My Math 601 page
Advanced Engineering Mathematics I
Fall 2018
University of Wisconsin, Milwaukee

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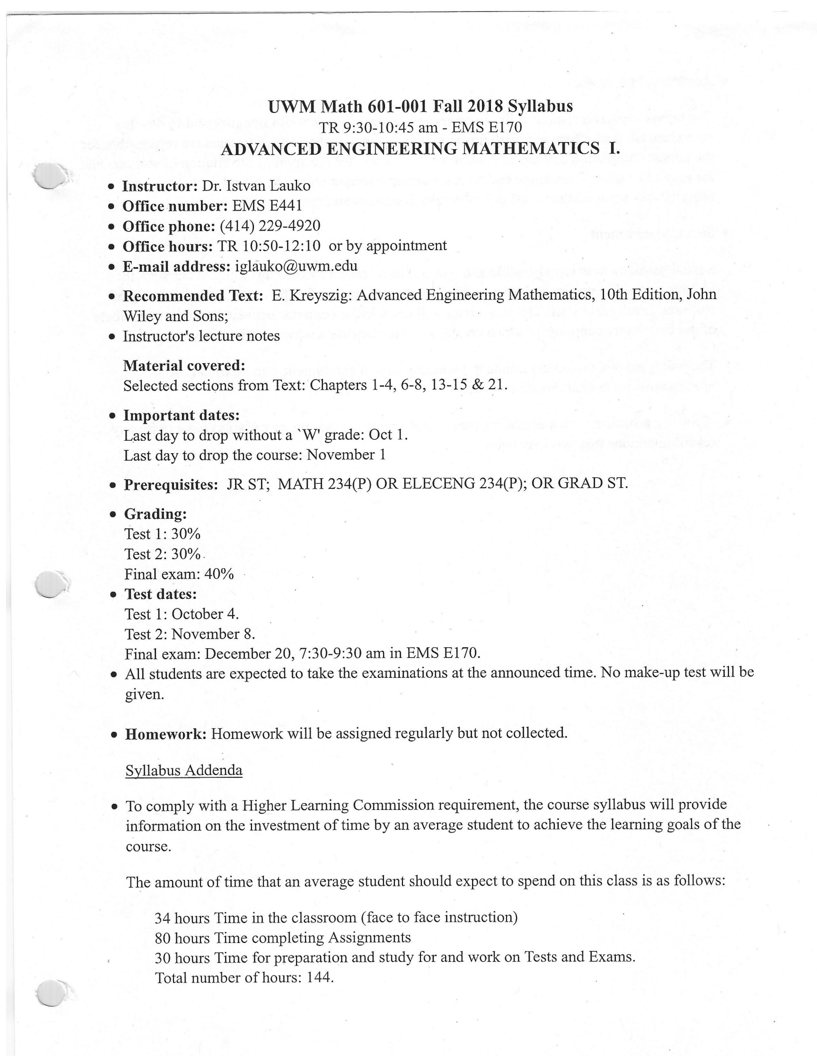
Fall 2018 Compiled on November 3, 2019 at 12:03 Noon [public]

Contents

Chapter 1

Introduction

1.0.1 syllabus



UWM Math 601-001 Fall 2018 Syllabus
TR 9:30-10:45 am - EMS E170
ADVANCED ENGINEERING MATHEMATICS I.

- **Instructor:** Dr. Istvan Lauko
- **Office number:** EMS E441
- **Office phone:** (414) 229-4920
- **Office hours:** TR 10:50-12:10 or by appointment
- **E-mail address:** iglauko@uwm.edu
- **Recommended Text:** E. Kreyszig: Advanced Engineering Mathematics, 10th Edition, John Wiley and Sons;
- Instructor's lecture notes

  Material covered:
  Selected sections from Text: Chapters 1-4, 6-8, 13-15 & 21.
- **Important dates:**
  Last day to drop without a `W' grade: Oct 1.
  Last day to drop the course: November 1
- **Prerequisites:** JR ST; MATH 234(P) OR ELECENG 234(P); OR GRAD ST.
- **Grading:**
  Test 1: 30%
  Test 2: 30%
  Final exam: 40%
- **Test dates:**
  Test 1: October 4.
  Test 2: November 8.
  Final exam: December 20, 7:30-9:30 am in EMS E170.
- All students are expected to take the examinations at the announced time. No make-up test will be given.
- **Homework:** Homework will be assigned regularly but not collected.

  Syllabus Addenda
- To comply with a Higher Learning Commission requirement, the course syllabus will provide information on the investment of time by an average student to achieve the learning goals of the course.

  The amount of time that an average student should expect to spend on this class is as follows:

  34 hours Time in the classroom (face to face instruction)
  80 hours Time completing Assignments
  30 hours Time for preparation and study for and work on Tests and Exams.
  Total number of hours: 144.

Chapter 2

HWs

2.1 HW 1

2.1.1 Problem set

PROBLEM SET 13.1

1. **(Powers of i)** Show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$ and $1/i = -i$, $1/i^2 = -1$, $1/i^3 = i, \cdots$.
2. **(Rotation)** Multiplication by i is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing z and iz and the angle of rotation for $z = 2 + 2i$, $z = -1 - 5i$, $z = 4 - 3i$.
3. **(Division)** Verify the calculation in (7).
4. **(Multiplication)** If the product of two complex numbers is zero, show that at least one factor must be zero.
5. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.
6. **(Laws for conjugates)** Verify (9) for $z_1 = 24 + 10i$, $z_2 = 4 + 6i$.

7–15 COMPLEX ARITHMETIC

Let $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$. Showing the details of your work, find (in the form $x + iy$):

7. $(5z_1 + 3z_2)^2$
8. $\bar{z}_1\bar{z}_2$
9. $\text{Re}\,(1/z_1^{\,2})$
10. $\text{Re}\,(z_2^{\,2})$, $(\text{Re}\,z_2)^2$
11. z_2/z_1
12. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$
13. $(4z_1 - z_2)^2$
14. $\bar{z}_1/z_1$, $z_1/\bar{z}_1$
15. $(z_1 + z_2)/(z_1 - z_2)$

16–19 Let $z = x + iy$. Find:

16. $\text{Im}\,z^3$, $(\text{Im}\,z)^3$
17. $\text{Re}\,(1/\bar{z})$
18. $\text{Im}\,[(1 + i)^8 z^2]$
19. $\text{Re}\,(1/\bar{z}^2)$

20. **(Laws of addition and multiplication)** Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1,\quad z_1z_2 = z_2z_1 \quad \textit{(Commutative laws)}$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3),$$

$$(z_1z_2)z_3 = z_1(z_2z_3) \quad \textit{(Associative laws)}$$

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \quad \textit{(Distributive law)}$$

$$0 + z = z + 0 = z,$$

$$z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

Problem Set 13.2

7. $\frac{-6 + 5i}{3i}$

8. $\frac{2 + 3i}{5 + 4i}$

9–15 PRINCIPAL ARGUMENT

Determine the principal value of the argument.

9. $-1 - i$
10. $-20 + i, -20 - i$
11. $4 \pm 3i$
12. $-\pi^2$
13. $7 \pm 7i$
14. $(1 + i)^{12}$
15. $(9 + 9i)^3$

16–20 CONVERSION TO $x + iy$

Represent in the form $x + iy$ and graph it in the complex plane.

16. $\cos \frac{1}{2}\pi + i \sin (\pm \frac{1}{2}\pi)$
17. $3(\cos 0.2 + i \sin 0.2)$
18. $4(\cos \frac{1}{3}\pi \pm i \sin \frac{1}{3}\pi)$
19. $\cos (-1) + i \sin (-1)$
20. $12(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi)$

21–25 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt{-i}$
22. $\sqrt[8]{1}$
23. $\sqrt[4]{-1}$
24. $\sqrt[3]{3 + 4i}$
25. $\sqrt[5]{-1}$

26. **TEAM PROJECT. Square Root.** (a) Show that $w = \sqrt{z}$ has the values

$$w_1 = \sqrt{r}\left[\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}\right],$$

$$(18) \quad w_2 = \sqrt{r}\left[\cos \left(\frac{\theta}{2} + \pi\right) + i \sin \left(\frac{\theta}{2} + \pi\right)\right] = -w_1.$$

(b) Obtain from (18) the often more practical formula

$$(19) \quad \sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + (\text{sign } y) i \sqrt{\tfrac{1}{2}(|z| - x)}\right]$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $4i$, $16 - 30i$, and $9 + 8\sqrt{7}i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

27–30 EQUATIONS

Solve and graph all solutions, showing the details:

27. $z^2 - (8 - 5i)z + 40 - 20i = 0$ (Use (19).)
28. $z^4 + (5 - 14i)z^2 - (24 + 10i) = 0$
29. $8z^2 - (36 - 6i)z + 42 - 11i = 0$
30. $z^4 + 16 = 0$. Then use the solutions to factor $z^4 + 16$ into quadratic factors with *real* coefficients.

31. **CAS PROJECT. Roots of Unity and Their Graphs.** Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

32–35 INEQUALITIES AND AN EQUATION

Verify or prove as indicated.

32. **(Re and Im)** Prove $|\text{Re } z| \leqq |z|$, $|\text{Im } z| \leqq |z|$.
33. **(Parallelogram equality)** Prove
$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$
Explain the name.
34. **(Triangle inequality)** Verify (6) for $z_1 = 4 + 7i$, $z_2 = 5 + 2i$.
35. **(Triangle inequality)** Prove (6).

PROBLEM SET 13.3

1–10 CURVES AND REGIONS OF PRACTICAL INTEREST

Find and sketch or graph the sets in the complex plane given by

1. $|z - 3 - 2i| = \frac{4}{3}$
2. $1 \leqq |z - 1 + 4i| \leqq 5$
3. $0 < |z - 1| < 1$
4. $-\pi < \text{Re } z < \pi$
5. $\text{Im } z^2 = 2$
6. $\text{Re } z > -1$
7. $|z + 1| = |z - 1|$
8. $|\text{Arg } z| \leqq \frac{1}{4}\pi$
9. $\text{Re } z \leqq \text{Im } z$
10. $\text{Re } (1/z) < 1$

PROBLEM SET 13.5

1. Using the Cauchy–Riemann equations, show that e^z is entire.

2–8 **Values of e^z.** Compute e^z in the form $u + iv$ and $|e^z|$, where z equals:

2. $3 + \pi i$
3. $1 + 2i$
4. $\sqrt{2} - \frac{1}{2}\pi i$
5. $7\pi i/2$
6. $(1 + i)\pi$
7. $0.8 - 5i$
8. $9\pi i/2$

9–12 **Real and Imaginary Parts.** Find Re and Im of:

9. e^{-2z}
10. e^{z^3}
11. e^{z^2}
12. $e^{1/z}$

13–17 **Polar Form.** Write in polar form:

13. $\sqrt{i}$
14. $1 + i$
15. $\sqrt[n]{z}$
16. $3 + 4i$
17. -9

18–21 **Equations.** Find all solutions and graph some of them in the complex plane.

18. $e^{3z} = 4$
19. $e^z = -2$
20. $e^z = 0$
21. $e^z = 4 - 3i$

22. **TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity.** Show that e^z is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos ky + i \sin ky)$? (Use the Cauchy–Riemann equations.)

    (b) Special values. Find all z such that (i) e^z is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

    (c) Harmonic function. Show that $u = e^{xy} \cos (x^2/2 - y^2/2)$ is harmonic and find a conjugate.

    (d) Uniqueness. It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this using the Cauchy–Riemann equations.

PROBLEM SET 13.6

1. Prove that $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are entire functions.
2. Verify by differentiation that Re $\cos z$ and Im $\sin z$ are harmonic.

3–6 **FORMULAS FOR HYPERBOLIC FUNCTIONS**

Show that

3. $$\cosh z = \cosh x \cos y + i \sinh x \sin y$$
$$\sinh z = \sinh x \cos y + i \cosh x \sin y.$$

4. $$\cosh (z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$
$$\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$$

5. $\cosh^2 z - \sinh^2 z = 1$
6. $\cosh^2 z + \sinh^2 z = \cosh 2z$

7–15 **Function Values.** Compute (in the form $u + iv$)

7. $\cos (1 + i)$
8. $\sin (1 + i)$
9. $\sin 5i$, $\cos 5i$
10. $\cos 3\pi i$
11. $\cosh (-2 + 3i)$, $\cos (-3 - 2i)$
12. $-i \sinh (-\pi + 2i)$, $\sin (2 + \pi i)$
13. $\cosh (2n + 1)\pi i$, $n = 1, 2, \cdots$
14. $\sinh (4 - 3i)$
15. $\cosh (4 - 6\pi i)$

16. **(Real and imaginary parts)** Show that

$$\text{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$

$$\text{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

17–21 **Equations.** Find all solutions of the following equations.

17. $\cosh z = 0$
18. $\sin z = 100$
19. $\cos z = 2i$
20. $\cosh z = -1$
21. $\sinh z = 0$

22. Find all z for which (a) $\cos z$, (b) $\sin z$ has real values.

23–25 **Equations and Inequalities.** Using the definitions, prove:

23. $\cos z$ is even, $\cos (-z) = \cos z$, and $\sin z$ is odd, $\sin (-z) = -\sin z$.
24. $|\sinh y| \leqq |\cos z| \leqq \cosh y$, $|\sinh y| \leqq |\sin z| \leqq \cosh y$. Conclude that the complex cosine and sine are not bounded in the whole complex plane.
25. $\sin z_1 \cos z_2 = \frac{1}{2}[\sin (z_1 + z_2) + \sin (z_1 - z_2)]$

PROBLEM SET 13.7

1–9 **Principal Value Ln z.** Find Ln z when z equals:

1. -10
2. $2 + 2i$
3. $2 - 2i$
4. $-5 \pm 0.1i$
5. $-3 - 4i$
6. -100
7. $0.6 + 0.8i$
8. $-ei$
9. $1 - i$

10–16 **All Values of ln z.** Find all values and graph some of them in the complex plane.

10. $\ln 1$
11. $\ln (-1)$
12. $\ln e$
13. $\ln (-6)$
14. $\ln (4 + 3i)$
15. $\ln (-e^{-i})$
16. $\ln (e^{3i})$

17. Show that the set of values of $\ln (i^2)$ differs from the set of values of $2 \ln i$.

18–21 **Equations.** Solve for z:

18. $\ln z = (2 - \frac{1}{2}i)\pi$
19. $\ln z = 0.3 + 0.7i$
20. $\ln z = e - \pi i$
21. $\ln z = 2 + \frac{1}{4}\pi i$

22–28 **General Powers.** Showing the details of your work, find the principal value of:

22. i^{2i}, $(2i)^i$ **23.** 4^{3+i}
24. $(1-i)^{1+i}$ **25.** $(1+i)^{1-i}$
26. $(-1)^{1-2i}$ **27.** $i^{1/2}$
28. $(3-4i)^{1/3}$

29. How can you find the answer to Prob. 24 from the answer to Prob. 25?

30. TEAM PROJECT. Inverse Trigonometric and Hyperbolic Functions. By definition, the **inverse sine** $w = \arcsin z$ is the relation such that $\sin w = z$. The **inverse cosine** $w = \arccos z$ is the relation such that $\cos w = z$. The **inverse tangent, inverse cotangent, inverse hyperbolic sine,** etc., are defined and denoted in a similar fashion. (Note that all these relations are *multivalued.*) Using $\sin w = (e^{iw} - e^{-iw})/(2i)$ and similar representations of $\cos w$, etc., show that

(a) $\arccos z = -i \ln (z + \sqrt{z^2-1})$

(b) $\arcsin z = -i \ln (iz + \sqrt{1-z^2})$

(c) $\operatorname{arccosh} z = \ln (z + \sqrt{z^2-1})$

(d) $\operatorname{arcsinh} z = \ln (z + \sqrt{z^2+1})$

(e) $\arctan z = \dfrac{i}{2} \ln \dfrac{i+z}{i-z}$

(f) $\operatorname{arctanh} z = \dfrac{1}{2} \ln \dfrac{1+z}{1-z}$

(g) Show that $w = \arcsin z$ is infinitely many-valued, and if w_1 is one of these values, the others are of the form $w_1 \pm 2n\pi$ and $\pi - w_1 \pm 2n\pi$, $n = 0, 1, \cdots$. (The *principal value of* $w = u + iv = \arcsin z$ is defined to be the value for which $-\pi/2 \leqq u \leqq \pi/2$ if $v \geqq 0$ and $-\pi/2 < u < \pi/2$ if $v < 0$.)

