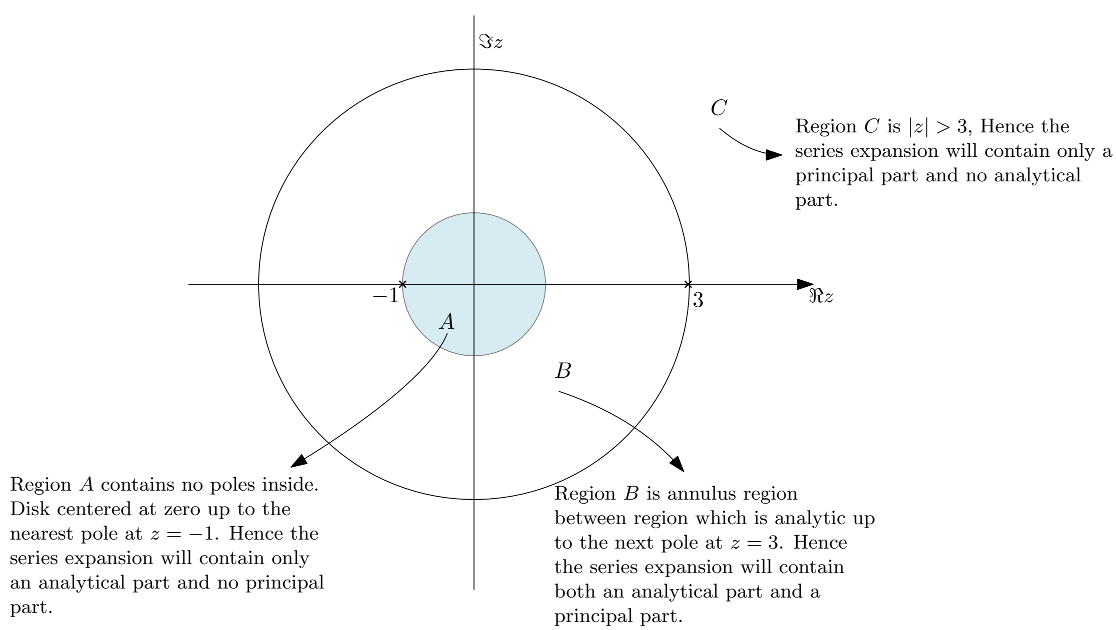

Figure 1: Laurent series regions

First the expression $\frac{z}{(z+1)(z-3)}$ is expanded using partial fractions

$$\frac{z}{(z+1)(z-3)}=\frac{A}{(z+1)}+\frac{B}{(z-3)} \tag{1}$$

Hence

$$\begin{aligned} z &= A(z-3)+B(z+1) \\ &= z(A+B)-3A+B \end{aligned}$$

The above gives two equations

$$\begin{aligned} A+B &= 1 \\ 0 &= -3A+B \end{aligned}$$

First equation gives $A=1-B$. Substituting in the second equation gives $0=-3(1-B)+B$ or $0=-3+4B$, hence $B=\frac{3}{4}$, which implies $A=1-\frac{3}{4}=\frac{1}{4}$, therefore (1) becomes

$$\frac{z}{(z+1)(z-3)}=\frac{1}{4}\frac{1}{(z+1)}+\frac{3}{4}\frac{1}{(z-3)}$$

Considering each term in turn. For $\frac{1}{4}\frac{1}{(z+1)}$, we can expand this as

$$\frac{1}{4}\frac{1}{(z+1)}=\frac{1}{4}\left(1-z+z^2-z^3+z^4+\cdots\right) \qquad |z|<1 \tag{2a}$$

$$\frac{1}{4}\frac{1}{(z+1)}=\frac{1}{4z}\frac{1}{\left(1+\frac{1}{z}\right)}=\frac{1}{4z}\left(1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\left(\frac{1}{z}\right)^4-\cdots\right) \qquad |z|>1 \tag{2b}$$

And for the term $\frac{3}{4}\frac{1}{(z-3)}$, we can expand this as

$$\frac{3}{4}\frac{1}{(z-3)}=-\frac{1}{4}\frac{1}{\left(1-\frac{z}{3}\right)}=-\frac{1}{4}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\left(\frac{z}{3}\right)^4+\cdots\right) \qquad |z|<3 \tag{3a}$$

$$\frac{3}{4}\frac{1}{(z-3)}=\frac{3}{4z}\frac{1}{\left(1-\frac{3}{z}\right)}=\frac{3}{4z}\left(1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right) \qquad |z|>3 \tag{3b}$$

Now that we expanded all the terms in the two possible ways for each each, we now consider each region of interest, and look at the above 4 expansions, and simply pick for each region the expansion which is valid in for that region of interest.

