HW 2, Physics 501
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Contents

1 Problem 1

Find all possible values for (put into $x + iy$ form)

1. $\log\left(1 + \sqrt{3}i\right)$
2. $\left(1 + \sqrt{3}i\right)^{2i}$

Answer

1.1 Part 1

Let $z = x + iy$, where here $x = 1, y = 3$, then $|z| = \sqrt{x^2 + y^2} = \sqrt{1+3} = 2$ and $\arg(z) = \theta_0 = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{6} = 60^0$. The function $\log(z)$ is infinitely multi-valued, given by

$$\log(z) = \ln|z| + i(\theta_0 + 2n\pi) \qquad n = 0, \pm 1, \pm 2, \cdots \tag{1}$$

Where θ_0 is the principal argument, which is 60^0 in this example, which is when $n = 0$. This is done to make $\log(z)$ single valued. This makes the argument of z restricted to $-\pi < \theta_0 < \pi$. This makes the negative real axis the branch cut, including the origin. To find all values, we simply use (1) for all possible n values other than $n = 0$. Each different n values gives different branch cut. This gives, where $\ln|z| = \ln(2)$ in all cases, the following

$$\begin{aligned}
\log(z) &= \ln(2) + i\left(\frac{\pi}{3}\right) && n = 0\\
&= \ln(2) + i\left(\frac{\pi}{3} + 2\pi\right) && n = 1\\
&= \ln(2) + i\left(\frac{\pi}{3} - 2\pi\right) && n = -1\\
&= \ln(2) + i\left(\frac{\pi}{3} + 4\pi\right) && n = 2\\
&= \ln(2) + i\left(\frac{\pi}{3} - 4\pi\right) && n = -2\\
&\vdots
\end{aligned}$$

Or

$$\begin{aligned}
\log(z) &= 0.693 + 1.047i\\
&= 0.693 + 7.330i\\
&= 0.693 - 5.236i\\
&= 0.693 + 13.614i\\
&= 0.693 - 11.519i\\
&\vdots
\end{aligned}$$

These are in $(x + iy)$ form. There are infinite number of values. Picking a specific branch cuts (i.e. specific n value), picks one of these values. The principal value is one associated with $n = 0$.

1.2 Part 2

Let $z = 1 + i\sqrt{3}$, hence

$$\begin{aligned}
f(z) &= z^{2i}\\
&= \exp(2i\log(z))\\
&= \exp(2i(\ln|z| + i(\theta_0 + 2n\pi))) \qquad n = 0, \pm 1, \pm 2, \cdots
\end{aligned}$$

Where in this example, as in first part, $\ln|z| = \ln(2) = 0.693$ and principal argument is $\theta_0 = \frac{\pi}{3} = 60^0$. Hence the above becomes

$$
\begin{aligned}
f(z) &= \exp\left(2i\left(\ln(2) + i\left(\frac{\pi}{3} + 2n\pi\right)\right)\right) \\
&= \exp\left(2i\ln(2) - \left(\frac{2\pi}{3} + 4n\pi\right)\right) \\
&= \exp\left(i\ln 4 - \left(\frac{2\pi}{3} + 4n\pi\right)\right) \\
&= \exp(i\ln 4)\exp\left(-\left(\frac{2\pi}{3} + 4n\pi\right)\right) \\
&= e^{-\left(\frac{2\pi}{3}+4n\pi\right)}\left(\cos(\ln 4) + i\sin(\ln 4)\right) \\
&= e^{-\left(\frac{2\pi}{3}+4n\pi\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}+4n\pi\right)}\sin(\ln 4)
\end{aligned}
$$

Which is now in the form of $x + iy$. First few values are

$$
\begin{aligned}
f(z) &= e^{-\left(\frac{2\pi}{3}\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}\right)}\sin(\ln 4) & n &= 0 \\
&= e^{-\left(\frac{2\pi}{3}+4\pi\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}+4\pi\right)}\sin(\ln 4) & n &= 1 \\
&= e^{-\left(\frac{2\pi}{3}-4\pi\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}-4\pi\right)}\sin(\ln 4) & n &= -1 \\
&= e^{-\left(\frac{2\pi}{3}+8\pi\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}+8\pi\right)}\sin(\ln 4) & n &= 2 \\
&= e^{-\left(\frac{2\pi}{3}-8\pi\right)}\cos(\ln 4) + ie^{-\left(\frac{2\pi}{3}-8\pi\right)}\sin(\ln 4) & n &= -2 \\
&\vdots
\end{aligned}
$$