2.1.2 key solution

Key Solution.

Math 601

13.1 1 By definition

$i^1 = i$
$i^2 = i \cdot i = -1$
$i^3 = i^2 \cdot i = -i$
$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$
$i^5 = i^4 \cdot i = i$
$i^6 = i^4 \cdot i^2 = -1$
$i^7 = i^4 \cdot i^3 = -i$
$i^8 = i^4 \cdot i^4 = 1$
$\vdots$

the sequence $\{i, -1, -i, 1\}$ repeats.

$\frac{1}{i} = -i$ since $1 = -i \cdot i$
$\frac{1}{i^2} = \frac{1}{-1} = -1$
$1/i^3 = \frac{1}{i^2} \cdot \frac{1}{i} = -1 \cdot (-i) = i$
$1/i^4 = \frac{1}{1} = 1$ $\{-i, -1, i, 1\}$ repeats
$\frac{1}{i^5} = \frac{1}{i^4} \cdot \frac{1}{i} = -i$
$\frac{1}{i^6} = \frac{1}{i^4} \cdot \frac{1}{i^2} = -1$
$\frac{1}{i^7} = \frac{1}{i^4} \cdot \frac{1}{i^3} = i$
$\frac{1}{i^8} = \frac{1}{i^4} \cdot \frac{1}{i^4} = 1$

7 $z_1 = 2 + 3i$, $z_2 = 4 - 5i$

$(5z_1 + 3z_2)^2 = (5(2+3i) + 3(4-5i))^2 =$
$= (10 + \cancel{15i} + 12 - \cancel{15i})^2 = 22^2 = 484$

10 $z_2 = 4 - 5i$ $(\operatorname{Re} z_2)^2 = 4^2 = 16$

$z_2^2 = (4-5i)(4-5i)$ $\Rightarrow$ $\operatorname{Re}(z_2^2) = -9$
$= 16 - 25 - 2 \cdot 4 \cdot 5i$
$= -9 - 40i$

12 $\dfrac{\bar z_1}{\bar z_2} = \dfrac{2-3i}{4+5i} = \dfrac{\bar z_1 z_2}{\bar z_2 z_2} = \dfrac{\bar z_1 z_2}{|z_2|^2} = \dfrac{-7-22i}{41} = -\dfrac{7}{41} - \dfrac{22}{41} i$

$|z_2|^2 = 4^2 + 5^2 = 41$
$\bar z_1 z_2 = (2-3i)(4-5i) = 8 - 15 - (10+12)i = -7 - 22i$

We know that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar z_1}{\bar z_2} = -\dfrac{7}{41} - \dfrac{22}{41} i$

$\boxed{13.2}$ $\boxed{8}$ polar form of $z = \frac{2+3i}{5+4i} = re^{i\theta} = r(\cos\theta + i\sin\theta)$

with $r = \sqrt{z\cdot\bar{z}}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$(z = x+iy)$

$$z = \frac{2+3i}{5+4i} = \frac{(2+3i)(5-4i)}{(5+4i)(5-4i)} = \frac{10+12+(15-8)i}{25+16} = \frac{22}{41} + \frac{7}{41}i = x + y\cdot i$$

thus $r = \sqrt{\left(\frac{22}{41}\right)^2 + \left(\frac{7}{41}\right)^2}$

and $\theta = \tan^{-1}\frac{7/41}{22/41} = \tan^{-1}\left(\frac{7}{22}\right)$

$\boxed{11}$

$z = x+iy$ $\qquad z = 4+3i \qquad -\pi < \operatorname{Arg} z \le \pi$ and $\tan \operatorname{Arg} z = \frac{y}{x} = \frac{3}{4}$

$\bar{z} = x-iy \qquad \bar{z} = 4-3i \qquad \operatorname{Arg} z = \tan^{-1}\frac{3}{4} \approx 0.64$

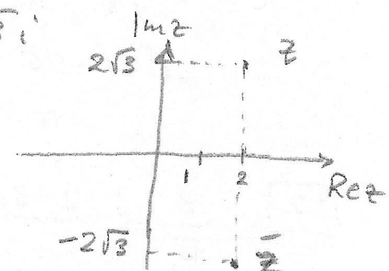
$\tan^{-1}\frac{-y}{x} = \operatorname{Arg}\bar{z} = -\operatorname{Arg} z = \tan^{-1}\frac{y}{x}$

(and $\tan^{-1}$ is odd function)

$\boxed{18}$ $z = 4\cdot\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{4}{2} + i\,\frac{4\sqrt{3}}{2} = 2 + 2\sqrt{3}\,i$

$\bar{z} = 4\cdot\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 2 - 2\sqrt{3}\,i$

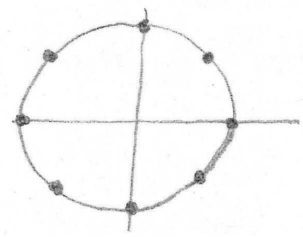
$\cos\frac{\pi}{3} = \frac{1}{2}, \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$



$\boxed{22}$ $\sqrt[8]{1} = \sqrt[8]{r}\left(\cos\frac{\theta+2k\pi}{8} + i\sin\frac{\theta+2k\pi}{8}\right) =$ $\quad k = 0, 1, 2, \dots, 7$

$r = |1| = 1$

$\theta = \operatorname{Arg} 1 = 0 \qquad = \cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4}$

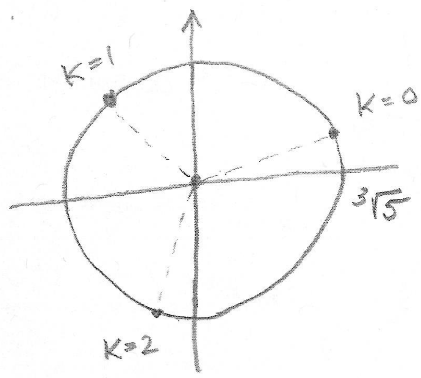


$\boxed{13.2}$ $\boxed{24}$ $\sqrt[3]{3+4i} = r^{1/3}\left(\cos\frac{\theta+2k\pi}{3} + i\sin\frac{\theta+2k\pi}{3}\right)$

$r = |3+4i| = \sqrt{9+16} = 5$ $\qquad = \sqrt[3]{5}\left[\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}k\right) + i\sin\left(\frac{\theta}{3}+\frac{2\pi}{3}k\right)\right]$

$\theta = \tan^{-1}\frac{4}{3} \approx 0.9273$

$k = 0, 1, 2$



the three roots.

Let $z_1 = x_1 + y_1 i$

$z_2 = x_2 + y_2 i$

$|z_1| = \sqrt{x_1^2+y_1^2}$

$|z_2| = \sqrt{x_2^2+y_2^2}$

$z_1+z_2 = x_1+x_2 + i(y_1+y_2)$

$|z_1+z_2|^2 =$

$= (x_1+x_2)^2 + (y_1+y_2)^2$

$= x_1^2+2x_1x_2+x_2^2 + y_1^2+2y_1y_2+y_2^2$

$\boxed{35}$ Show that $|z_1+z_2| \le |z_1| + |z_2|$

or equivalently

$$|z_1+z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

thus we need to show:

$$\cancel{x_1^2} + 2x_1x_2 + \cancel{x_2^2} + \cancel{y_1^2} + 2y_1y_2 + \cancel{y_2^2} \le$$
$$\le \cancel{x_1^2} + \cancel{y_1^2} + \cancel{x_2^2} + \cancel{y_2^2} + 2\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}$$

or $\quad x_1x_2 + y_1y_2 \le \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$

(Note: RHS non-negative) to show this, square each sides

$$\cancel{x_1^2x_2^2} + 2x_1x_2y_1y_2 + \cancel{y_1^2y_2^2} \overset{?}{\le} (x_1^2+y_1^2)(x_2^2+y_2^2)$$
$$= \cancel{x_1^2x_2^2} + x_1^2y_2^2 + x_2^2y_1^2 + \cancel{y_1^2y_2^2}$$

or

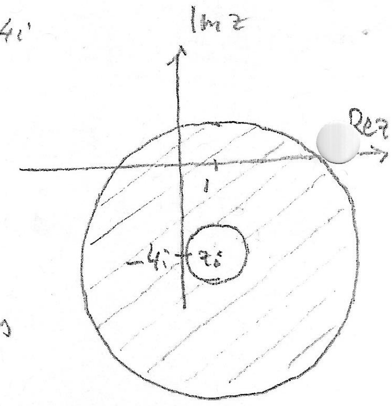
$$0 \le x_1^2y_2^2 + x_2^2y_1^2 - 2x_1x_2y_1y_2 =$$
$$= (x_1y_2 - x_2y_1)^2$$ which is true for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

$\boxed{13.3}$ $\boxed{2}$ $1 \le |z-(1-4i)| \le 5$ take $z_0 = 1-4i$

$|z-z_0| = 1$: circle of radius 1 around z_0

$|z-z_0| = 5$ — " — radius 5

annulus



with $xy \Rightarrow y = \frac{1}{x}$ easier!

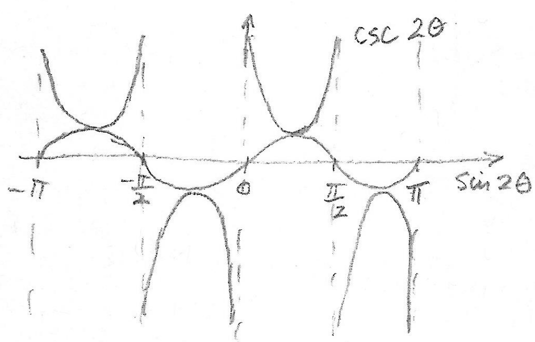
$\boxed{5}$ $\operatorname{Im} z^2 = 2$ $z = r \cdot e^{i\theta}$ $z^2 = r^2 \cdot e^{i2\theta} =$

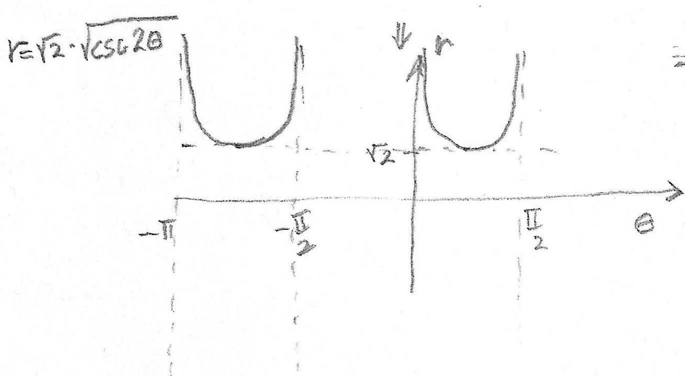
$r^2 \cos 2\theta + i r^2 \sin 2\theta$

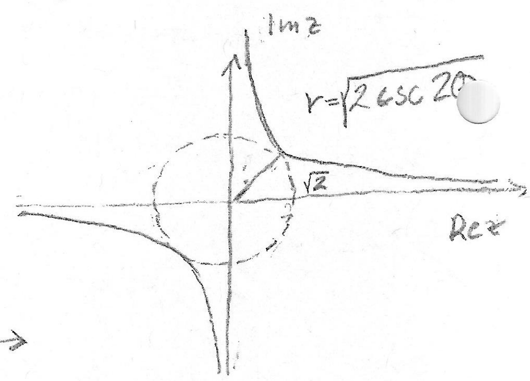
$\operatorname{Im} z^2 = r^2 \sin 2\theta$

$= 2$

$\Rightarrow r^2 = 2 \csc 2\theta$



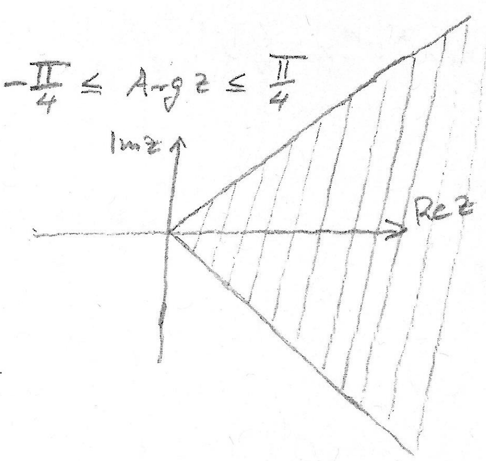




$\boxed{8}$ $|\operatorname{Arg} z| \le \frac{1}{4}\pi \Rightarrow -\frac{\pi}{4} \le \operatorname{Arg} z \le \frac{\pi}{4}$

$z = r \cdot e^{\operatorname{Arg} z}$

with $-\pi < \operatorname{Arg} z \le \pi$



$\boxed{13.5}$ $\boxed{2}$

$$e^{3+\pi i} = e^3 e^{\pi i} = e^3(\underbrace{\cos\pi}_{-1} + i\underbrace{\sin\pi}_{0}) = -e^3$$

$\boxed{12}$

$$z = r\cdot e^{i\theta} \Rightarrow z^{-1} = r^{-1}\cdot e^{-i\theta} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) =$$

$$= \frac{1}{r}(\cos\theta - i\sin\theta)$$

$r = |z|$

$\theta = \arg z$

if $z = x + iy$

$|z| = \sqrt{x^2+y^2}$

$\arg z = \tan^{-1}\frac{y}{x}$

$$e^{\frac{1}{z}} = e^{\frac{1}{r}\cos\theta - \frac{1}{r}\sin\theta\, i} =$$

$$= e^{\frac{1}{r}\cos\theta}\left(\cos\left(-\frac{1}{r}\sin\theta\right) + i\sin\left(-\frac{1}{r}\sin\theta\right)\right) =$$

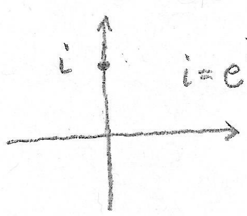
$$= e^{\frac{1}{r}\cos\theta}\cdot\cos\left(\frac{1}{r}\sin\theta\right)$$

$$- i\, e^{\frac{1}{r}\cos\theta}\cos\left(\frac{1}{r}\sin\theta\right)$$

Thus $\operatorname{Re} e^{\frac{1}{z}} = e^{1/r\cos\theta}\cdot\cos\left(\frac{1}{r}\sin\theta\right)$

$\operatorname{Im} e^{1/z} = e^{\frac{1}{r}\cos\theta}\cdot\cos\left(\frac{1}{r}\sin\theta\right)$

$\boxed{13}$ $\sqrt{i}$:



$i = e^{\frac{\pi}{2}i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$|i| = 1$

$\operatorname{Arg} i = \frac{\pi}{2}$

thus $\sqrt{i} = \sqrt{1}\cdot\left(\cos\frac{\pi/2 + k\cdot 2\pi}{2} + i\sin\frac{\pi/2 + k\cdot 2\pi}{2}\right)$

with $k=0, k=1$

1) • either $(k=0)$ $\sqrt{i} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

2) • or $k=1$ $\sqrt{i} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

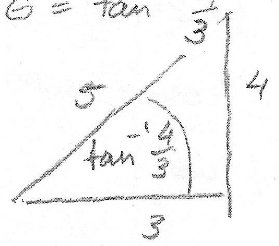
i.e $\sqrt{i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

or $\sqrt{i} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$\boxed{13.5}$ $\boxed{16}$ $3+4i = r\cdot e^{i\theta} = 5\cdot\left(\cos\left(\tan^{-1}\frac{4}{3}\right) + i\sin\left(\tan^{-1}\frac{4}{3}\right)\right)$

$r = \sqrt{3^2+4^2} = 5$ $\qquad /= 5\left(\frac{3}{5} + i\frac{4}{5}\right)/$

$\theta = \tan^{-1}\frac{4}{3}$



$\Rightarrow$ $\cos\left(\tan^{-1}\frac{4}{3}\right) = \frac{3}{5}$

$\sin\left(\tan^{-1}\frac{4}{3}\right) = \frac{4}{5}$

$\boxed{20}$ $e^z = e^{|z|}\cdot e^{\operatorname{Arg} z\, i} \Rightarrow |e^z| = |e^r\cdot e^{i\theta}| =$

$|e^r|\cdot|\cos\theta + i\sin\theta|$

$r = |z|$ $\qquad \underset{>0}{|e^r|} \qquad \underset{=1}{}$

$\theta = \operatorname{Arg} z$

thus $|e^{|z|}| = 0$

so $e^{|z|} = 0$ has no solution.