For (i), region A: In this region, we want $|z|<1$. From (2,3) we see that (2a) and (3a) are valid expansions in $|z|<1$. Hence

$$\begin{aligned} \frac{z}{(z+1)(z-3)} &= \frac{1}{4}\left(1-z+z^2-z^3+z^4+\cdots\right)-\frac{1}{4}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\left(\frac{z}{3}\right)^4+\cdots\right) \\ &= \frac{1}{4}\left(1-z+z^2-z^3+z^4-\cdots\right)-\frac{1}{4}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\frac{z^4}{81}+\cdots\right) \\ &= \left(\frac{1}{4}-\frac{1}{4}z+\frac{1}{4}z^2-\frac{1}{4}z^3+\frac{1}{4}z^4-\cdots\right)-\left(\frac{1}{4}+\frac{z}{12}+\frac{z^2}{36}+\frac{z^3}{108}+\frac{z^4}{324}+\cdots\right) \\ &= -\frac{1}{4}z-\frac{z}{12}+\frac{1}{4}z^2-\frac{z^2}{36}-\frac{1}{4}z^3-\frac{z^3}{108}+\frac{1}{4}z^4-\frac{z^4}{324}-\cdots \\ & -\frac{1}{3}z+\frac{2}{9}z^2-\frac{7}{27}z^3+\frac{20}{81}z^4-\cdots \end{aligned}$$

For (ii), region B: This is for $1 < |z| < 3$. From equations (2,3) we see that (2b) and (3a) are valid in this region. Hence

$$
\begin{aligned}
\frac{z}{(z+1)(z-3)} &= \frac{1}{4z}\left(1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\left(\frac{1}{z}\right)^4-\cdots\right)-\frac{1}{4}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\left(\frac{z}{3}\right)^4+\cdots\right)\\
&= \frac{1}{4z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}-\cdots\right)-\frac{1}{4}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\frac{z^4}{81}+\cdots\right)\\
&= \left(\frac{1}{4z}-\frac{1}{4z^2}+\frac{1}{4z^3}-\frac{1}{4z^4}+\frac{1}{4z^5}-\cdots\right)-\left(\frac{1}{4}+\frac{z}{12}+\frac{z^2}{36}+\frac{z^3}{108}+\frac{z^4}{324}+\cdots\right)\\
&= \overbrace{\cdots+\frac{1}{4z^5}-\frac{1}{4z^4}+\frac{1}{4z^3}-\frac{1}{4z^2}+\frac{1}{4z}}^{\text{principal part}}-\overbrace{\frac{1}{4}-\frac{z}{12}-\frac{z^2}{36}-\frac{z^3}{108}-\frac{z^4}{324}-\cdots}^{\text{analytical part}}
\end{aligned}
$$

The residue is $\frac{1}{4}$ by looking at the above. The value for the residue can be verified as follows. Using

$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}dz$$

Where in the above z_0 is the location of the pole and n is the coefficient of the $\frac{1}{z^n}$ is the principal part. Since we want the residue, then $n=1$ and the above becomes

$$b_1 = \frac{1}{2\pi i}\oint_C f(z)\,dz$$

In the above, the contour C is circle somewhere inside the annulus $1 < |z| < 3$. It does not matter that the radius is, as long as it is located in this range. For example, choosing radius 2 will work. The above then becomes

$$b_1 = \frac{1}{2\pi i}\oint_C \frac{z}{(z+1)(z-3)}dz \tag{5}$$

However, since $f(z)$ is analytic in this region, then $\oint_C f(z)\,dz = 2\pi i\sum$ (residues inside). There is only one pole now inside C, which is at $z=-1$. So all what we have to do is find the residue at $z=-1$.