Or

$$
\begin{aligned}
f(z) &= 0.0226 + i0.121 \\
&= 7.878 \times 10^{-8} + i4.222 \times 10^{-7} \\
&= 6478 + i34713 \\
&= 2.748 \times 10^{-13} + i1.472 \times 10^{-12} \\
&= 1.858 \times 10^{9} + i9.954 \times 10^{9} \\
&\vdots
\end{aligned}
$$

2 Problem 2

Given that $u(x,y) = 3x^2y - y^3$ find $v(x,y)$ such that $f(z)$ is analytic. Do the same for $u(x,y) = \frac{y}{x^2+y^2}$

Solution

2.1 Part (1)

$u(x,y) = 3x^2y - y^3$. The function $f(z)$ is analytic if it satisfies Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

Applying the first equation gives

$$6xy = \frac{\partial v}{\partial y}$$

Hence, solving for v by integrating, gives

$$v(x,y) = 3xy^2 + f(x) \tag{3}$$

Is the solution to (3) where $f(x)$ is the constant of integration since it is a partial differential equation. We now use equation (2) to find $f(x)$. From (2)

$$\begin{aligned} -\left(3x^2 - 3y^2\right) &= \frac{\partial v}{\partial x} \\ -3x^2 + 3y^2 &= \frac{\partial v}{\partial x} \end{aligned}$$

But (3) gives $\frac{\partial v}{\partial x} = 3y^2 + f'(x)$, hence the above becomes

$$\begin{aligned} -3x^2 + 3y^2 &= 3y^2 + f'(x) \\ f'(x) &= -3x^2 + 3y^2 - 3y^2 \\ &= -3x^2 \end{aligned}$$

Integrating gives

$$\begin{aligned} f(x) &= \int -3x^2 dx \\ &= -x^3 + C \end{aligned}$$

Therefore, (3) becomes

$$v(x,y) = 3xy^2 + f(x)$$

Or

$$\boxed{v(x,y) = 3xy^2 - x^3 + C}$$

Where C is arbitrary constant. To verify, we apply CR again. Equation (1) now gives

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ 6xy &= 6yx \end{aligned}$$

Verified. Equation (2) gives

$$\begin{aligned} -\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} \\ -3x^2 + 3y^2 &= -3x^2 + 3y^2 \end{aligned}$$

Verified.

2.2 Part (2)

$u(x,y) = \frac{y}{x^2+y^2}$. The function $f(z)$ is analytic if it satisfies Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

Applying the first equation gives

$$-\frac{2xy}{\left(x^2+y^2\right)^2} = \frac{\partial v}{\partial y}$$

Hence, solving for v by integrating, gives

$$\begin{aligned} v &= -2x \int \frac{y}{\left(x^2+y^2\right)^2} dy \\ &= \frac{x}{x^2+y^2} + f(x) \end{aligned} \tag{3}$$

Is the solution to (3) where $f(x)$ is the constant of integration since it is a partial differential equation. equation (2) gives

$$-\frac{1}{x^2+y^2} + \frac{2y^2}{\left(x^2+y^2\right)^2} = \frac{\partial v}{\partial x}$$

But (3) gives $\frac{\partial v}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{\left(x^2+y^2\right)^2} + f'(x)$, hence the above becomes

$$\begin{aligned} -\frac{1}{x^2+y^2} + \frac{2y^2}{\left(x^2+y^2\right)^2} &= \frac{1}{x^2+y^2} - \frac{2x^2}{\left(x^2+y^2\right)^2} + f'(x) \\ f'(x) &= -\frac{2}{x^2+y^2} + \frac{2\left(y^2+x^2\right)}{\left(x^2+y^2\right)^2} \\ &= -\frac{2}{x^2+y^2} + \frac{2}{\left(x^2+y^2\right)} \\ &= 0 \end{aligned}$$

Hence

$$f(x) = C$$

where C is arbitrary constant. Therefore, (3) becomes

$$\boxed{v(x,y) = \frac{x}{x^2+y^2} + C}$$

To verify, CR is applied again. Equation (1) now gives

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{-2xy}{\left(x^2+y^2\right)^2} &= \frac{-2xy}{\left(x^2+y^2\right)^2} \end{aligned}$$

Hence verified. Equation (2) gives

$$\begin{aligned} -\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} \\ -\frac{1}{x^2+y^2} + \frac{2y^2}{\left(x^2+y^2\right)^2} &= \frac{1}{x^2+y^2} - \frac{2x^2}{\left(x^2+y^2\right)^2} \\ \frac{-\left(x^2+y^2\right) + 2y^2}{\left(x^2+y^2\right)^2} &= \frac{x^2+y^2-2x^2}{\left(x^2+y^2\right)^2} \\ \frac{-x^2+y^2}{\left(x^2+y^2\right)^2} &= \frac{-x^2+y^2}{\left(x^2+y^2\right)^2} \end{aligned}$$

Verified.