* typo $\qquad (4-3i)$

$\boxed{21}$ $e^z = 3-4i \Rightarrow \operatorname{Ln} e^z = z = \operatorname{Ln}(3-4i) =$

$\quad x+yi$

$= \ln|3-4i| + i\operatorname{Arg} z =$

$|3-4i| = 5$

$= \ln 5 - i\tan^{-1}\frac{4}{3}$

$\operatorname{Arg} z = \tan^{-1} -\frac{4}{3}$

$\boxed{13.6}$ $\boxed{8}$ Compute as $u+iv$: $\sin(1+i)$

$z = 1+i$

$iz = -1+i$

$-iz = 1-i$

$|iz| = |-iz| = |z| = \sqrt{2}$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$e^{iz} = e^{-1+i} = \frac{1}{e}(\cos 1 + i\sin 1)$

$e^{-iz} = e^{1-i} = e(\cos 1 - i\sin 1)$

$$\sin(1+i) = \frac{e^{iz} - e^{-iz}}{2i} =$$

$$= \frac{-i}{2}\left(\cos 1 \cdot \left(\frac{1}{e} - e\right) + i\sin 1 \cdot \left(\frac{1}{e} + e\right)\right)$$

$$= \underbrace{\frac{\sin 1}{2}\left(\frac{1}{e} + e\right)}_{u} - i\underbrace{\frac{\cos 1}{2}\left(\frac{1}{e} - e\right)}_{v} = u + iv$$

$\boxed{23}$ $$\cos(-z) = \frac{1}{2}\left(e^{i(-z)} + e^{-i(-z)}\right) = \frac{1}{2}\left(e^{-iz} + e^{iz}\right) = \cos z$$

$$\sin(-z) = \frac{1}{2i}\left(e^{i(-z)} - e^{-i(-z)}\right) = \frac{1}{2i}\left(e^{-iz} - e^{iz}\right) =$$

$$= -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin z$$

thus $\cos(z)$ is even

$\sin(z)$ is odd

$\boxed{13.7}$ $\boxed{14}$ See $\boxed{13.5}$ $\boxed{21}$

$\boxed{19}$ solve $\ln z = 0.3 + 0.7i \Rightarrow z = e^{\ln z} = e^{\frac{3}{10} + \frac{7}{10}i} =$

$$= e^{3/10}\left(\cos\frac{7}{10} + i\sin\frac{7}{10}\right) =$$

$$e^{0.3}\cos 0.7 + i\, e^{3/10}\sin 0.7$$

$\boxed{13.7}$ $\boxed{25}$ and $\boxed{29}$ $\quad (1+i)^{(1-i)} = e^{\ln(1+i)^{(1-i)}} = e^{(1-i)\ln(1+i)}$

$\ln(1+i) = \ln\sqrt{2} + i\frac{\pi}{4}$ thus $(1-i)(\ln\sqrt{2} + i\frac{\pi}{4}) =$

$|1+i| = \sqrt{2}$ $\quad \ln\sqrt{2} + \frac{\pi}{4} + i\left(\frac{\pi}{4} - \ln\sqrt{2}\right)$

$\text{Arg}(1+i) = \tan^{-1} 1 = \frac{\pi}{4}$

So

$$(1+i)^{1-i} = e^{\ln\sqrt{2}+\frac{\pi}{4}} \cdot \cos\left(\frac{\pi}{4} - \ln\sqrt{2}\right) + i\, e^{\ln\sqrt{2}+\frac{\pi}{4}} \cdot \sin\left(\frac{\pi}{4} - \ln\sqrt{2}\right)$$

consider

$z^{\bar{z}}$ and $\bar{z}^{z}$

① ②

$$\overline{z^{\bar{z}}} = \overline{e^{\bar{z}\cdot\ln z}} = e^{\bar{\bar{z}}\cdot\overline{\ln z}} = e^{z} e^{\ln\bar{z}} = e^{z\cdot\ln\bar{z}} = \bar{z}^{z}$$

note: $z = x+iy$

$\overline{e^{z}} = \overline{e^{x}(\cos y + i\sin y)} = e^{x}(\cos y - i\sin y) = e^{\bar{z}}$

also

$\overline{\text{Ln}\, z} = \overline{\ln|z| + i\,\text{Arg}\, z}$
$= \ln|z| - i\,\text{Arg}\, z$
$= \ln|z| + i\,\text{Arg}\, \bar{z}$
$= \text{Ln}\, \bar{z}$

thus (as $\overline{1+i} = 1-i$) taking $z = 1+i$ we have that

$$\overline{(1+i)^{1-i}} = (1-i)^{1+i}$$

2.2 HW 2

2.2.1 Problem set

PROBLEM SET 15.1

1–10 SEQUENCES

Are the following sequences $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find their limit points. (Show the details of your work.)

1. $z_n = (-1)^n + i/2^n$
2. $z_n = e^{-n\pi i/4}$
3. $z_n = (-1)^n/(n + i)$
4. $z_n = (1 + i)^n$
5. $z_n = \text{Ln}\,((2 + i)^n)$
6. $z_n = (3 + 4i)^n/n!$
7. $z_n = \sin(n\pi/4) + i^n$
8. $z_n = [(1 + 3i)/\sqrt{10}]^n$
9. $z_n = (0.9 + 0.1i)^{2n}$
10. $z_n = (5 + 5i)^{-n}$

11. Illustrate Theorem 1 by an example of your own.

12. (Uniqueness of limit) Show that if a sequence converges, its limit is unique.

13. (Addition) If $z_1, z_2, \cdots$ converges with the limit l and $z_1^*, z_2^*, \cdots$ converges with the limit l^*, show that $z_1 + z_1^*, z_2 + z_2^*, \cdots$ converges with the limit $l + l^*$.

14. (Multiplication) Show that under the assumptions of Prob. 13 the sequence $z_1z_1^*, z_2z_2^*, \cdots$ converges with the limit ll^*.

15. (Boundedness) Show that a complex sequence is bounded if and only if the two corresponding sequences of the real parts and of the imaginary parts are bounded.

16–24 SERIES

Are the following series convergent or divergent? (Give a reason.)

16. $\sum_{n=0}^{\infty} \frac{(10 - 15i)^n}{n!}$
17. $\sum_{n=0}^{\infty} \frac{(-1)^n(1 + 2i)^{2n+1}}{(2n + 1)!}$

18. $\sum_{n=0}^{\infty} \frac{i^n}{n^2 - 2i}$
19. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

20. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$
21. $\sum_{n=1}^{\infty} \frac{i^n}{n}$

22. $\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}(1 + i)^n$
23. $\sum_{n=0}^{\infty} \frac{n - i}{3n + 2i}$

24. $\sum_{n=1}^{\infty} n^2\left(\frac{i}{3}\right)^n$

25. What is the difference between (7) and just stating $|z_{n+1}/z_n| < 1$?

26. Illustrate Theorem 2 by an example of your choice.

27. For what n do we obtain the term of greatest absolute value of the series in Example 4? About how big is it? First guess, then calculate it by the Stirling formula in Sec. 24.4.

28. Give another example showing that Theorem 7 is more general than Theorem 8.

29. CAS PROJECT. Sequences and Series. (a) Write a program for graphing complex **sequences.** Apply it to sequences of your choice that have interesting "geometrical" properties (e.g., lying on an ellipse, spiraling toward its limit, etc.).

(b) Write a program for computing and graphing numeric values of the first n partial sums of a **series** of complex numbers. Use the program to experiment with the rapidity of convergence of series of your choice.

30. TEAM PROJECT. Series. (a) Absolute convergence. Show that if a series converges absolutely, it is convergent.

(b) Write a short report on the basic concepts and properties of series of numbers, explaining in each case whether or not they carry over from real series (discussed in calculus) to complex series, with reasons given.

PROBLEM SET 15.2

1. (Powers missing) Show that if $\Sigma a_n z^n$ has radius of convergence R (assumed finite), then $\Sigma a_n z^{2n}$ has radius of convergence $\sqrt{R}$. Give examples.

2. (Convergence behavior) Illustrate the facts shown by Examples 1–3 by further examples of your own.

3–18 RADIUS OF CONVERGENCE

Find the center and the radius of convergence of the following power series. (Show the details.)

3. $\sum_{n=1}^{\infty} \frac{(z + i)^n}{n^2}$
4. $\sum_{n=0}^{\infty} \frac{n^n}{n!}(z + 2i)^n$

5. $\sum_{n=0}^{\infty} \frac{n!}{n^n}(z + 1)^n$
6. $\sum_{n=0}^{\infty} \frac{2^{100n}}{n!} z^n$

7. $\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n z^n$
8. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} z^{2n}$

9. $\sum_{n=0}^{\infty} (n - i)^n z^n$
10. $\sum_{n=0}^{\infty} \frac{(2z)^{2n}}{(2n)!}$

11. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$
12. $\sum_{n=0}^{\infty} \frac{4^n}{(1 + i)^n}(z - 5)^n$

678 CHAP. 15 Power Series, Taylor Serie:

13. $\sum_{n=2}^{\infty} n(n-1)(z-3+2i)^n$

14. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$

15. $\sum_{n=0}^{\infty} 2^n(z-i)^{4n}$

16. $\sum_{n=0}^{\infty} \left(\frac{2+3i}{5-i}\right)^n (z-\pi)^n$

17. $\sum_{n=0}^{\infty} \frac{n^4}{2^n} z^{2n}$

18. $\sum_{n=0}^{\infty} \frac{(4n)!}{2^n(n!)^4} (z+\pi i)^n$

2.2.2 Problem 2, section 15.1

Is sequence $z_n = e^{-\frac{n\pi i}{4}}$ bounded? convergent? Find their limit points.

Solution

Sequence is bounded, since each element has modulus 1. It does not converge, since sequence repeats. $2\pi = \frac{n\pi}{4}$, hence $n = 8$. So only 8 elements are unique. Each of these is limit point. These are roots of $\sqrt[8]{1}$.

2.2.3 Problem 6, section 15.1

Is sequence $z_n = \frac{(3+4i)^n}{n!}$ bounded? convergent? Find their limit points.

Solution

$$z_n = \frac{\left(re^{i\theta_0}\right)^n}{n!}$$

But $r = 5$ and $\theta_0 = \arctan\left(\frac{4}{3}\right)$. The above becomes

$$\begin{aligned} z_n &= \frac{5^n e^{in\theta_0}}{n!} \\ &= \frac{5^n}{n!} e^{in\theta_0} \end{aligned}$$

Since modulus of $e^{in\theta_0} = 1$, then we just need to look at $\frac{5^n}{n!}$ to see if it is bounded or not. $\lim_{n\to\infty} \frac{5^n}{n!} = 0$. So it is bounded. Since n^{th} term goes to zero as $n \to \infty$ it converges. The terms are $\frac{5^n}{n!}(\cos n\theta_0 + i \sin n\theta_0)$. It converges to zero, since $\lim_{n\to\infty} \frac{5^n}{n!} = 0$.

2.2.4 Problem 13, section 15.1

If $z_1, z_2, \cdots$ converges to L, and $\bar{z}_1, \bar{z}_2, \cdots$ converges to $\bar{L}$, show that $z_1 + \bar{z}_1, z_2 + \bar{z}_2, \cdots$ converges to $L + \bar{L}$

Solution

This problem seems to be based on the idea that if sequence is convergent to L, then for any ε no matter how small we can find an n, such that $|z_n - L| < \varepsilon$. So let us pick

$$|z_n - L| < \frac{1}{2}\varepsilon$$
$$\left|\bar{z}_n - \bar{L}\right| < \frac{1}{2}\varepsilon$$

Where in the above, we did the same for the other sequence. Now by triangle inequality $|A+B| \leq |A| + |B|$, where now we treat A as $(z_n - L)$ and B as $\left(\bar{z}_n - \bar{L}\right)$, we have

$$\left|(z_n - L) + \left(\bar{z}_n - \bar{L}\right)\right| \leq |z_n - L| + \left|\bar{z}_n - \bar{L}\right|$$
$$\left|(z_n + \bar{z}_n) - \left(L + \bar{L}\right)\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$$

The above is $\left|(z_n + \bar{z}_n) - \left(L + \bar{L}\right)\right| < \varepsilon$. But this is the definition of a limit. It says that $(z_n + \bar{z}_n)$ has limit $L + \bar{L}$, which is what we are asked to show.

2.2.5 Problem 18, section 15.1

Are the following series convergent or divergent? Give a reason. $\sum_{n=0}^{\infty} \frac{i^n}{n^2-2i}$

Solution

The numerator has modulus 1. So we just need to consider $\sum_{n=0}^{\infty} \frac{1}{|n^2-2i|}$. Since $\frac{1}{n^2}$ converges and since $\left|n^2 - 2i\right| > n^2$ (vectors, Argand diagram), then $\frac{1}{|n^2-2i|} < \frac{1}{n^2}$, therefore it converges. We could also use the ratio test, but this is simpler.

2.2.6 Problem 19, section 15.1

Are the following series convergent or divergent? Give a reason. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Solution

Since terms are $\frac{1}{n^\alpha}$ where $|\alpha| < 1$, since $\alpha = \frac{1}{2}$ here. Then we know it is divergent. It series becomes convergent for $\alpha > 1$. To show this, we can try the ratio test. But this gives the limit of 1, so ratio test is inconclusive. Using the integral test is best here. (notice that only upper limit is needed in this test, no need to use lower limit). We can use the integral test because the terms $\frac{1}{\sqrt{n}}$ are monotonically decreasing.

$$\begin{aligned}\lim_{N\to\infty} \int^N \frac{1}{x^{\frac{1}{2}}} dx &= \lim_{N\to\infty} \left(2\sqrt{x}\right)^N \\ &= \lim_{N\to\infty} 2\sqrt{N} \\ &= \infty\end{aligned}$$

Hence diverges.