$$
\begin{aligned}
\text{Residue}(-1) &= \lim_{z\to -1}(z+1)f(z)\\
&= \lim_{z\to -1}(z+1)\frac{z}{(z+1)(z-3)}\\
&= \lim_{z\to -1}\frac{z}{(z-3)}\\
&= \frac{-1}{(-1-3)}\\
&= \frac{1}{4}
\end{aligned}
$$

Using this in (5) gives

$$\begin{aligned} b_1 &= \frac{1}{2\pi i}\left(2\pi i \frac{1}{4}\right) \\ &= \frac{1}{4} \end{aligned}$$

Which agrees with what we found in (4) above.

<u>For (iii), region C</u>: This is for $|z| > 3$. From (2,3) we see that (2b) and (3b) are valid expansions <u>in $z > 3$, Hence</u>

$$\begin{aligned} \frac{z}{(z+1)(z-3)} &= \frac{1}{4z}\left(1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4 - \cdots\right) + \frac{3}{4z}\left(1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \cdots\right) \\ &= \left(\frac{1}{4z} - \frac{1}{4z^2} + \frac{1}{4z^3} - \frac{1}{4z^4} + \frac{1}{4z^5} - \cdots\right) + \frac{3}{4z}\left(1 + \frac{3}{z} + \frac{9}{z^2} + \frac{27}{z^3} + \cdots\right) \\ &= \left(\frac{1}{4z} - \frac{1}{4z^2} + \frac{1}{4z^3} - \frac{1}{4z^4} + \frac{1}{4z^5} - \cdots\right) + \left(\frac{3}{4z} + \frac{9}{4z^2} + \frac{27}{4z^3} + \frac{81}{4z^4} + \cdots\right) \\ &= \cdots + \frac{20}{z^4} + \frac{7}{z^3} + \frac{2}{z^2} + \frac{1}{z} \end{aligned}$$

This is as expected contains only a principal part and no analytical part. The <u>residue</u> is 1. This above value for the residue can be verified as follows. Using

$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

Where in the above z_0 is the location of the pole and n is the coefficient of the $\frac{1}{z^n}$ is the principal part. Since we want the residue, then $n = 1$ and the above becomes

$$b_1 = \frac{1}{2\pi i}\oint_C f(z)\, dz$$

In the above, the contour C is circle somewhere in $|z| > 3$. It does not matter that the radius is. The above integral then becomes

$$b_1 = \frac{1}{2\pi i}\oint_C \frac{z}{(z+1)(z-3)} dz \tag{7}$$

However, since $f(z)$ is analytic in $|z| > 3$, then $\oint_C f(z)\, dz = 2\pi i \sum$ (residues inside). There are now two poles inside C, one at $z = -1$ and one at $z = 3$. So all what we have to do is find the residues at each. We found earlier that Residue $(-1) = \frac{1}{4}$. Now

$$\begin{aligned} \text{Residue}(3) &= \lim_{z\to 3}(z-3) f(z) \\ &= \lim_{z\to 3}(z-3)\frac{z}{(z+1)(z-3)} \\ &= \lim_{z\to 3}\frac{z}{(z+1)} \\ &= \frac{3}{4} \end{aligned}$$

Therefore the sum of residues is 1. Using this result in (7) gives

$$\begin{aligned} b_1 &= \frac{1}{2\pi i}\left(2\pi i\left(\frac{1}{4}+\frac{3}{4}\right)\right) \\ &= 1 \end{aligned}$$

Which agrees with what result from (6) above.

Summary of results

$$f(z)=\frac{z}{(z+1)(z-3)}=\begin{cases} -\frac{1}{3}z+\frac{2}{9}z^2-\frac{7}{27}z^3+\frac{20}{81}z^4-\cdots & |z|<1 \\ \cdots+\frac{1}{4z^5}-\frac{1}{4z^4}+\frac{1}{4z^3}-\frac{1}{4z^2}+\frac{1}{4z}-\frac{1}{4}-\frac{z}{12}-\frac{z^2}{36}-\frac{z^3}{108}-\frac{z^4}{324}-\cdots & 1<|z|<3 \\ \cdots+\frac{20}{z^4}+\frac{7}{z^3}+\frac{2}{z^2}+\frac{1}{z} & |z|>3 \end{cases}$$

3 Problem 3

3.1 Part (a)

Use residue theorem to evaluate $\oint_C \frac{e^{-2z}}{z^2}dz$ on contour C which is circle $|z|=1$ in positive sense.

