HW 1, Physics 501
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Contents

1 Problem 1

1.1 Part a

For what values of x does the following series converge. $f(x) = 1 + \frac{9}{x^2} + \frac{81}{x^2} + \frac{729}{x^3} + \cdots$
answer
The general term of the series is

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{x}\right)^{2n}$$

The ratio test can be used to determine convergence. Since all the terms are positive (powers are even), then the absolute value is not needed.

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{\left(\frac{3}{x}\right)^{2(n+1)}}{\left(\frac{3}{x}\right)^{2n}} \\
&= \lim_{n\to\infty} \frac{\left(\frac{3}{x}\right)^{2n}}{\left(\frac{3}{x}\right)^{2n}} \left(\frac{3}{x}\right)^2 \\
&= \lim_{n\to\infty} \frac{3^2}{x^2} \\
&= \frac{9}{x^2}
\end{aligned}$$

The series converges when $L < 1$, which means $\frac{9}{x^2} < 1$ or $x^2 > 9$. Therefore it convergence for

$$|x| > 3$$

1.2 Part b

Does the following series converges or diverges? $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$
answer
The terms are $\{\ln(2), \ln\left(2 + \frac{1}{2}\right), \ln\left(2 + \frac{1}{3}\right), \ln\left(2 + \frac{1}{4}\right), \cdots\}$.
Since the terms become monotonically decreasing after some n (after the second term in this case), the integral test could be used. Let

$$I = \int \ln\left(1 + \frac{1}{x}\right) dx$$

The above indefinite integral is first evaluated. Since $\ln\left(1+\frac{1}{x}\right)=\ln\left(\frac{1+x}{x}\right)=\ln(1+x)-\ln(x)$, the above can be written as

$$I=\int\ln(1+x)\,dx-\int\ln(x)\,dx \tag{1}$$

To evaluate the first integral in (2) $\int\ln(1+x)\,dx$, let $u=1+x$, then $du=dx$, therefore

$$\begin{aligned}\int\ln(1+x)\,dx&=\int\ln(u)\,du\\&=u\ln(u)-u\end{aligned}$$

Hence

$$\int\ln(1+x)\,dx=(1+x)\ln(1+x)-(1+x) \tag{2}$$

The second integral in (1) is

$$\int\ln(x)\,dx=x\ln(x)-x \tag{3}$$

Using (2,3) back into (1) gives

$$\begin{aligned}I&=((1+x)\ln(1+x)-(1+x))-(x\ln(x)-x)\\&=(1+x)\ln(1+x)-1-x-x\ln(x)+x\\&=(1+x)\ln(1+x)-x\ln(x)-1 \qquad (4)\end{aligned}$$

Now that the indefinite integral is evaluated, the limit is taken using

$$R=\lim_{N\to\infty}\int^{N}\ln\left(1+\frac{1}{x}\right)dx$$

Only upper limit is used following the book method[1]. Using the result found in (4), the above becomes

$$R=\lim_{N\to\infty}\left[(1+x)\ln(1+x)-x\ln(x)-1\right]^{N}$$

The above becomes

$$\begin{aligned}R&=\lim_{N\to\infty}\left[(1+x)\ln(1+x)-x\ln(x)-1\right]^{N}\\&=\lim_{N\to\infty}\left[(1+N)\ln(1+N)-N\ln(N)-1\right]\\&=\lim_{N\to\infty}\left[\ln(1+N)+N\ln(1+N)-N\ln(N)-1\right]\\&=\lim_{N\to\infty}\left[\ln(1+N)+N\ln\left(\frac{1+N}{N}\right)-1\right]\\&=\lim_{N\to\infty}\ln(1+N)+\lim_{N\to\infty}N\ln\left(\frac{1+N}{N}\right)-1 \qquad (5)\end{aligned}$$

But

$$\lim_{N\to\infty}N\ln\left(\frac{1+N}{N}\right)=\lim_{N\to\infty}\frac{\ln\left(\frac{1+N}{N}\right)}{\frac{1}{N}}$$

This gives indeterminate form $1/0$. So using L'Hospital's rule, by taking derivatives of numerator and denominator gives

$$\lim_{N\to\infty}\frac{\ln\left(\frac{1+N}{N}\right)}{\frac{1}{N}}=\lim_{N\to\infty}\frac{\frac{\left(1-\frac{1+N}{N}\right)}{1+N}}{-\frac{1}{N^2}}=\lim_{N\to\infty}-\frac{\left(1-\frac{1+N}{N}\right)N^2}{1+N}=\lim_{N\to\infty}-\frac{(N-1-N)N}{1+N}=\lim_{N\to\infty}\frac{N}{1+N}=1$$

Therefore $\lim_{N\to\infty}N\ln\left(\frac{1+N}{N}\right)=1$. Using this result in (5) results in

$$\begin{aligned}R&=\lim_{N\to\infty}\ln(1+N)+1-1\\&=\lim_{N\to\infty}\ln(1+N)\\&=\infty\end{aligned}$$

Therefore, by the integral test the series diverges.