2.2.7 Problem 24, section 15.1

Are the following series convergent or divergent? Give a reason. $\sum_{n=1}^{\infty} n^2 \left(\frac{i}{3}\right)^n$

Solution

Trying ratio test gives

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\frac{\left(\frac{i}{3}\right)^{n+1}}{\left(\frac{i}{3}\right)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\right|\left|\frac{\left(\frac{i}{3}\right)^{n+1}}{\left(\frac{i}{3}\right)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\left(\frac{i}{3}\right)^{n+1}}{\left(\frac{i}{3}\right)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{i^{n+1}3^n}{i^n3^{n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{i}{3}\right| \\
&= \frac{1}{3}
\end{aligned}
$$

Since limit is smaller than 1, then converges.

2.2.8 Problem 7, section 15.2

Find center and radius of convergence of series $\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^n z^n$

Solution

For these type of problem, always compare it to standard form $\sum_{n=0}^{\infty} A_n (z-z_0)^n$. Where z_0 is the center of disk. So we see that here z_0 is the origin. Now to find R (the radius of convergence), it is given by the inverse of $L = \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$. Therefore we start by finding L

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{a^{n+1}b^n}{a^n b^{+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{a}{b}\right| \\
&= \left|\frac{a}{b}\right|
\end{aligned}
$$

Hence $R = \left|\frac{b}{a}\right|$

2.2.9 Problem 9, section 15.2

Find center and radius of convergence of series $\sum_{n=0}^{\infty}(n-i)^n z^n$

Solution

The center is $z_0 = 0$ by comparing to $\sum_{n=0}^{\infty} A_n (z-z_0)^n$. To find L

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{(n-i)^{n+1}}{(n-i)^n}\right| \\
&= 1
\end{aligned}
$$

Hence $R = 1$.

2.2.10 Problem 11, section 15.2

Find center and radius of convergence of series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} z^n$

Solution

The center is $z_0 = 0$ by comparing to $\sum_{n=0}^{\infty} A_n (z - z_0)^n$. To find L

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}}{n+2}}{\frac{(-1)^{n+1}}{n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+2} n}{(-1)^n (n+2)} \right| \\
&= \lim_{n\to\infty} \left| \frac{n}{(n+2)} \right| \\
&= 1
\end{aligned}$$

Hence $R = 1$.

2.2.11 Problem 12, section 15.2

Find center and radius of convergence of series $\sum_{n=1}^{\infty} \frac{4^n}{(1+i)^n} (z-5)^n$

Solution

The center is $z_0 = 5$ by comparing to $\sum_{n=0}^{\infty} A_n (z - z_0)^n$. To find L

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{\frac{4^{n+1}}{(1+i)^{n+1}}}{\frac{4^n}{(1+i)^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{4^{n+1} (1+i)^n}{4^n (1+i)^{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{4 (1+i)^n}{(1+i)^{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{4}{1+i} \right| \\
&= \lim_{n\to\infty} \frac{4}{|1+i|} \\
&= \lim_{n\to\infty} \frac{4}{\sqrt{2}}
\end{aligned}$$

Hence

$$R = \frac{\sqrt{2}}{4}$$

2.2.12 Problem 18, section 15.2

Find center and radius of convergence of series $\sum_{n=1}^{\infty} \frac{(4n)!}{2^n (n!)^4} (z + \pi i)^n$

Solution

The center is $z_0 = -\pi i$ by comparing to $\sum_{n=0}^{\infty} A_n (z - z_0)^n$. To find L

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{\frac{(4(n+1))!}{2^{(n+1)}((n+1)!)^4}}{\frac{(4n)!}{2^n(n!)^4}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(4(n+1))!2^n (n!)^4}{(4n)!2^{(n+1)} ((n+1)!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4(n+1))! (n!)^4}{(4n)! ((n+1)!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4n+4)! (n!)^4}{(4n)! ((n+1)!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)(4n)! (n!)^4}{(4n)! ((n+1)!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)(n!)^4}{((n+1)!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)(n!)^4}{((n+1)\, n!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{256n^4 + 640n^3 + 560n^2 + 200n + 24}{n^4 + 4n^3 + 6n^2 + 4n + 1} \right|
\end{aligned}$$

Hence

$$\begin{aligned}
L &= \frac{1}{2} \lim_{n\to\infty} \left| \frac{256 + 640\frac{1}{n} + 560\frac{1}{n^2} + 200\frac{1}{n^3} + \frac{24}{n^4}}{1 + 4\frac{1}{n} + 6\frac{1}{n^2} + 4\frac{1}{n^3} + \frac{1}{n^4}} \right| \\
&= \frac{1}{2}(256) \\
&= 128
\end{aligned}$$

Hence

$$R = \frac{1}{128}$$

2.2.13 key solution

2.3 HW 3

2.3.1 Problem set

HW: MATH 601 sections 15.2 – 15.5 Eds 9,10 Sept 24, 201
Tuesday

[1] Consider the power series below. Give the center and find the radius of convergence for each.

a) $\sum_{n=1}^{\infty} n(z+i\sqrt{2})^n$ b) $\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n (z-\pi i)^n$ c) $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$

d) $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}(z+2-i)^n$

[2] Find the radius of convergence using both

① The formula $R = \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$ (Hadamard)

② The termwise differentiation/integration properties of power series

for the series: a) $\sum_{n=1}^{\infty} \frac{6^n}{n}(z-i)^n$; b) $\sum_{n=0}^{\infty} \frac{3^n (n+1)n}{5^n} z^{2n}$

[3] Show that $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n$

a) using the Cauchy product

b) differentiating a suitable series.

[4] If $f(z)$ is an even function (i.e. $f(-z)=f(z)$) where $f(z)=\sum_{n=0}^{\infty} a_n z^n$, show that $a_n=0$ when n is odd.

If $f(z)$ is odd function (i.e. $f(-z)=-f(z)$) show that $a_n=0$ for n even.

[5] Develop the functions below in a Maclaurin series and determine the radius of convergence R for each

a) $\cos(2z^2)$ b) $\frac{z+2}{1-z^2}$

$\boxed{6}$ Develop

a) $f(z) = \frac{1}{z}$ in a Taylor series with center $z_0 = i$

b) $g(z) = e^z$ —— " —— $z_0 = a$.

What is the radius of convergence for each?

$\boxed{7}$ Show that $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ converges uniformly in $|z| \leq 3$

$\boxed{8}$ Where does the series $\sum_{n=1}^{\infty} \left(\frac{n+2}{5n-3}\right)^n z^n$ converges uniformly?

2.3.2 Problem 1

Consider the power series below. Given the center and find radius of convergence for each.

1. $\sum_{n=1}^{\infty} n \left(z + i\sqrt{2}\right)^n$
2. $\sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^n (z - i\pi)^n$
3. $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$
4. $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} (z + 2 - i)^n$

<u>Solution</u>

1) Comparing to form $\sum_{n=1}^{\infty} a_n (z-z_0)^n$ then center is $-i\sqrt{2}$. Now,

$$\begin{aligned} L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{(n+1)}{n}\right| \\ &= 1 \end{aligned}$$

Hence $R = 1$

2) Comparing to form $\sum_{n=1}^{\infty} a_n (z-z_0)^n$ then center is $i\pi$. Now,

$$\begin{aligned} L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^{n}}\right| \\ &= \lim_{n\to\infty} \left|\left(\frac{a}{b}\right)\right| \\ &= \frac{a}{b} \end{aligned}$$

Hence $R = \frac{b}{a}$

3) Comparing $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$ to $\sum_{n=1}^{\infty} a_n (z-z_0)^n$ then center is 0. Now

$$\begin{aligned} L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{\frac{(3(n+1))!}{2^{n+1}((n+1)!)^3}}{\frac{(3n)!}{2^n (n!)^3}}\right| \\ &= \lim_{n\to\infty} \left|\frac{(3(n+1))!2^n (n!)^3}{(3n)!2^{n+1} ((n+1)!)^3}\right| \\ &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{(3n+3)! (n!)^3}{(3n)! ((n+1)!)^3}\right| \\ &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)(3n)! (n!)^3}{(3n)! ((n+1)!)^3}\right| \\ &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)(n!)^3}{((n+1)!)^3}\right| \\ &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)(n!)^3}{((n+1)n!)^3}\right| \\ &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)(n!)^3}{(n+1)^3 (n!)^3}\right| \\ &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)}{(n+1)^3}\right| \\ &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)}{(n+1)^3}\right| \end{aligned}$$

Hence the above becomes

$$\begin{aligned} L &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{27n^3 + 36n^2 + 15n + 2}{n^3 + 3n^2 + 3n + 1}\right| \\ &= \frac{1}{2} \lim_{n\to\infty} \left|\frac{27 + 36\frac{1}{n} + 15\frac{1}{n^2} + \frac{2}{n^3}}{1 + 3\frac{1}{n} + 3\frac{1}{n^2} + \frac{1}{n^3}}\right| \\ &= \frac{27}{2} \end{aligned}$$

Hence $R = \frac{2}{27}$.

4) Comparing $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} (z - (-2+i))^n$ to $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ shows that center is $z_0 = -2 + i$. Now

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{1}{(1+i)^{n+1}}}{\frac{1}{(1+i)^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(1+i)^n}{(1+i)^{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{1}{(1+i)} \right| \\
&= \left| \frac{1}{(1+i)} \right| \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

Hence $R = \sqrt{2}$

2.3.3 Problem 2

Find radius of convergence using both 1) $R = \frac{1}{L}$ where $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ and 2) the termwise differentiation/integration properties of power series. Do this for

1. $\sum_{n=1}^{\infty} \frac{6^n}{n} (z-i)^n$
2. $\sum_{n=0}^{\infty} \frac{3^n (n+1) n}{5^n} z^{2n}$

<u>Solution</u>

1) <u>First method</u>. The center is i. And

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{6^{n+1}}{n+1}}{\frac{6^n}{n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{\left(6^{n+1}\right) n}{6^n (n+1)} \right| \\
&= \lim_{n\to\infty} \left| \frac{6n}{(n+1)} \right| \\
&= 6 \lim_{n\to\infty} \left| \frac{n}{(n+1)} \right| \\
&= 6
\end{aligned}$$

Hence $R = \frac{1}{6}$.

<u>Second method:</u> Taking termwise differentiation gives

$$\begin{aligned}
f'(z) &= \sum_{n=1}^{\infty} \frac{6^n}{n} n (z-i)^{n-1} \\
&= 6 \sum_{n=1}^{\infty} 6^{n-1} (z-i)^{n-1}
\end{aligned}$$

Changing the indexing gives

$$\begin{aligned}
f'(z) &= 6 \sum_{n=0}^{\infty} 6^n (z-i)^n \\
&= 6 \sum_{n=0}^{\infty} (6(z-i))^n
\end{aligned}$$

Comparing to Binomial series $\sum_{n=0}^{\infty} r^n$, the above is $6\frac{1}{1-r}$ where $r = 6(z-i)$. Hence this converges for $|r| < 1$ or $|6(z-i)| < 1$ or $|(z-i)| < \frac{1}{6}$ and diverges for $|z-i| > \frac{1}{6}$. Since termwise differentiated series has same radius of convergence, then $R = \frac{1}{6}$ as using first method.

2) First method. Comparing $\sum_{n=0}^{\infty} \frac{3^n(n+1)n}{5^n}\left(z^2\right)^n$ to $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ then center is zero. And TODO

2.3.4 Problem 3

Show that $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n$, using (a) the Cauchy product. (b) By differentiating a suitable series.

Solution (a)

$$
\begin{aligned}
\frac{1}{(1-z)^2} &= \frac{1}{(1-z)}\frac{1}{(1-z)} \\
&= \left(1+z+z^2+z^3+\cdots\right)\left(1+z+z^2+z^3+\cdots\right) \\
&= \left(1+z+z^2+z^3+\cdots\right) + z\left(1+z+z^2+z^3+\cdots\right) + z^2\left(1+z+z^2+z^3+\cdots\right) + \cdots \\
&= \left(1+z+z^2+z^3+\cdots\right) + \left(z+z^2+z^3+\cdots\right) + \left(z^2++z^3+z^4+\cdots\right) + \cdots \\
&= 1+2z+3z^2+4z^4+\cdots \qquad |z|<1
\end{aligned}
$$

But $\sum_{n=0}^{\infty} (n+1) z^n = 1+2z+3z^2+4z^4+\cdots$. Hence the same.

Solution (b) Observing that

$$
(n+1) z^n = \frac{d}{dz} z^{n+1}
$$

Then

$$
\begin{aligned}
\sum_{n=0}^{\infty} (n+1) z^n &= \sum_{n=0}^{\infty} \frac{d}{dz} z^{n+1} \\
&= \frac{d}{dz}\sum_{n=0}^{\infty} z^{n+1} \\
&= \frac{d}{dz}\sum_{n=1}^{\infty} z^{n} \\
&= \frac{d}{dz}\left(z+z^2+z^3+\cdots\right) \\
&= \frac{d}{dz}\left(z\left(1+z+z^2+\cdots\right)\right) \\
&= \frac{d}{dz}\left(\frac{z}{1-z}\right)
\end{aligned}
$$

But $\frac{d}{dz}\frac{A(z)}{B(z)} = \frac{A'B-AB'}{B^2}$, hence the above becomes, where $A = z, B = 1-z$

$$
\begin{aligned}
\sum_{n=0}^{\infty} (n+1) z^n &= \frac{(1-z) - z(-1)}{(1-z)^2} \\
&= \frac{1-z+z}{(1-z)^2} \\
&= \frac{1}{(1-z)^2}
\end{aligned}
$$

2.3.5 Problem 4

If $f(z)$ is an even function, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, show that $a_n = 0$ when n is odd. And if $f(z)$ is odd function, show that $a_n = 0$ when n is even.

Solution

If $f(z)$ is even, then $f(-z) = f(z)$. Therefore

$$\sum_{n=0}^{\infty} a_n (-z)^n = \sum_{n=0}^{\infty} a_n z^n$$
$$a_0 - a_1 z + a_2 z^2 - a_3 z^3 + a_4 z^4 - \cdots = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

Since power series is unique, then we must have $a_1 = -a_1$ which means $a_1 = 0$, the same for $a_3 = -a_3$, which gives $a_3 = 0$ and so on for all odd a_n.

If $f(z)$ is odd, then $f(-z) = -f(z)$. Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} a_n (-z)^n &= -\sum_{n=0}^{\infty} a_n z^n \\ a_0 - a_1 z + a_2 z^2 - a_3 z^3 + a_4 z^4 - \cdots &= -\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots\right) \\ &= -a_0 - a_1 z - a_2 z^2 - a_3 z^3 - a_4 z^4 + \cdots\end{aligned}$$

Since power series is unique, then we must have $a_0 = -a_0$ which means $a_0 = 0$, the same for $a_2 = -a_2$, which gives $a_2 = 0$ and so on for all even a_n.

2.3.6 Problem 5

Develop the functions below in Maclaurin's series and determine the radius of convergence R for each. (a) $\cos\left(2z^2\right)$, (b) $\frac{z+2}{1-z^2}$

Solution (a)

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Replacing $x = 2z^2$ gives

$$\begin{aligned}\cos\left(2z^2\right) &= 1 - \frac{\left(2z^2\right)^2}{2!} + \frac{\left(2z^2\right)^4}{4!} - \frac{\left(2z^2\right)^6}{6!} + \cdots \\ &= 1 - \frac{2^2 z^4}{2!} + \frac{2^4 z^8}{4!} - \frac{2^6 z^{12}}{6!} + \cdots \\ &= 1 - \frac{4z^4}{2!} + \frac{4^2 z^8}{4!} - \frac{4^3 z^{12}}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{4^n z^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\left(4z^2\right)^n}{(2n)!}\end{aligned}$$

Hence

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{\left(4z^2\right)^{n+1}}{(2(n+1))!}}{\frac{\left(4z^2\right)^n}{(2n)!}}\right| \\
&= \lim_{n\to\infty}\left|\frac{\left(4z^2\right)^{n+1}(2n)!}{\left(4z^2\right)^n(2(n+1))!}\right| \\
&= \lim_{n\to\infty}\left|\frac{4z^2(2n)!}{(2n+2)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{4z^2(2n)!}{(2n+2)(2n+1)(2n)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{4z^2}{(2n+2)(2n+1)}\right| \\
&= \lim_{n\to\infty}\left|\frac{4z^2}{4n^2+6n+2}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{4}{n^2}z^2}{4+6\frac{1}{n}+\frac{2}{n}}\right| \\
&= \left|z^2\right|\lim_{n\to\infty}\left|\frac{0}{4}\right| \\
&= 0
\end{aligned}$$

Hence $R=\frac{1}{L}=\infty$

(b) $\frac{z+2}{1-z^2}$. Apply partial fractions. Obtain two binomial series and combine.