3 Problem 3

Evaluate the integral (i) $\oint\limits_C |z|^2 \, dz$ and (ii) $\oint\limits_C \frac{1}{z^2} dz$ along two contours. These contours are

1. Line segment with initial point 1 and fixed point i
2. Arc of unit circle with $\mathrm{Im}\,(z) \geq 0$ with initial point 1 and final point i

Solution

3.1 Part (1)

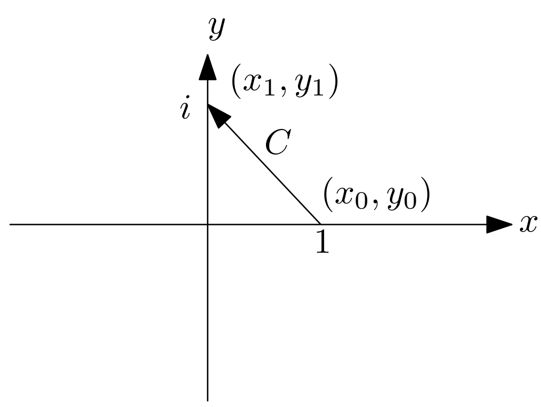

Figure 1: Integration path

First integral We start by finding the parameterization. For line segments that starts at (x_0, y_0) and ends at (x_1, y_1), the parametrization is given by

$$\begin{aligned} x\,(t) &= (1-t)\,x_0 + t x_1 \\ y\,(t) &= (1-t)\,y_0 + t y_1 \end{aligned}$$

For $0 \leq t \leq 1$. Hence for $z = x + iy$, it becomes $z\,(t) = x\,(t) + iy\,(t)$. In this case, $x_0 = 1, y_0 = 0, x_1 = 0, y_1 = 1$, therefore

$$\begin{aligned} x\,(t) &= (1-t) \\ y\,(t) &= t \end{aligned}$$

Using these, $z\,(t)$ is found from

$$\begin{aligned} z\,(t) &= x\,(t) + iy\,(t) \\ &= (1-t) + it \end{aligned}$$

And

$$z'\,(t) = -1 + i$$

Since $|z|^2 = x^2 + y^2$, then in terms of t it becomes

$$|z\,(t)|^2 = (1-t)^2 + t^2$$

Hence the line integral now becomes

$$\begin{aligned}\int_C |z|^2 \, dz &= \int_0^1 |z(t)|^2 \, z'(t) \, dt \\ &= \int_0^1 \left((1-t)^2 + t^2\right)(-1+i) \, dt \\ &= (-1+i) \int_0^1 (1-t)^2 + t^2 dt \\ &= (-1+i) \int_0^1 1 + t^2 - 2t + t^2 dt \\ &= (-1+i) \int_0^1 1 + 2t^2 - 2t \, dt \\ &= (-1+i) \left(\int_0^1 dt + \int_0^1 2t^2 dt - \int_0^1 2t \, dt \right) \\ &= (-1+i) \left((t)_0^1 + 2 \left(\frac{t^3}{3} \right)_0^1 - 2 \left(\frac{t^2}{2} \right)_0^1 \right) \\ &= (-1+i) \left(1 + \frac{2}{3} - 2 \left(\frac{1}{2} \right) \right)\end{aligned}$$

Hence

$$\boxed{\int_C |z|^2 \, dz = \tfrac{2}{3}(i-1)}$$

second integral

Using the same parameterization above. But here the integrand is

$$\frac{1}{z^2} = \frac{1}{((1-t)+it)^2}$$

Hence the integral becomes

$$\begin{aligned}\int_C \frac{1}{z^2} dz &= \int_0^1 \frac{1}{((1-t)+it)^2} z'(t) \, dt \\ &= (i-1) \int_0^1 \frac{1}{((1-t)+it)^2} dt \\ &= (i-1)(-i)\end{aligned}$$