Solution

For $f(z)$ which is analytic on and inside C, the Cauchy integral formula says

$$\oint_C f(z)\,dz = 2\pi i\sum_j \text{Residue}\left(z=z_j\right) \tag{1}$$

Where the sum is over all residues located inside C. for $f(z)=\frac{e^{-2z}}{z^2}$ there is a simple pole at $z=0$ of order 2. To find the residue, we use the formula for pole or order m given by

$$\text{Residue}(z_0)=\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\frac{(z-z_0)^m}{(m-1)!}f(z)$$

Hence for $m=2$ and $z_0=0$ the above becomes

$$\begin{aligned} \text{Residue}(0) &= \lim_{z\to 0}\frac{d}{dz}z^2 f(z) \\ &= \lim_{z\to 0}\frac{d}{dz}z^2\frac{e^{-2z}}{z^2} \\ &= \lim_{z\to 0}\frac{d}{dz}e^{-2z} \\ &= \lim_{z\to 0}\left(-2e^{-2z}\right) \\ &= -2 \end{aligned}$$

Therefore (1) becomes

$$\oint_C \frac{e^{-2z}}{z^2} dz = 2\pi i (-2)$$
$$= -4\pi i$$

3.2 Part (b)

Use residue theorem to evaluate $\oint_C z e^{\frac{1}{z}} dz$ on contour C which is circle $|z| = 1$ in positive sense.

Solution

The singularity is at $z = 0$, but we can not use the simple pole residue finding method here, since this is an essential singularity now due to the $e^{\frac{1}{z}}$ term. To find the residue, we expand $f(z)$ around $z = 0$ in Laurent series and look for the coefficient of $\frac{1}{z}$ term.

$$\begin{aligned} f(z) &= z e^{\frac{1}{z}} \\ &= z\left(1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right) \\ &= z + 1 + \frac{1}{2}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \cdots \end{aligned}$$

Hence residue is $\frac{1}{2}$. Therefore

$$\oint_C z e^{\frac{1}{z}} dz = 2\pi i \left(\frac{1}{2}\right)$$
$$= \pi i$$

3.3 Part (c)

Use residue theorem to evaluate $\oint_C \frac{z+2}{z^2 - \frac{z}{2}} dz$ on contour C which is circle $|z| = 1$ in positive sense.

Solution

$$\begin{aligned} f(z) &= \frac{z+2}{z^2 - \frac{z}{2}} \\ &= \frac{z+2}{z\left(z - \frac{1}{2}\right)} \end{aligned}$$

Hence there is a simple pole at $z = 0$ and simple pole at $z = \frac{1}{2}$

$$\begin{aligned}
\text{Residue}\,(0) &= \lim_{z\to 0}(z)\,f(z) \\
&= \lim_{z\to 0} z\frac{z+2}{z\left(z-\frac{1}{2}\right)} \\
&= \lim_{z\to 0} \frac{z+2}{\left(z-\frac{1}{2}\right)} \\
&= \frac{2}{-\frac{1}{2}} \\
&= -4
\end{aligned}$$

And

$$\begin{aligned}
\text{Residue}\left(\frac{1}{2}\right) &= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z) \\
&= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z+2}{z\left(z-\frac{1}{2}\right)} \\
&= \lim_{z\to\frac{1}{2}}\frac{z+2}{z} \\
&= \frac{\frac{1}{2}+2}{\frac{1}{2}} \\
&= 5
\end{aligned}$$

Therefore

$$\begin{aligned}
\oint_C \frac{z+2}{z^2-\frac{z}{2}}dz &= 2\pi i\,(5-4) \\
&= 2\pi i
\end{aligned}$$