2.3.7 Problem 6

Develop (a) $f(z)=\frac{1}{z}$ in Taylor series around $z_0=i$. (b) $g(z)=e^z$ around $z_0=a$. What is radius of convergence?

Solution (a)

$$f(z)=f(i)+(z-i)f'(i)+\frac{(z-i)^2f''(i)}{2!}+\frac{(z-i)^3f'''(i)}{3!}+\cdots$$

But $f'(z)=-\frac{1}{z^2}, f''(z)=\frac{2}{z^3}, f'''(z)=-\frac{(2)(3)}{z^4},\cdots$, hence the above becomes

$$\begin{aligned}
f(z) &= \frac{1}{i}-(z-i)\frac{1}{i^2}+\frac{(z-i)^2}{2!}\frac{2}{i^3}+\frac{(z-i)^3}{3!}\left(-\frac{2(3)}{i^4}\right)+\cdots \\
&= -i+(z-i)+2i\frac{(z-i)^2}{2!}-2(3)\frac{(z-i)^3}{3!}+\cdots \\
&= -i+(z-i)+i(z-i)^2-(z-i)^3+\cdots \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{i^{n+1}}(z-i)^n
\end{aligned}$$

Hence this convergence for $|z-i|<1$.

Solution (b)

$$g(z)=g(a)+(z-a)g'(a)+\frac{(z-a)^2g''(a)}{2!}+\frac{(z-a)^3g'''(a)}{3!}+\cdots$$

But $g'(z)=e^z, g''(z)=e^z, g'''(z)=e^z,\cdots$, hence the above becomes

$$
\begin{aligned}
g(z) &= e^a + (z-a)e^a + \frac{(z-a)^2 e^a}{2!} + \frac{(z-a)^3 e^a}{3!} + \cdots \\
&= e^a \left(1 + (z-a) + \frac{(z-a)^2}{2!} + \frac{(z-a)^3}{3!} + \cdots\right) \\
&= e^a \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!}
\end{aligned}
$$

Where $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty} \left|\frac{n!}{(1+n)!}\right| = \lim_{n\to\infty} \frac{n!}{n!(1+n)} = \lim_{n\to\infty} \frac{1}{1+n} = 0$. Hence $R = \frac{1}{L} = \infty$. Converges everywhere.

2.3.8 Problem 7

Show that $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ converges uniformly in $|z| \le 3$

Solution:

To find if it converges uniformly for $|z| \le 3$, we need to find R, the radius of converges using normal method, then it $R > 3$, then it will converge uniformly for $|z| \le 3$.

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}}\right| \\
&= \lim_{n\to\infty} \left|\frac{((n+1)!)^2 (2n)!}{(n!)^2 (2(n+1))!}\right| \\
&= \lim_{n\to\infty} \left|\frac{((n+1)n!)^2 (2n)!}{(n!)^2 (2(n+1))!}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)^2 (2n)!}{(2n+2)!}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)^2 (2n)!}{(2n+2)(2n+1)(2n)!}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| \\
&= \lim_{n\to\infty} \left|\frac{n^2+2n+1}{4n^2+6n+2}\right| \\
&= \lim_{n\to\infty} \left|\frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{6}{n^2}+\frac{2}{n^2}}\right| \\
&= \frac{1}{4}
\end{aligned}
$$

Hence Radius of convergence $R = 4$. Since $3 < 4$, then it converges uniformly for $R < 3$.

2.3.9 Problem 8

Where does $\sum_{n=1}^{\infty} \left(\frac{n+2}{5n-3}\right)^n z^n$ converges uniformly?

Solution We first find R. Since the series of the form $\sum_{n=1}^{\infty} A^n z^n$ then it is easier to use

$$\begin{aligned}
L &= \lim_{n\to\infty} \sqrt[\frac{1}{}]{|A^n|} \\
&= \lim_{n\to\infty} \sqrt[\frac{1}{}]{|A^n|} \\
&= \lim_{n\to\infty} \sqrt[\frac{1}{n}]{\left|\frac{n+2}{5n-3}\right|^n} \\
&= \lim_{n\to\infty} \sqrt[\frac{1}{n}]{\left|\frac{1+\frac{2}{n}}{5-\frac{3}{n}}\right|^n} \\
&= \frac{1}{5}
\end{aligned}$$

Hence $R = 5$. Therefore it converges uniformly for $|z| \leq r < 5$

2.3.10 key solution

-- Solutions:

$\boxed{1}$ (a) $\sum_{n=1}^{\infty} n\,(z+i\sqrt{2})^n = \sum_{n=1}^{\infty} n\,(z-(-i\sqrt{2}))^n$ $\quad a_n = n$

center $z_0 = -i\sqrt{2}$ $\quad R = \frac{1}{L^*} = 1$

$L^* = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n} = 1$

(d) $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}(z-(i-2))^n$ $\quad a_n = \frac{1}{(1+i)^n}$ $\quad z_0 = i-2$

$L^* = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(1+i)^n}{(1+i)^{n+1}}\right| = \frac{1}{|1+i|} = \frac{1}{\sqrt{2}}$; $R = \sqrt{2}$

(c) $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$ $\quad$ center $z_0 = 0$ $\quad a_n = \frac{(3n)!}{2^n (n!)^3}$

$L^* = \lim_{n\to\infty} \frac{(3n+3)!}{2^{n+1}(n+1)!^3} \Big/ \frac{3n!}{2^n (n!)^3} = \lim \frac{1\cdots 3n\,(3n+1)(3n+2)(3n+3)\,2^n (n!)^3}{1\cdots 3n\;2^{n+1}\,[(n+1)!]^3}$

$= \lim_{n\to\infty} \frac{(3n+1)(3n+2)(3n+3)}{2\cdot(n+1)(n+1)(n+1)} = \frac{3^3}{2} = \frac{27}{2}$

$R = \frac{1}{L^*} = \frac{2}{27}$

(b) $L^* = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^n} = \frac{a}{b}$; $R = \frac{1}{L^*} = \frac{b}{a}$

$\boxed{2}$ (a) $\sum_{n=1}^{\infty} \frac{6^n}{n}(z-i)^n \qquad a_n = 6^n/n$

1, $L^* = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{6^{n+1}}{6^n}\frac{n}{n+1} = 6 \cdot \lim_{n\to\infty} \frac{n}{n+1} = 6 \; ; \; R = \frac{1}{6}$

2, $\sum_{n=1}^{\infty} \frac{6^n}{n}(z-i)^n$ termwise differentiated gives

$$6 \sum_{n=1}^{\infty} 6^{n-1}(z-i)^{n-1} = 6 \sum_{m=0}^{\infty} 6^m (z-i)^m = 6 \cdot \frac{1}{1-6(z-i)}$$

for $|6(z-i)|<1$ or $|z-i| < \frac{1}{6}$ convergent

for $|z-i| > \frac{1}{6}$ divergent.

Termwise differentiated series has same rad. of conv. $R = \frac{1}{6}$

(b) $\sum_{n=0}^{\infty} \frac{3^n(n+1)n}{5^n} z^{2n}$

1.) the coefficient of series are

$a_0 = 0 \qquad a_1 = 0$

$a_2 = \frac{3^1(1+1)1}{5^1} \qquad a_3 = 0$

$a_4 = \frac{3^2(2+1)2}{5^2} \qquad a_5 = 0$

$\vdots \qquad \vdots$

$\lim_{n\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ or $\lim_{n\to\infty} \sqrt[k]{|a_k|}$ does not exist.

z^k k odd: $\lim_{n\to\infty} \sqrt[k]{|a_k|} = 0$

k even: $\lim_{n\to\infty} \sqrt[k]{|a_k|} = \lim \sqrt[2n]{\frac{3^n(n+1)n}{5^n}} = \lim \sqrt[2]{\sqrt[n]{\frac{3^n}{5^n}(n+1)n}} =$
$k=2n$

$\left(\lim_{n\to\infty} \frac{3}{5} \sqrt[n]{n}\, \sqrt[n]{n+1}\right)^{1/2} = \sqrt{3/5}$

$\sqrt[k]{|a_k|}$ has two cluster points 0 and $\sqrt{3/5}$

$\overline{\lim} \sqrt[k]{|a_k|} = \sqrt{3/5} \qquad R = \frac{1}{\overline{\lim} \sqrt[k]{|a_k|}} = \sqrt{5/3}$

$\boxed{2}$ cont.

2) Let $v = z^2$ new complex variable

in terms of v the series can be written as

$$\sum_{n=0}^{\infty} \frac{3^n (n+1) n}{5^n} v^n = \underbrace{v}_{\text{everywhere convergent}} \underbrace{\sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1) n \, v^{n-1}}_{} \qquad R_v$$

integrate termwise w.r.t. v

(same radius of conv)

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1) v^n \qquad R_v$$

integrate w.r.t. v

(same R)

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^{n+1} = \underbrace{v}_{\text{convergent everywhere}} \underbrace{\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^n}_{\text{geom series}}$$

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^n \quad \text{convegent when}$$

$$\left|\frac{3}{5} v\right| < 1 \qquad |v| < \frac{5}{3}$$

$$R_v = \frac{5}{3}$$

thus $$\sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1) n \, z^{2n} = \sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1) n \cdot v^n$$

converges when $|z^2| = |v| < R_v = \frac{5}{3}$

or $|z|^2 < \frac{5}{3}$ $\quad |z| < \sqrt{5/3}$ $\quad \boxed{R = \sqrt{\frac{5}{3}}}$

$\boxed{3}$ a) $$\frac{1}{(1-z)^2} = \frac{1}{1-z} \cdot \frac{1}{1-z} = \left(\sum_{n=0}^{\infty} z^n\right) \cdot \left(\sum_{n=0}^{\infty} z^n\right) \qquad \text{if } |z| < 1$$

$$= \underbrace{(1 + z + z^2 + \dots)}_{a_n = 1} \underbrace{(1 + z + z^2 + \dots)}_{b_n = 1} = \sum_{n=0}^{\infty} (\underset{1}{a_0} b_n + \underset{1}{a_1} b_{n-1} + \dots + \underset{1}{a_{n-1}} b_1 + \underset{1}{a_n} b_0) z^n$$

$$= \sum_{0}^{\infty} (n+1) z^n \qquad (\text{if } |z| < 1)$$

b) $$\frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) \qquad \text{if } |z|<1$$

termwise derivative of series converges to derivative:

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^{n-1} = \underline{\sum_{\ell=0}^{\infty} (\ell+1) z^{\ell}}$$

$\ell = n-1$
$n = \ell+1$

$\boxed{4}$ Let $f(z)$ be even: $f(-z) = f(z)$ for all z. in $|z|<R$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ Show that $a_n = 0$ when n is odd.

$$f(-z) = \sum_{n=0}^{\infty} a_n (-z)^n = \sum_{n=0}^{\infty} a_n (-1)^n z^n$$

$$0 = f(z) - f(-z) = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n (-1)^n z^n = \sum_{n=0}^{\infty} a_n \underbrace{(1-(-1)^n)}_{\substack{0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even}}} z^n$$

thus

$$\underset{\text{for all } z}{0} = \sum_{n=0}^{\infty} 0 \cdot z^n \qquad \underset{\text{for } |z|<R}{0} = 2a_0 + 2a_2 z^2 + 2a_4 z^4 + \ldots$$

By theorem 2 in $|z|<R$ /where both series converge/ the coefficients of the two series have to coincide.

Thus $2a_0 = 0$, $2a_2 = 0$, $2a_4 = 0, \ldots$

or $a_0 = 0$, $a_2 = 0$, $a_4 = 0, \ldots$

[5] $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$ for any $z \in \mathbb{C}$

(a) $\cos(2z^2) = 1 - \frac{(2z^2)^2}{2!} + \frac{(2z^2)^4}{4!} - \frac{(2z^2)^6}{6!} + \dots$

$= 1 - \frac{2^2 z^4}{2!} + \frac{2^4 z^8}{4!} - \frac{2^6 z^{12}}{6!} + \dots$ for any $2z^2 \in \mathbb{C}$ or $z \in \mathbb{C}$

$= \sum_{n=0}^{\infty} \frac{4^n z^{4n}}{2n!} \cdot (-1)^n$ radius of conv. $R = \infty$

(b) $\frac{z+2}{1-z^2} = \frac{z+2}{(1-z)(z+1)} = \frac{A}{1-z} + \frac{B}{z+1} = \frac{Az + A - Bz + B}{(1-z)(z+1)}$

$\Rightarrow \begin{array}{l} A - B = 1 \\ A + B = 2 \end{array} \Rightarrow \begin{array}{l} A = 3/2 \\ B = 1/2 \end{array}$ $= \frac{3}{2}\frac{1}{1-z} + \frac{1}{2}\frac{1}{z+1}$

$\frac{3}{2}\frac{1}{1-z} = \frac{3}{2}\left[1 + z + z^2 + z^3 + \dots\right]$ for $|z| < 1$

$\frac{1}{2}\frac{1}{1+z} = \frac{1}{2}\left[\frac{1}{1-(-z)}\right] = \frac{1}{2}\left[1 - z + z^2 - z^3 + \dots\right]$ if $|-z| = |z| <$

$\frac{z+2}{1-z^2} = \left[\frac{3}{2} + \frac{1}{2}\right] + \left(\frac{3}{2} - \frac{1}{2}\right)z + \left(\frac{3}{2} + \frac{1}{2}\right)z^2 + \left(\frac{3}{2} - \frac{1}{2}\right)z^3 + \dots$

$= 2 + z + 2z^2 + z^3 + 2z^4 + z^5$ $(R = 1)$

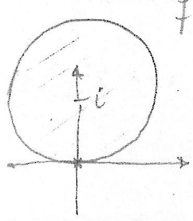
$\sum_{n=0}^{\infty} \left(\frac{3}{2} + \frac{1}{2}(-1)^n\right) z^n$ if $|z| < 1$

$\boxed{6}$

(a) $f(z) = \frac{1}{z} \qquad z_0 = i \qquad f'(z) = -\frac{1}{z^2}, \; f''(z) = \frac{2\cdot 1}{z^3},$

$f'''(z) = -\frac{3\cdot 2\cdot 1}{z^4} \qquad f^{(n)}(z) = \frac{(-1)^n \, n!}{z^{n+1}}$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n =$$



$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(i)^{n+1}}(z-i)^n = -i + (z-i) - i(z-i)^2 + (z-i)^3$$

$$-i(z-i)^4 + (z-i)^5 - \dots$$

$|z-i| < 1$

(b) $f(z) = e^z \qquad z_0 = a$

$f^{(n)} = e^z$

$$f(z) = \sum_{n=0}^{\infty} \frac{e^a}{n!}(z-a)^n = e^a \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!}$$

$\sqrt[n]{\frac{e^a}{n!}} \to 0$

$$= e^a + e^a(z-a) + \frac{e^a}{2!}(z-a)^2 + \frac{e^a}{3!}(z-a)^3 + \dots$$

$R = \infty$

converges for all z

7) Power series $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ is uniformly convergent in $|z| < r < R$ where R is radius of convergence

we need to show that $R > 3$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)! \cdot (n+1)!}{(2n+2)!} \cdot \frac{2n!}{n! \cdot n!} = \frac{(n+1)^2}{(2n+1)(2n+2)} \longrightarrow \frac{1}{4} = L^*$$

$$R = \frac{1}{L^*} = 4$$

8) $\sum_{n=1}^{\infty} \left[\frac{n+2}{5n-3}\right]^n z^n$ $\quad R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty} \frac{1}{\frac{n+2}{5n-3}} = 5$

$a_n \qquad z_0 = 0$

Series converges uniformly in any disk $|z| < r < 5$

HW 3 MATH 601

1 a) Verify that $y(x) = -\sin x + ax^2 + bx + c$ for any a, b, c constants is a solution to $y''' = \cos x$

b) Show that $y(x) = \tan(x+c)$ solves $y' = 1+y^2$ for any constant c

2 Show that the given function solves the specified initial value problem

a) $y(x) = Ce^{x/2}$ $\begin{cases} y' = \frac{1}{2}y \\ y(2) = 2 \end{cases}$

b) $y(x) = Ce^{-x^2}$ $\begin{cases} y' + 2xy = 0 \\ y(1) = 1/e \end{cases}$

3 $y = Cx - C^2$ is a solution to the ODE $(y')^2 - xy' + y = 0$ for any C const.