Hence

$$\boxed{\int_C \tfrac{1}{z^2} dz = 1 + i}$$

3.2 Part (2)

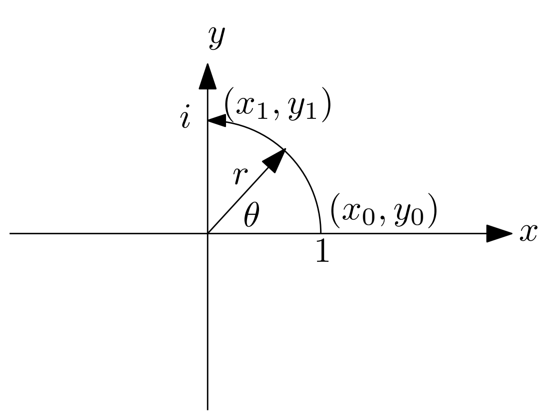

Figure 2: Integration path

First integral Let $z = re^{i\theta}$ then $\frac{dz}{d\theta} = rie^{i\theta}$. When $z = 1$ then $\theta = 0$. When $z = i$ then $\theta = \frac{\pi}{2}$, hence we can parameterize the contour integral using θ and it becomes

$$\begin{aligned}
\int_C |z|^2 \, dz &= \int_0^{\frac{\pi}{2}} r^2 \left(rie^{i\theta} \right) d\theta \\
&= ir^3 \int_0^{\frac{\pi}{2}} e^{i\theta} d\theta \\
&= ir^3 \left[\frac{e^{i\theta}}{i} \right]_0^{\frac{\pi}{2}} \\
&= r^3 \left[e^{i\theta} \right]_0^{\frac{\pi}{2}} \\
&= r^3 \left[e^{i\frac{\pi}{2}} - e^0 \right] \\
&= r^3 \left[i - 1 \right]
\end{aligned}$$

But $r = 1$, therefore the above becomes

$$\boxed{\int_C |z|^2 \, dz = i - 1}$$

second integral

Using the same parameterization above. But here the integrand now

$$\frac{1}{z^2} = \frac{1}{r^2 e^{i2\theta}}$$

Therefore

$$\begin{aligned}
\int_C \frac{1}{z^2} dz &= \int_0^{\frac{\pi}{2}} \frac{1}{r^2 e^{i2\theta}} \left(rie^{i\theta} \right) d\theta \\
&= \frac{i}{r} \int_0^{\frac{\pi}{2}} e^{-i\theta} d\theta \\
&= \frac{i}{r} \left(\frac{e^{-i\theta}}{-i} \right)_0^{\frac{\pi}{2}} \\
&= \frac{-1}{r} \left(e^{-i\theta} \right)_0^{\frac{\pi}{2}} \\
&= \frac{-1}{r} \left(e^{-i\frac{\pi}{2}} - 1 \right) \\
&= \frac{-1}{r} \left(-i - 1 \right)
\end{aligned}$$

But $r = 1$, hence

$$\boxed{\int_C \frac{1}{z^2} dz = 1 + i}$$

4 Problem 4

Use the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$

To evaluate

$$\oint_C \frac{1}{(z+1)(z+2)}\,dz$$

Where C is the circular contour $|z+1| = R$ with $R < 1$. Note that if $R > 1$ then a different result is found. Why can't the Cauchy integral formula above be used for $R > 1$?

Solution

The disk $|z+1| = R$ is centered at $z = -1$ with $R < 1$. The function

$$g(z) = \frac{1}{(z+1)(z+2)}$$

has pole at $z = -1$ and at $z = -2$.

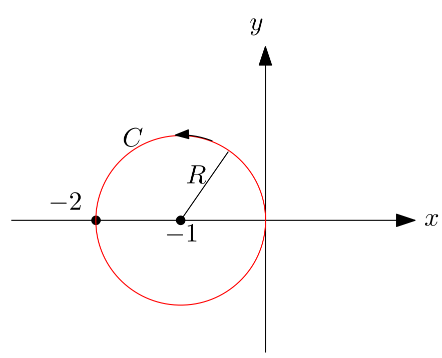

Figure 3: Showing location of pole

In the Cauchy integral formula, the function $f(z)$ is analytic on C and inside C. Hence, to use Cauchy integral formula, we need to convert $g(z) = \frac{1}{(z+1)(z+2)}$ to look like $\frac{f(z)}{z-z_0}$ where $f(z)$ is analytic inside C. This is done as follows

$$\begin{aligned}\frac{1}{(z+1)(z+2)} &= \frac{\frac{1}{(z+2)}}{z-(-1)} \\ &= \frac{f(z)}{z-(-1)}\end{aligned}$$

Where now $f(z) = \frac{1}{(z+2)}$. This has pole at $z = -2$. Since this pole is outside C then $f(z)$ is analytic on and inside C and can be used for the purpose of using Cauchy integral formula, which now can be written as

$$\begin{aligned}\oint_C \frac{1}{(z+1)(z+2)}\,dz &= \oint_C \frac{\frac{1}{(z+2)}}{z-(-1)}\,dz \\ &= \oint_C \frac{f(z)}{z-(-1)}\,dz \\ &= (2\pi i)\,f(-1)\end{aligned}$$