[1] See page 131, second edition. Mathematical methods for physics and engineering. Riley, Hobson and Bence.

2 Problem 2

Find closed form for the series $f(x)=\sum_{n=0}^{\infty}n^2x^{2n}$ by taking derivatives of variant of $\frac{1}{1-x}$. For what values of x does the series converge?
answer

$$f(x)=x^2+4x^4+9x^6+16x^8+25x^{10}+\cdots$$

Observing that

$$n^2x^{2n}=\frac{x}{2}\frac{d}{dx}\left(nx^{2n}\right)$$

Therefore the sum can be written as

$$\begin{aligned}f(x)&=\frac{x}{2}\sum_{n=0}^{\infty}\frac{d}{dx}\left(nx^{2n}\right)\\&=\frac{x}{2}\frac{d}{dx}\sum_{n=0}^{\infty}nx^{2n}\\&=\frac{x}{2}\frac{d}{dx}\left(x^2+2x^4+3x^6+4x^8+5x^{10}+\cdots\right)\\&=\frac{x}{2}\frac{d}{dx}\left(x^2\left(1+2x^2+3x^4+4x^6+5x^8+\cdots\right)\right)\end{aligned}\tag{1}$$

To find what $1+2x^2+3x^4+4x^6+\cdots$ sums to, we compare it to the binomial series

$$(1+z)^{\alpha}=1+\alpha z+\frac{\alpha(\alpha-1)z^2}{2!}+\frac{\alpha(\alpha-1)(\alpha-2)z^3}{3!}+\cdots=1+2x^2+3x^4+4x^6+5x^8+\cdots$$

Hence, by setting

$$\begin{aligned}z&=-x^2\\\alpha&=-2\end{aligned}$$

shows they are the same. Therefore

$$\begin{aligned}\left(1-x^2\right)^{-2}&=1+(-2)\left(-x^2\right)+\frac{(-2)(-3)\left(-x^2\right)^2}{2!}+\frac{(-2)(-3)(-4)\left(-x^2\right)^3}{3!}+\cdots\\&=1+2x^2+3x^2+3x^4+\cdots\end{aligned}$$

The above is valid for $|z|<1$ which implies $x^2<1$ or $|x|<1$. Hence

$$1+2x^2+3x^4+4x^6+\cdots=\left(1-x^2\right)^{-2}$$

Using the above result in (1) gives

$$\begin{aligned}f(x)&=\frac{x}{2}\frac{d}{dx}\left(\frac{x^2}{(1-x^2)^2}\right)\\&=\frac{x}{2}\left(\frac{4x^3}{(1-x^2)^3}+\frac{2x}{(1-x^2)^2}\right)\\&=\frac{2x^4}{(1-x^2)^3}+\frac{x^2}{(1-x^2)^2}\\&=\frac{2x^4+x^2\left(1-x^2\right)}{(1-x^2)^3}\\&=\frac{2x^4+x^2-x^4}{(1-x^2)^3}\\&=\frac{x^4+x^2}{(1-x^2)^3}\end{aligned}$$

Therefore

$$\boxed{f(x)=\frac{x^2(x^2+1)}{(1-x^2)^3}}$$

Where the above converges for $|x|<1$, from above, where we used Binomial expansion which is valid for $|x|<1$. This result could also be obtained by using the ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^2x^{2(n+1)}}{n^2x^{2n}}\right|$$

Since all powers are even, the absolute value is not needed. The above becomes

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)^2 x^2}{n^2} \\
&= x^2 \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \\
&= x^2
\end{aligned}$$

Therefore for the series to converge, we know that $\frac{a_{n+1}}{a_n}$ must be less than 1. Hence $x^2 < 1$ or $|x| < 1$, which is the same result as above.

3 Problem 3

3.1 Part a

Find the sum of $1 + \frac{1}{4} - \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} - \cdots$

solution

We would like to combine each two consecutive negative terms and combine each two consecutive positive terms in the series in order to obtain an alternating series which is easier to work with. but to do that, we first need to check that the series is absolutely convergent. The $|a_n|$ term is $\frac{1}{4^n}$, therefore

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{\frac{1}{4^{n+1}}}{\frac{1}{4^n}}\right| \\
&= \lim_{n\to\infty} \left|\frac{4^n}{4^{n+1}}\right| \\
&= \left|\frac{1}{4}\right|
\end{aligned}$$

Since $|L| < 1$ then the series is absolutely convergent so we are allowed now to group (or rearrange) terms as follows

$$\begin{aligned}
S &= \left(1 + \frac{1}{4}\right) - \left(\frac{1}{16} + \frac{1}{64}\right) + \left(\frac{1}{256} + \frac{1}{1024}\right) - \left(\frac{1}{4096} + \frac{1}{16384}\right) + \cdots \\
&= \frac{5}{4} - \frac{5}{64} + \frac{5}{1024} - \frac{5}{16\,384} + \cdots \\
&= \frac{5}{4}\left(1 - \frac{1}{16} + \frac{1}{256} - \frac{1}{4096} + \cdots\right) \\
&= \frac{5}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n}} \\
&= \frac{5}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{16}\right)^n \qquad (1)
\end{aligned}$$