Find a singular solution to the ODE (not given by $y = cx - c^2$) by rewriting the ODE using the quadratic formula.

4 Find all solutions of the following differential equations.

a) $yy' + 25x = 0$ b) $y' = ky^2$

c) $xy' = x + y$ (Hint $u = y/x$)

5 Solve the IVPs:

a) $\begin{cases} y' = 1 + 4y^2 \\ y(0) = 0 \end{cases}$ b $\begin{cases} y' = -x/y \\ y(1) = \sqrt{3} \end{cases}$

c) $\begin{cases} e^x y' = 2(x+1)y^2 \\ y(0) = 1/6 \end{cases}$

2.4 HW 4

2.4.1 problems description

HW5 second ODE HW   MATH 601   HW 1 ODEs   oct 4, 2018

HW on ODE

1. a) Verify that $y(x) = -\sin x + ax^2 + bx + c$ for any a, b, c constants is a solution to $y''' = \cos x$

   b) Show that $y(x) = \tan(x + c)$ solves $y' = 1 + y^2$ for any constant c

2. Show that the given function solves the specified initial value problem

   a) $y(x) = Ce^{x/2}$ $\quad \begin{cases} y' = \frac{1}{2} y \\ y(2) = 2 \end{cases}$

   b) $y(x) = ce^{-x^2}$ $\quad \begin{cases} y' + 2xy = 0 \\ y(1) = 1/e \end{cases}$

3. $y = cx - c^2$ is a solution to the ODE $(y')^2 - xy' + y = 0$ for any c const.
   Find a singular solution to the ODE (not given by $y = cx - c^2$) by rewriting the ODE using the quadratic formula.

4. Find all solutions of the following differential equations.

   a) $yy' + 25x = 0$ $\qquad$ b) $y' = ky^2$

   c) $xy' = x + y$ (Hint $u = y/x$)

5. Solve the IVPs.

   a) $\begin{cases} y' = 1 + 4y^2 \\ y(0) = 0 \end{cases}$ $\qquad$ b $\begin{cases} y' = -x/y \\ y(1) = \sqrt{3} \end{cases}$

   c) $\begin{cases} e^x y' = 2(x+1) y^2 \\ y(0) = 1/6 \end{cases}$

16 a, Show that the solution curves of $y' = -\frac{x}{y}$ lie on circles

b, Consider the hyperbolas $x \cdot y = c$.
Give a differential equation for which all these curves are solutions

c, Find an ODE that has the straight lines throught as solutions (except $x=0$ line).

d, Note for $y' = -\frac{x}{y} = f_1(x,y)$ and $y' = \frac{y}{x} = f_2(x,y)$

$f_1(x,y) \cdot f_2(x,y) = -1$, also the respective solution curves intersect each other at right angle.
Explane why is this always the case if $f_1(x,y) \cdot f_2(x,y) = -1$

2.4.2 Problem 1

part a

$$
\begin{aligned}
y' &= -\cos x + 2ax + b \\
y'' &= \sin x + 2a \\
y''' &= \cos x
\end{aligned}
$$

Substituting into the ODE $y''' = \cos x$ shows it satsifies it. Hence this is true for any a, b, c.

part b

Since $\tan(x+c) = \frac{\sin(x+c)}{\cos(x+c)}$ then

$$y' = 1 + \tan^2(x+c)$$

Substituting this into the ode $y' = 1 + y^2$ gives

$$1 + \tan^2(x+c) = 1 + \tan^2(x+c)$$

Which is true for any c

2.4.3 Problem 2

see Key.

2.4.4 Problem 3

see Key

2.4.5 Problem 4

(a) Find all solutions to $yy' + 25x = 0$ (b) $y' = ky^2$ (c) $xy' = x + y$

Part a

$$
\begin{aligned}
y\frac{dy}{dx} &= -25x \\
ydy &= -25xdx \\
\frac{y^2}{2} &= -\frac{25}{2}x^2 + C \\
y^2 &= -25x^2 + C_1
\end{aligned}
$$

Hence

$$y = \pm\sqrt{C_1 - 25x^2}$$

For real solution, we want $C_1 > 25x^2$.

Part b

$$
\begin{aligned}
\frac{1}{y^2}\frac{dy}{dx} &= k \\
\frac{1}{y^2}dy &= kdx \\
\frac{-1}{y} &= kx + C \\
y &= \frac{-1}{kx + C}
\end{aligned}
$$

Part c

$$\frac{dy}{dx} = 1 + \frac{y}{x} \qquad x \neq 0$$

Let $u = \frac{y}{x}$ or $y = ux$. Hence $\frac{dy}{dx} = u'x + u$ and the above ODE becomes

$$
\begin{aligned}
u'x + u &= 1 + u \\
u' &= \frac{1}{x} \\
du &= \frac{1}{x}dx \\
u &= \ln|x| + C
\end{aligned}
$$

Hence

$$y = x(\ln|x| + C)$$

2.4.6 Problem 5

(a) Solve the IVP $y'(x) = 1 + 4y^2$ with $y(0) = 0$. (b) $y' = -\frac{x}{y}$ with $y(1) = \sqrt{3}$ (c) $e^x y' = 2(x+1)y^2$ with $y(0) = \frac{1}{6}$

Part a

$$
\begin{aligned}
y'(x) &= 1 + 4y^2 \\
\frac{dy}{1 + 4y^2} &= dx \\
\frac{1}{2}\arctan\left(2y\right) &= x + C \\
\arctan\left(2y\right) &= 2x + C_1 \\
y &= \frac{\tan(2x + C_1)}{2}
\end{aligned}
$$

Applying IC gives

$$0 = \frac{1}{2}\tan(C_1)$$

Hence $C_1 = 0$. Therefore the solution is

$$y = \frac{1}{2}\tan(2x)$$

Part b

$$\begin{aligned} y' &= -\frac{x}{y} \\ ydy &= -xdx \\ \frac{1}{2}y^2 &= -\frac{1}{2}x^2 + C \\ y^2 &= -x^2 + C_1 \end{aligned}$$

Applying IC gives

$$\begin{aligned} 3 &= -1 + C_1 \\ C_1 &= 4 \end{aligned}$$

Hence solution is

$$\begin{aligned} y^2 &= -x^2 + 4 \\ y &= \pm\sqrt{4 - x^2} \end{aligned}$$

For real solution $4 - x^2 > 0$.

Part c

$$\begin{aligned} e^x y' &= 2(x+1)y^2 \\ \frac{y'}{y^2} &= 2(x+1)e^{-x} \\ y^{-2}dy &= 2(x+1)e^{-x} \\ -\frac{1}{y} &= \int 2(x+1)e^{-x}dx \\ &= -2(x+2)e^{-x} + C \end{aligned}$$

Hence

$$\begin{aligned} y &= \frac{1}{2(x+2)e^{-x} + C_1} \\ &= \frac{1}{2xe^{-x} + 4e^{-x} + C_1} \end{aligned}$$

Applying IC gives

$$\begin{aligned} \frac{1}{6} &= \frac{1}{4 + C_1} \\ 4 + C_1 &= 6 \\ C_1 &= 2 \end{aligned}$$

Hence solution is

$$y = \frac{1}{2xe^{-x} + 4e^{-x} + 2}$$

2.4.7 Key solution

Math 601 Solutions HW1 ODEs ①

1. a) $y = -\sin x + ax^2 + bx + c$ $\quad$ a, b, c constants

$y' = -\cos x + a \cdot 2x + b \cdot 1 + 0$

$y'' = -(-\sin x) + 2a + 0$

$y''' = \cos x$

b) $y(x) = \tan(x + c)$ $\quad$ c const.

$$y' = \frac{1}{\cos^2(x+c)} = \frac{\sin^2(x+c) + \cos^2(x+c)}{\cos^2(x+c)} = \tan^2(x+c) + 1$$

$$= 1 + y^2$$

2. ⓐ $\begin{cases} y' = \frac{1}{2} y \\ y(2) = 2 \end{cases}$ $\quad$ $y = Ce^{x/2}$

$y' = C\frac{1}{2}e^{x/2} = \frac{1}{2}y$ solves ODE

initial condition $2 = y(2) = Ce^{2/2} = C \cdot e$

only if $C = 2/e$

$y = \frac{2}{e}e^{x/2}$ is solution

ⓑ $\begin{cases} y' + 2xy = 0 \\ y(1) = 1/e \end{cases}$ $\quad$ $y = Ce^{-x^2}$

$\underbrace{-2xCe^{-x^2}}_{y'(x)} + 2xCe^{-x^2} = 0$

for all x

initial condition

solves ODE for all C

$\frac{1}{e} = y(1) = Ce^{-1} = \frac{C}{e} \Rightarrow C = 1$ solution to IVP only if $C = 1$ $\quad$ $y = e^{-x^2}$

3. $y = cx - c^2$ $\quad$ $(y')^2 - xy' + y = c^2 - x(c) + cx - c^2 = 0$

$y' = c$

$$y' = \frac{x \pm \sqrt{x^2 - 4y}}{2} \Rightarrow y' = \frac{x}{2} \quad \text{if } 4y = x^2$$

$\Downarrow$

$y = \frac{x^2}{4} + c$ $\quad$ $\boxed{y = \frac{x^2}{4}}$

singular solution

(2)

4 a) $yy' = -25x$ separable

$$\frac{d}{dx}\left(\frac{y^2}{2}\right) = -25x$$

$$\frac{y^2}{2} = -\frac{25x^2}{2} + c \qquad D = 2c$$

$$y = \pm\sqrt{D - 25x^2} \quad \text{where } D - 25x^2 \geq 0 \text{ only if } D \geq 0$$

and $|5x| \leq \sqrt{D}$

$$-\frac{1}{5}\sqrt{D} \leq x \leq \frac{1}{5}\sqrt{D}$$

b) $y' = ky^2$ if $y \neq 0$ on I then separable

$$\frac{y'}{y^2} = k$$

$y = 0 \Rightarrow y' = 0$

$y \equiv 0$ solution for all x for any k

$$-y^{-1} = kx + c$$

$$y = \frac{-1}{kx+c} \quad \text{if } x \neq -\frac{c}{k}$$

sol. on $(-\infty, -\frac{c}{k})$ & on $(-\frac{c}{k}, \infty)$

c) $xy' = x + y$ if $x \neq 0$ $u = y/x$ $y = ux$

$$y' = 1 + y/x \qquad y' = u'x + u$$

$$xu' + \cancel{u} = 1 + \cancel{u}$$

$$u' \cdot x = 1$$

$$u' = \frac{1}{x} \qquad u = \ln|x| + c \qquad \underline{y(x) = \frac{\ln|x|}{x} + \frac{c}{x}}$$

5 $\begin{cases} y' = 1 + 4y^2 \\ y(0) = 0 \end{cases}$

$1 + 4y^2 > 0 \Rightarrow y' = 1 + 4y^2$ and $\frac{y'}{1+4y^2} = 1$ have same sols. separable

$$\int \frac{1}{1+4y^2}\,dy = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u = \frac{\tan^{-1}2y}{2}$$

$u = 2y$

$dy = du/2$

$$\tan^{-1}2y = 2t + c$$

$$\boxed{y = \frac{\tan(2t+c)}{2}} \text{ for all } t$$

I.C.: $0 = y(0) = \frac{\tan(c)}{2} \Rightarrow \boxed{c = 0}$

b) $\begin{cases} y' = -\dfrac{x}{y} \\ y(1) = \sqrt{3} \end{cases} \overset{y\neq 0:}{\Rightarrow} \quad yy' = -x \quad \Rightarrow \quad \dfrac{y^2}{2} = -\dfrac{x^2}{2} + c \qquad D = 2C$ ③

Solution: $\boxed{y = \pm\sqrt{D - x^2}}$

$\boxed{y \neq 0} \Rightarrow x \neq \pm\sqrt{D}$

$D - x^2 \geq 0 \Rightarrow$

on: $\boxed{-\sqrt{D} \leq x \leq \sqrt{D}}$

c) $e^x y' = 2(x+1)y^2$ separable

$\begin{cases} e^x y' = 2(x+1)y^2 \\ y(0) = 1/6 \end{cases}$

$\dfrac{y'}{y^2} = 2e^{-x}(x+1) = 2\underbrace{xe^{-x}}_{\text{in by parts}} + 2e^{-x}$

$\int \underset{u}{x}\, \underset{v'}{e^{-x}}\, dx = -xe^{-x} - \int -e^{-x}\, dx$

$v = -e^{-x} \qquad = -xe^{-x} - e^{-x}$

$-\dfrac{1}{y} = 2[-xe^{-x} - e^{-x}] - 2e^{-x} + c$

$y = \dfrac{1}{e^{-x}(2x+4) + C} \qquad D = -c$

I.C.: $\dfrac{1}{6} = y(0) = \dfrac{1}{4 + D} \Rightarrow D = 2$

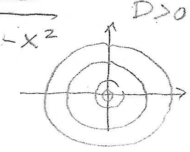
solution $y = \dfrac{1}{2 + e^{-x}(2x+4)}$

for all $x \in \mathbb{R}$ except $(2x+4)e^{-x} = -2$ at $x = x^* < -2$

solution on (x^*, ∞)

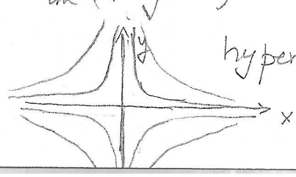
6 a) $y' = -\dfrac{x}{y} \underset{y\neq 0}{\Rightarrow} yy' = -x \qquad \dfrac{y^2}{2} = -\dfrac{x^2}{2} + c \qquad D = 2c \qquad \boxed{y^2 + x^2 = D}$ circles $D > 0$

$y = \pm\sqrt{D - x^2}$



b) $x \cdot y = c \qquad y = y(x)$

$\dfrac{d}{dx}(x \cdot y = c) \Rightarrow y + xy' = 0 \underset{\text{if } y \neq 0}{\Rightarrow} \boxed{y' = -\dfrac{y}{x}}$



hyperbolas

4

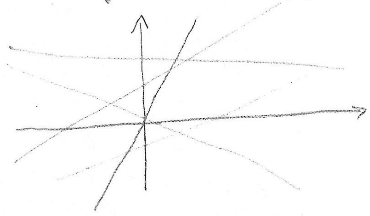
c) straight lines: linear function graphs

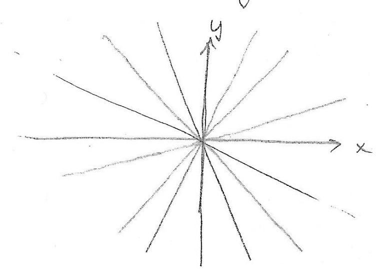
$y = mx + b \qquad y' = m$ constant

straight lines through origin $\qquad y = mx \qquad b = 0 \qquad \begin{cases} y' = m \\ y(0) = 0 \end{cases}$ or $y' = m \cdot x$

$= y/x$

$x \neq 0$





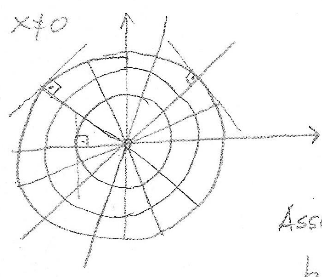
d) $y' = -\frac{x}{y}$ circles

$y' = y/x \Rightarrow \frac{1}{y} y' = \frac{1}{x} \Rightarrow \ln|y| = \ln|x| + c$

$y = \pm x \cdot e^c = Cx \qquad C \neq 0$

straight lines through origin

$x \neq 0$



Assume that $y' = f_1(x,y)$ and $y' = f_2(x,y)$ have two solutions $y_1(x)$ & $y_2(x)$ respectively intersecting at (x_0, y_0) point.