Therefore, we just need to evaluate $f(-1)$ which is seen as 1. Hence

$$\boxed{\oint_C \frac{1}{(z+1)(z+2)}\,dz = 2\pi i} \tag{1}$$

To verify, we can solve this again using the residue theorem

$$\oint_C g(z)\,dz = 2\pi i\,(\text{sum of residues of } g(z) \text{ inside } C)$$

But $g(z) = \frac{1}{(z+1)(z+2)}$ has only one pole inside C, which is at $z = -1$. Therefore the above becomes

$$\oint_C \frac{1}{(z+1)(z+2)} = 2\pi i\,(\text{residue of } g(z) \text{ at } -1) \tag{2}$$

To find residue at -1, we can use one of the short cuts to do that. Where we write $\frac{1}{(z+1)(z+2)} = \frac{\Phi(z)}{z+1}$ where $\Phi(z)$ is analytic at $z = -1$ and $\Phi(-1) \neq 0$. Therefore we see that $\Phi(z) = \frac{1}{z+2}$. Hence residue of $\frac{1}{(z+1)(z+2)} = \Phi(z_0) = \frac{1}{(-1)+2} = 1$. Equation (2) becomes

$$\oint_C \frac{1}{(z+1)(z+2)} = 2\pi i$$

Which is same result obtained in (1) by using Cauchy integral formula directly.

To answer last part, when $R > 1$, then now both poles $z = -1$ and $= -2$, are inside C. Therefore, we can't split $\frac{1}{(z+1)(z+2)}$ into one part that is analytic (the $f(z)$ in the above), in order to obtain expression $\frac{f(z)}{z-z_0}$ in order to apply Cauchy integral formula directly. Therefore when $R > 1$ we should use

$$\oint_C g(z)\,dz = 2\pi i\,(\text{sum of residues of } g(z) \text{ inside } C)$$

5 Problem 5

Evaluate the integral

$$\oint_C e^{z^2}\left(\frac{1}{z^2}-\frac{1}{z^3}\right)dz$$

Where he contour is the unit circle around origin (counter clockwise direction).

Solution

$$\begin{aligned}\oint_C e^{z^2}\left(\frac{1}{z^2}-\frac{1}{z^3}\right)dz &= \oint_C e^{z^2}\left(\frac{z-1}{z^3}\right)dz\\ &= \oint_C \frac{f(z)}{(z-z_0)^3}dz\end{aligned}$$

Where $z_0 = 0$ and where

$$f(z) = e^{z^2}(z-1)$$

But $f(z)$ is analytic on C and inside, since e^{z^2} is analytic everywhere and $z-1$ has no poles. Hence we can use Cauchy integral formula for pole of higher order given by

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(z_0)$$

Where $n = 2$ in this case. Therefore, since $z_0 = 0$ the above reduces to

$$\oint_C \frac{f(z)}{z^3}dz = \frac{2\pi i}{2}f''(0) \tag{1}$$

Now we just need to find $f''(z)$ and evaluate the result at $z_0 = 0$

$$\begin{aligned}f'(z) &= 2ze^{z^2}(z-1)+e^{z^2}\\ f''(z) &= 2e^{z^2}(z-1)+2z\left(2ze^{z^2}(z-1)+e^{z^2}\right)+2ze^{z^2}\end{aligned}$$

Hence

$$f''(0) = -2$$

Therefore (1) becomes

$$\oint_C \frac{e^{z^2}(z-1)}{z^3}dz = -2\pi i \tag{2}$$

To verify, we will do the same integration by converting it to line integration using parameterization on θ. Let $z(\theta) = re^{i\theta}$, but $r = 1$, therefore $z(\theta) = e^{i\theta}$, $dz = ie^{i\theta}d\theta$. Therefore the integral becomes

$$\begin{aligned}\oint_C e^{z^2}\left(\frac{z-1}{z^3}\right)dz &= \int_0^{2\pi} e^{e^{2i\theta}}\left(\frac{e^{i\theta}-1}{e^{3i\theta}}\right)ie^{i\theta}d\theta\\ &= i\int_0^{2\pi} e^{e^{2i\theta}}\left(\frac{e^{i\theta}-1}{e^{2i\theta}}\right)d\theta\end{aligned}$$

This is a hard integral to solve by hand. Using computer algebra software, it also gave $-2\pi i$. This verified the result. Clearly using the Cauchy integral formula to solve this problem was much simpler that using parameterization.