But $\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{16}\right)^n$ has the form $\sum_{n=0}^{\infty} (-1)^n r^n$ where $r = \frac{1}{16}$ and since $|r| < 1$ then by the binomial series

$$\begin{aligned}
\sum_{n=0}^{\infty} (-1)^n r^n &= 1 - r + r^2 - r^3 + \cdots \\
&= \frac{1}{1+r}
\end{aligned}$$

Therefore the sum in (1) becomes, using $r = \frac{1}{16}$

$$\begin{aligned}
S &= \frac{5}{4}\left(\frac{1}{1 + \frac{1}{16}}\right) \\
&= \frac{5}{4}\left(\frac{16}{17}\right)
\end{aligned}$$

Hence

$$\boxed{S = \frac{20}{17}}$$

Or

$$S \approx 1.176$$

3.2 Part b

Find the sum of $\frac{1}{1!}+\frac{8}{2!}+\frac{16}{3!}+\frac{64}{4!}+\cdots$

solution

The sum can be written as

$$\begin{aligned} S &= \sum_{n=1}^{\infty}\frac{n^3}{n!} \\ &= \sum_{n=1}^{\infty}\frac{nn^2}{n!} \end{aligned}$$

But $\frac{n}{n!}=\frac{n}{(n-1)!n}=\frac{1}{(n-1)!}$ and the above reduces to

$$S=\sum_{n=1}^{\infty}\frac{n^2}{(n-1)!}$$

Let $n-1=m$ or $n=m+1$. The above becomes

$$\begin{aligned} S &= \sum_{m=0}^{\infty}\frac{(m+1)^2}{m!} \\ &= \sum_{n=0}^{\infty}\frac{(n+1)^2}{n!} \\ &= \sum_{n=0}^{\infty}\frac{n^2+1+2n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{n^2}{n!}+\sum_{n=0}^{\infty}\frac{1}{n!}+2\sum_{n=0}^{\infty}\frac{n}{n!} \qquad (1) \end{aligned}$$

Considering the first term $\sum_{n=0}^{\infty}\frac{n^2}{n!}$ which can be written as

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{n^2}{n!} &= \sum_{n=1}^{\infty}\frac{n^2}{n!} \\ &= \sum_{n=1}^{\infty}\frac{n}{(n-1)!} \qquad (2) \end{aligned}$$

Again, letting Let $n-1=m$ then $\sum_{n=1}^{\infty}\frac{n}{(n-1)!}$ becomes $\sum_{m=0}^{\infty}\frac{m+1}{m!}$. Hence (2) becomes

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{n^2}{n!} &= \sum_{m=0}^{\infty}\frac{m+1}{m!} \\ &= \sum_{n=0}^{\infty}\frac{n+1}{n!} \\ &= \sum_{n=0}^{\infty}\frac{n}{n!}+\sum_{n=0}^{\infty}\frac{1}{n!} \\ &= \sum_{n=1}^{\infty}\frac{n}{n!}+\sum_{n=0}^{\infty}\frac{1}{n!} \end{aligned}$$

But $\sum_{n=1}^{\infty}\frac{n}{n!}=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$. Letting $n-1=m$, this becomes $\sum_{m=0}^{\infty}\frac{1}{m!}=\sum_{n=0}^{\infty}\frac{1}{n!}$and the above reduces to

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{n^2}{n!} &= \sum_{n=0}^{\infty}\frac{1}{n!}+\sum_{n=0}^{\infty}\frac{1}{n!} \\ &= e+e \\ &= 2e \qquad (3) \end{aligned}$$

The above takes care of the first term in (1). Therefore (1) can now be written as

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\frac{n^2}{n!}+\sum_{n=0}^{\infty}\frac{1}{n!}+2\sum_{n=0}^{\infty}\frac{n}{n!} \\ &= 2e+e+2\sum_{n=0}^{\infty}\frac{n}{n!} \\ &= 3e+2\sum_{n=0}^{\infty}\frac{n}{n!} \end{aligned}$$

But $\sum_{n=0}^{\infty}\frac{n}{n!}=\sum_{n=1}^{\infty}\frac{n}{n!}$ and $\sum_{n=1}^{\infty}\frac{n}{n!}$ was calculated above. It can be written as $\sum_{n=0}^{\infty}\frac{1}{n!}$. The above now becomes

$$\begin{aligned} S &= 3e+2\left(\sum_{n=0}^{\infty}\frac{1}{n!}\right) \\ &= 3e+2e \end{aligned}$$

Therefore

$$\boxed{S = 5e}$$

or

$$S \approx 13.5914$$