$y_1'(x_0) = f_1(x_0, y_1(x_0)) = f_1(x_0, y_0)$ is the slope of tangent to y_1 at (x_0, y_0)

$y_2'(x_0) = f_2(x_0, y_2(x_0)) = f_2(x_0, y_0)$ is the slope of tangent to y_2 at (x_0, y_0)

intersecting tangent lines $\quad y = f_1(x_0, y_0)\, x + y_0$

$y = f_2(x_0, y_0)\, x + y_0$

with finite slopes $m_1 = f_1(x_0, y_0)$ and $m_2 = f_2(x_0, y_0)$ are intersecting exactly when

$m_1 = -\frac{1}{m_2}$ or $f_1(x_0, y_0) f_2(x_0, y_0) = -1$

2.5 HW 5

2.5.1 problems description

MATH 601 HW #2 ODEs given out oct 10, 2018

[1] Solve the initial value problems

a) $y'+4y=20$, $y(0)=2$

b) $y'+3y=\sin x$, $y(\pi/2)=0.3$

c) $y'-y(1+\frac{3}{x})=x+2$, $y(1)=e-1$

[2] Solve the Bernoulli equation

$$y'+\frac{y}{3}=\frac{1-2x}{3}y^4$$

[3] Consider the model $\begin{cases} m\frac{dv}{dt}=W-B-kv \\ v(0)=0 \end{cases}$

for the sinking of a container in the ocean.

/ $v=v(t)$ - velocity. W - weight, B buoyancy force, k - drag coefficient, W,B,k constants /

If container hits bottom at $v_c=12$ m/sec critical velocity or less it will not break. Determine the critical time t_c when the container reaches critical velocity, assuming $W=2254$ N, $B=2090$ N, $k=0.637$ kg/sec. $(N=\frac{kg\ m}{sec^2})$

What is the critical depth beyond which the container might break up?

[4] Consider the Ricatti equation

$$y'=x^3(y-x)^2+y/x$$

and solve it

/ Hint consider a substitution $w(x)=y(x)-x$ /

[5] Under what conditions for the constants A,B,C,D is $(Ax+By)+(Cx+Dy)y'=0$ exact diff. eq. Solve the equation!

2.5.2 problem 1

part a

The ODE to solve is

$$\frac{d}{dx}y(x)+4y(x)=20$$

with initial conditions $y(0)=2$.

Trying separable ODE.

In canonical form, the ODE is written as

$$\begin{aligned} y' &= F(x,y) \\ &= -4y+20 \end{aligned}$$

The ODE $\frac{dy}{dx}=-4y+20$, is separable. It can be written as

$$\frac{dy}{dx}=f(x)g(y)$$

Where $f(x)=1$ and $g(y)=-4y+20$. Therefore

$$\frac{dy}{dx}=-4y+20$$

Hence

$$\begin{aligned}(-4y+20)^{-1}\,dy &= dx\\ \int(-4y+20)^{-1}\,dy &= \int dx\\ -1/2\ln(2)-1/4\ln(|y-5|) &= x+C_1\end{aligned}$$

Solving for y gives

$$y=-1/4\,e^{-4x-4C_1}+5$$

The solution above can be written as

$$\boxed{y=-1/4\,C_1e^{-4x}+5} \tag{2.1}$$

Initial conditions are now used to solve for C_1. Substituting $x=0$ and $y=2$ in the above solution gives an equation to solve for the constant of integration.

$$\begin{aligned}2 &= -1/4\,C_1e^0+5\\ &= -1/4\,C_1+5\end{aligned}$$

Hence

$$C_1=12\,(e^0)^{-1}$$

Which is simplified to

$$C_1=12$$

Substituting C_1 found above back in the solution gives

$$y(x)=-3\,e^{-4x}+5$$

part b

The ODE to solve is

$$\frac{d}{dx}y(x)+3y(x)=\sin(x)$$

with initial conditions $y(\pi/2)=3/10$.

Trying Linear ODE.

In canonical form, the ODE is written as

$$\begin{aligned}y' &= F(x,y)\\ &= -3y+\sin(x)\end{aligned}$$

The ODE is linear in y and has the form

$$y'=yf(x)+g(x)$$

Where $f(x)=-3$ and $g(x)=\sin(x)$.

Writing the ODE as

$$\begin{aligned}y'-(-3y) &= \sin(x)\\ y'+3y &= \sin(x)\end{aligned}$$

Therefore the integrating factor μ is

$$\mu=e^{\int 3\,dx}=e^{3x}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu y &= \mu(\sin(x))\\ \frac{d}{dx}\left(ye^{3x}\right) &= \sin(x)\,e^{3x}\\ d\left(ye^{3x}\right) &= \left(\sin(x)\,e^{3x}\right)dx\end{aligned}$$

Integrating both sides gives

$$ye^{3x} = -1/10\cos(x)e^{3x} + 3/10\sin(x)e^{3x} + C_1$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = \frac{-1/10\cos(x)e^{3x} + 3/10\sin(x)e^{3x}}{e^{3x}} + \frac{C_1}{e^{3x}}$$

Simplifying the solution gives

$$\boxed{y = 3/10\sin(x) - 1/10\cos(x) + C_1e^{-3x}}$$

Initial conditions are now used to solve for C_1. Substituting $x = \pi/2$ and $y = 3/10$ in the above solution gives an equation to solve for the constant of integration.

$$\begin{aligned}3/10 &= 3/10\sin(\pi/2) - 1/10\cos(\pi/2) + C_1e^{-3/2\pi}\\ &= 3/10 + C_1e^{-3/2\pi}\end{aligned}$$

Hence

$$C_1 = -1/10\,\frac{3\sin(\pi/2) - \cos(\pi/2) - 3}{e^{-3/2\pi}}$$

Which is simplified to

$$C_1 = 0$$

Substituting C_1 found above back in the solution gives

$$y(x) = 3/10\sin(x) - 1/10\cos(x)$$

part c

The ODE to solve is

$$\frac{d}{dx}y(x) - y(x)\left(1 + 3x^{-1}\right) = x + 2$$

with initial conditions $y(1) = e - 1$.

Trying Linear ODE.

In canonical form, the ODE is written as

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{x^2 + xy + 2x + 3y}{x}\end{aligned}$$

The ODE is linear in y and has the form

$$y' = yf(x) + g(x)$$

Where $f(x) = \frac{x+3}{x}$ and $g(x) = \frac{x^2+2x}{x}$.

Writing the ODE as

$$\begin{aligned}y' - \left(\frac{(x+3)y}{x}\right) &= \frac{x^2 + 2x}{x}\\ y' - \frac{(x+3)y}{x} &= \frac{x^2 + 2x}{x}\end{aligned}$$

Therefore the integrating factor μ is

$$\mu = e^{\int -\frac{x+3}{x}dx} = e^{-x-3\ln(x)}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu y &= \mu\left(\frac{x^2 + 2x}{x}\right)\\ \frac{d}{dx}\left(ye^{-x-3\ln(x)}\right) &= \frac{\left(x^2 + 2x\right)e^{-x-3\ln(x)}}{x}\\ d\left(ye^{-x-3\ln(x)}\right) &= \left(\frac{\left(x^2 + 2x\right)e^{-x-3\ln(x)}}{x}\right)dx\end{aligned}$$

Integrating both sides gives

$$ye^{-x-3\ln(x)} = -e^{-x-3\ln(x)}x + C_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x-3\ln(x)}$ results in

$$y = -x + \frac{C_1}{\mathrm{e}^{-x-3\ln(x)}}$$

Simplifying the solution gives

$$\boxed{y = -x + C_1 x^3 \mathrm{e}^x}$$

Initial conditions are now used to solve for C_1. Substituting $x = 1$ and $y = \mathrm{e} - 1$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} - 1 = -1 + C_1 \mathrm{e}$$

Hence

$$C_1 = 1$$

Substituting C_1 found above back in the solution gives

$$y(x) = -x + x^3 \mathrm{e}^x$$

2.5.3 problem 2

The ODE to solve is

$$\frac{d}{dx}y(x) + 1/3\, y(x) = 1/3\, (1 - 2x) \left(y(x)\right)^4$$

Trying Bernoulli ODE.

In canonical form, the ODE is written as

$$\begin{aligned} y' &= F(x, y) \\ &= -y/3 - 2/3\, y^4 x + 1/3\, y^4 \end{aligned}$$

This is a Bernoulli ODE. Comparing the ODE to solve

$$y' = -y/3 - 2/3\, y^4 x + 1/3\, y^4$$

With Bernoulli ODE standard form

$$y' = f_0(x) y + f_1(x) y^n$$

Shows that $f_0(x) = -1/3$ and $f_1(x) = -2/3\, x + 1/3$ and $n = 4$.

Dividing the ODE by y^4 gives

$$y' y^{-4} = -1/3 y^{-3} + -2/3\, x + 1/3 \tag{1}$$

Let

$$v = y^{-3} \tag{2}$$

Taking derivative of (2) w.r.t x gives

$$\begin{aligned} v' &= -3\, y^{-4} y' \\ y^{-4} &= \frac{v'}{-3\, y'} \end{aligned} \tag{3}$$

Substituting (3) into (1) gives

$$\begin{aligned} \frac{v'}{(-3)} &= (-1/3)\, v + -2/3\, x + 1/3 \\ v' &= (-3)\,(-1/3)\, v + (-3)\,(-2/3\, x + 1/3) \\ &= v + 2x - 1 \end{aligned}$$

The above now is a linear ODE in $v(x)$ which can be easily solved using an integrating factor.

In canonical form, the ODE is written as

$$\begin{aligned} v' &= F(x, v) \\ &= v + 2x - 1 \end{aligned}$$

The ODE is linear in v and has the form

$$v' = v f(x) + g(x)$$

Where $f(x) = 1$ and $g(x) = 2x - 1$.

Writing the ODE as

$$\begin{aligned} v' - (v) &= 2x - 1 \\ v' - v &= 2x - 1 \end{aligned}$$

Therefore the integrating factor μ is

$$\mu = e^{\int -1 \,\mathrm{d}x} = \mathrm{e}^{-x}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu v &= \mu (2x - 1) \\ \frac{\mathrm{d}}{\mathrm{d}x}(v\mathrm{e}^{-x}) &= (2x - 1)\,\mathrm{e}^{-x} \\ \mathrm{d}\,(v\mathrm{e}^{-x}) &= ((2x - 1)\,\mathrm{e}^{-x})\,\mathrm{d}x \end{aligned}$$

Integrating both sides gives

$$v\mathrm{e}^{-x} = -(2x + 1)\,\mathrm{e}^{-x} + C_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$v = -2x - 1 + \frac{C_1}{\mathrm{e}^{-x}}$$

Simplifying the solution gives

$$\boxed{v = -2x - 1 + C_1\mathrm{e}^{x}}$$

Replacing v in the above by y^{-3} from equation (2), gives the final solution.

$$y^{-3} = -2x - 1 + C_1\mathrm{e}^{x}$$

Solving for y gives

$$\begin{aligned} y &= \frac{1}{\sqrt[3]{-2x - 1 + C_1\mathrm{e}^{x}}} \\ y &= -1/2\,\frac{1}{\sqrt[3]{-2x - 1 + C_1\mathrm{e}^{x}}} + \frac{i/2\sqrt{3}}{\sqrt[3]{-2x - 1 + C_1\mathrm{e}^{x}}} \\ y &= -1/2\,\frac{1}{\sqrt[3]{-2x - 1 + C_1\mathrm{e}^{x}}} - \frac{i/2\sqrt{3}}{\sqrt[3]{-2x - 1 + C_1\mathrm{e}^{x}}} \end{aligned}$$

2.5.4 problem 3

The ODE to solve is

$$m\frac{\mathrm{d}}{\mathrm{d}x}v(x) = w - B - kv(x)$$

with initial conditions $v(0) = 0$.

Trying separable ODE.

In canonical form, the ODE is written as

$$\begin{aligned} v' &= F(x, v) \\ &= -\frac{kv + B - w}{m} \end{aligned}$$

The ODE $\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{kv+B-w}{m}$, is separable. It can be written as

$$\frac{\mathrm{d}v}{\mathrm{d}x} = f(x)g(v)$$

Where $f(x) = 1$ and $g(v) = \frac{-kv-B+w}{m}$. Therefore

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-kv - B + w}{m}$$

Hence

$$\left(\frac{m}{-kv-B+w}\right)\mathrm{d}v = \mathrm{d}x$$
$$\int\left(\frac{m}{-kv-B+w}\right)\mathrm{d}v = \int \mathrm{d}x$$
$$-\frac{m\ln\left(|kv+B-w|\right)}{k} = x+C_1$$

Solving for v gives

$$v = \frac{1}{k}\left(-\mathrm{e}^{-\frac{k(x+C_1)}{m}}-B+w\right)$$

Initial conditions are now used to solve for C_1. Substituting $x=0$ and $v=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{k}\left(-\mathrm{e}^{-\frac{kC_1}{m}}-B+w\right)$$

Hence

$$C_1 = -\frac{m\ln\left(-B+w\right)}{k}$$

Substituting C_1 found above back in the solution gives

$$v(x) = \frac{1}{k}\left(-\mathrm{e}^{-\frac{k}{m}\left(x-\frac{m\ln(-B+w)}{k}\right)}-B+w\right)$$

The solution $\frac{1}{k}\left(-\mathrm{e}^{-\frac{k}{m}\left(x-\frac{m\ln(-B+w)}{k}\right)}-B+w\right)$ can be simplified to

$$\boxed{v(x) = \frac{1}{k}\left(-\mathrm{e}^{\frac{m\ln(-B+w)-xk}{m}}-B+w\right)} \tag{2.2}$$

2.5.5 problem 4

The ODE to solve is

$$\frac{\mathrm{d}}{\mathrm{d}x}y(x) = x^3\left(y(x)-x\right)^2+\frac{y(x)}{x}$$

Trying Riccati ODE.

In canonical form, the ODE is written as

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^6-2x^5y+x^4y^2+y}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^5-2x^4y+x^3y^2+\frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=x^5$, $f_1(x)=\frac{-2x^5+1}{x}$ and $f_2(x)=x^3$.

Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{ux^3} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification) in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x)-\left(f_2'+f_1f_2\right)u'(x)+f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 3\,x^2 \\ f_1 f_2 &= \left(-2\,x^5+1\right)x^2 \\ f_2^2 f_0 &= x^{11} \end{aligned}$$

Substituting the above terms back in (2) gives

$$x^3 \frac{d^2}{dx^2} u(x) - \left(3\,x^2 + \left(-2\,x^5+1\right)x^2\right) \frac{d}{dx} u(x) + x^{11} u(x) = 0$$

Solving the above ODE gives

$$u(x) = e^{-1/5\,x^5}\left(x^5 C_2 + C_1\right)$$

The above shows that

$$u'(x) = -x^4 e^{-1/5\,x^5}\left(x^5 C_2 + C_1 - 5\,C_2\right)$$

Hence, using the above in (1) gives the solution

$$y(x) = \frac{x\left(x^5 C_2 + C_1 - 5\,C_2\right)}{x^5 C_2 + C_1}$$

Dividing both numerator and denominator by C_2 gives, after renaming the constant $\frac{C_1}{C_2} = C_0$ the following

$$\boxed{y(x) = \frac{x\left(x^5 + C_0 - 5\right)}{x^5 + C_0}}$$

2.5.6 Key solution

MATH 601 HW#2 ODEs Solutions

$\boxed{1}$ a) $\begin{cases} y' + 4y = 20 \\ y(0) = 2 \end{cases}$ $\quad | \cdot e^{4x} \qquad [e^{4x} \cdot y]' = 20e^{4x}$

$$e^{4x} y = 5e^{4x} + C$$

$$y = 5 + Ce^{-4x}$$

$$2 = y(0) = 5 + C \Rightarrow C = -3$$

$$\boxed{y(t) = 5 - 3e^{-4x}}$$

b) $\begin{cases} y' + 3y = \sin x \\ y(\pi/2) = 0.3 \end{cases}$ $\quad | \cdot e^{3x} \qquad [e^{3x} y]' = e^{3x} \sin x$

$$y = e^{-3x} \int e^{3x} \sin x \, dx$$

$$\int \underbrace{e^{3x}}_{u'} \underbrace{\sin x}_{v} dx = \frac{1}{3} e^{3x} \sin x - \int \underbrace{\frac{1}{3} e^{3x}}_{u'} \underbrace{\cos x}_{v} dx$$

$u = \frac{1}{3} e^{3x}$

$$= \frac{e^{3x}}{3} \sin x - \left[\frac{1}{9} e^{3x} \cos x + \frac{1}{9} \int e^{3x} \sin x \, dx \right] \Rightarrow$$

$$\int e^{3x} \sin x \, dx \left(1 + \frac{1}{9}\right) = \frac{e^{3x}}{3} \left(\sin x - \frac{\cos x}{3} \right)$$

$$\int e^{3x} \sin x \, dx = \frac{9}{10} \frac{e^{3x}}{3} \left(\sin x - \frac{\cos x}{3} \right) + C$$

$$y(x) = \frac{3}{10} \sin x - \frac{1}{10} \cos x + Ce^{-3x}$$

$$\frac{3}{10} = y\left(\frac{\pi}{2}\right) = \frac{3}{10} \cdot 1 - \frac{1}{10} \cdot 0 + Ce^{-3\pi/2} \Rightarrow C = 0$$

solution $\quad y(x) = \frac{3}{10} \sin x - \frac{1}{10} \cos x$

c) $y' - y\left(1 + \frac{3}{x}\right) = x + 2 \quad | \cdot e^{\int 1 + 3/x \, dx} = e^{x + 3 \ln x} = e^x \cdot x^3$

$$[y \cdot e^x \cdot x^3]' = (x+2) x^3 e^x$$

$$y \cdot e^x \cdot x^3 = \int x^4 e^x dx + 2 \int x^3 e^x dx$$ integration by parts (next page)

$$e^x \left[x^4 - \frac{4!}{3!} x^3 + \frac{4!}{2!} x^2 - \frac{4!}{1!} x + 4! \right]$$

$$+ e^x \left[2x^3 - 2 \cdot \frac{3!}{2!} x^2 + 2 \frac{3!}{1!} x - 2 \cdot 3! \right] + C$$

$$e^x \left[x^4 + (-4+2) x^3 + (4 \cdot 3 - 2 \cdot 3) x^2 + (-4 \cdot 3 \cdot 2 + 2 \cdot 3 \cdot 2) x + (4! - 2 \cdot 3!) \right] + C$$

$$\int \underset{v}{x^4}\underset{u'}{e^x}\,dx = e^x x^4 - 4\int x^3 e^x dx = x^4e^x - 4x^3e^x + 4\cdot 3x^2e^x - 4\cdot 3\cdot 2\,xe^x + 4\cdot 3\cdot 2\cdot 1\,e^x + c$$

$$\int x^3 e^x dx = x^3e^x - 3\int x^2 e^x dx = x^3e^x - 3x^2e^x + 3!\,xe^x - 3!\,e^x + c$$

$$\int x^2 e^x dx = x^2e^x - 2\int xe^x dx = x^2e^x - 2xe^x + 2e^x + c$$

$$\int x e^x dx = xe^x - \int e^x dx = xe^x - e^x + c$$

$$y = \frac{1}{\cancel{e^x}\cdot x^3}\cdot\left[\cancel{e^x}\left[x^4 - 2x^3 + 6x^2 - 12x + 12\right] + c\right] =$$

$$= x - 2 + \frac{6}{x} - \frac{12}{x^2} + \frac{12}{x^3} + \frac{c}{e^x x^3}$$

$$y(1) = 1 - 2 + 6 - \cancel{12} + \cancel{12} + \frac{c}{e} = e - 1 \Rightarrow c = e(e-6)$$

$$\boxed{y = x - 2 + \frac{6}{x} - \frac{12}{x^2} + \frac{12}{x^3} + \frac{(e^2-6e)}{e^x x^3}}$$

$\boxed{2}$

$$y' + \frac{y}{3} = \frac{1-2x}{3}y^4$$

$$u = \frac{1}{y^{4-1}} = y^{-3}\ ;\ u' = -3y^{-4}y' = -\frac{3}{y^4}\left(\frac{1-2x}{3}y^4 - \frac{y}{3}\right) =$$

$$y = \frac{1}{\sqrt[3]{u}}$$

$$2x - 1 + y^{-3} = 2x - 1 + u$$

$$u' - u = 2x - 1$$

$$(ue^{-x})' = (2x-1)e^{-x}$$

$$ue^{-x} = 2\int xe^{-x}dx - \int e^{-x}dx$$

$$\boxed{\begin{aligned}\int \underset{u}{x}\underset{v'}{e^{-x}}dx &= -xe^{-x} + \int e^{-x}dx \\ v = -e^{-x} \quad &= -e^{-x}(x+1)\end{aligned}}$$

$$= -2e^{-x}(x+1) + e^{-x} + c =$$

$$= -e^{-x}(2x+2-1) + c \Rightarrow u = -e^x e^{-x}(2x+1) + ce^x$$

$$\underline{u = -1 - 2x + ce^x}$$

$$y = \frac{1}{\sqrt[3]{ce^x - 1 - 2x}}$$

$\boxed{3}$

$$\begin{cases} m v' = W - B - kv \\ v(0) = 0 \end{cases}$$

$$v' + \frac{k}{m}v = \frac{W}{m} - \frac{B}{m} \quad \Big|\cdot e^{k/m\,x}$$

$$\left(v e^{k/m\,x}\right)' = \left(\frac{W}{m} - \frac{B}{m}\right)e^{k/m\,x}$$

$$v = \cancel{e^{-\frac{k}{m}x}}\left(\frac{W-B}{m}\,\frac{m}{k}\right)\cancel{e^{k/m\,x}} + c\,e^{-k/m\,x}$$

$$0 = v(0) = \frac{W-B}{k} + c \Rightarrow c = \frac{B-W}{k}$$

$$v < V_c = 12\ \text{m/sec}$$

$$\frac{W-B}{k}\left(1 - e^{-k/m\,x}\right) \le 12$$

weight : $W = m \cdot g$ $\boxed{g = 9.89 \ m/sec^2}$

$m = g/W$

x – time in seconds

at $t = x$

Solve $\left(1 - e^{-\frac{kW}{g}t}\right) < \frac{12k}{W-B}$

$e^{-\frac{kW}{g}t} > 1 - \frac{12k}{W-B}$ $\left(\begin{matrix} k>0 \\ W-B>0 \end{matrix}\right)$

$-\frac{kW}{g}t > \ln \frac{W-B-12k}{W-B}$

$t < -\frac{g}{kW} \ln \frac{W-B-12k}{W-B}$

Thus $\boxed{t_{crit} = -\frac{g}{kW} \ln \frac{W-B-12k}{W-B}}$

y(t) position (depth) $y = \int \frac{W-B}{k}\left(1 - e^{-k/m\, t}\right) dt =$

$= \frac{W-B}{k} t + \frac{W-B}{m} e^{-k/m\, t} + D$

use $y(0) = 0$ start at surface

$0 = \frac{W-B}{m} + D \Rightarrow D = -\frac{(W-B)W}{g}$

$y = \frac{W-B}{k} \cdot t + \frac{(W-B)W}{g} e^{-\frac{kW}{g}t} - \frac{W-B}{g} W$

critical depth

$\boxed{y_{crit} = \frac{W-B}{k} t_{crit} + \frac{(W-B)W}{g} e^{-\frac{kW}{g}t_{crit}} - \frac{W-B}{g} W}$

substitute data

$W = 2254 \ \frac{kg\,m}{sec^2}$ $g = 9.89 \ m/sec^2$ (in KMS) SI units.

$B = 2090 \ \frac{kg\,m}{sec^2}$ $K = 0.637 \ kg/sec$

$\boxed{4}$ $y' = x^3(y-x)^2 + y/x$

$w = y - x$; $y' = w' + 1$

$\frac{w}{x} + 1 = y/x$

$w' + 1 = x^3 w^2 + \frac{w}{x} \cancel{+1}$

$w' - \frac{1}{x} w = x^3 w^2$

Bernoulli eq.

$u = \frac{1}{w^{2-1}} = \frac{1}{w}$ $\quad u' = -\frac{1}{w^2} w' =$

$= -\frac{1}{w^2}\left(\frac{1}{x} w + x^3 w^2\right) =$

$= -\frac{1}{x}\frac{1}{w} - x^3 = -\frac{1}{x} u - x^3$

$u' + \frac{1}{x} u = -x^3 \quad \Big| \cdot e^{\ln x} = x$

$u'x + u = -x^4$

$(ux)' = -x^4$

$u = \frac{1}{x}\left(-\frac{x^5}{5} + c\right) = -\frac{x^4}{5} + \frac{c}{x}$

$W = \frac{1}{u} = \frac{5x}{5C - x^5} = \frac{5x}{D - x^5}$

$\underline{y(x) = w(x) + x = \frac{5x}{D - x^5} + x}$

$\boxed{5}$ $\underbrace{(Ax + By)}_{M(x,y)} dx + \underbrace{(Cx + Dy)}_{N(x,y)} dy = 0$ $\quad M$

Equation exact when $M_y = N_x$ or $B = C$

Take $B = C$, $M(x,y) = \frac{\partial u(x,y)}{\partial x}$ $\quad N(x,y) = \frac{\partial u(x,y)}{\partial y}$

with $u(x,y) = r$ const implicit solution.

$u(x,y) = \int M(x,y)\,dx + k(y)$

$u_y = \left(\frac{Ax^2}{2} + Bxy + k(y)\right)_y = \cancel{Bx} + k'(y) = N(x,y) = Cx + Dy = \cancel{Bx} + Dy$

$\Rightarrow k'(y) = Dy$

Thus $u(x,y) = \frac{Ax^2}{2} + Bxy + \frac{Dy^2}{2} + q$ $\quad k(y) = \frac{Dy^2}{2} + q$ ← const.

implicit solution $\quad \frac{Ax^2}{2} + Bxy + \frac{Dy^2}{2} = \tilde{C} = r - q$ const.

You can use quadratic formula to obtain explicit expressions for y.

2.6 HW 6

2.6.1 problems description

Homework #3 | ODEs MATH 601 oct 23, 2018.

1. Show that the initial value problem $\begin{cases} xy' = 4y \\ y(0) = 1 \end{cases}$ has no solution. Does this contradict the existence theorem we considered?

2. State the definition of a Lipschitz condition. Show that $\begin{cases} y' = |\sin y| + x \\ y(x_0) = y_0 \end{cases}$ satisfies the Lipschitz condition with Lipschitz constant 1 on the whole x-y plane.

3. Apply the Picard iteration to $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$ and show that the aproximation approaches the exact solution $y = e^x$.

4. Solve the following initial value problems

a) $\begin{cases} y'' + 2y' + y = 0 \\ y(0) = 1, \ y'(0) = 0 \end{cases}$

b) $\begin{cases} y'' + 9y = 0 \\ y(0) = 4, \ y'(0) = -6 \end{cases}$

5. Solve the following problem (catenary)

$\begin{cases} y'' = \sqrt{1 + (y')^2} \\ y(-1) = y(1) = 0 \end{cases}$ and plot the solution.

2.6.2 Key solution

Homework #3 solutions Math 601 (1)

$\boxed{1}$ $\begin{cases} xy' = 4y \\ y(0) = 1 \end{cases}$ $\qquad xy' = 4y$ separable eq.

$\frac{1}{y} y' = \frac{4}{x}$ integrate

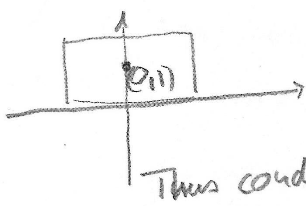
$\ln|y| = 4\ln|x| + c$

$|y| = |x|^4 e^c$

$y = Cx^4$ general solution

$1 = y(0) = C \cdot 0 = 0$ no such solution.

$y' = f(x,y) = \frac{4y}{x}$ not continuous at $x=0$ i.e not continuous in any R rectangle around the $(x,y) = (0,1)$ point in the x-y plane



Thus conditions of the existence theorem are not satisfied.

$\boxed{2}$ $y' = f(x,y) = |\sin y| + x$

Lipschitz continuity for $f(x,y)$: $|f(x,y_2) - f(x,y_1)| \leq L|y_2 - y_1|$ for L constant

$$|f(x,y_2) - f(x,y_1)| = \big||\sin y_2| \cancel{+x} - |\sin y_1| \cancel{-x}\big| \leq |\sin y_2 - \sin y_1| \underset{\text{Mean value T.}}{\leq} |\cos z| \cdot |y_2 - y_1| \leq 1 \cdot |y_2 - y_1|$$

(Note $|a| = |a-b+b| \leq |a-b| + |b| \Rightarrow |a| - |b| \leq |a-b|$

& $|b| = |b-a+a| \leq |b-a| + |a| \Rightarrow |b| - |a| \leq |b-a| = |a-b|$

so $\big||a| - |b|\big| \leq |a-b|$)

(with z in between y_2 and y_1) $\overset{L=1}{=}$

(2)

$\boxed{3}$ Picard iteration:

$$\begin{cases} y' = y = f(x,y) \\ y(0) = 1 \end{cases}$$

$$\Updownarrow$$

$$y(x) = y(0) + \int_0^x f(s,y)\,ds = 1 + \int_0^x y\,ds$$

$$\phi_0(x) = 1$$

$$\phi_1(x) = 1 + \int_0^x 1\,ds = 1 + x$$

$$\phi_2(x) = 1 + \int_0^x 1 + s\,ds = 1 + x + \frac{x^2}{2}$$

$$\phi_3(x) = 1 + \int_0^x 1 + s + \frac{s^2}{2}\,ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3}$$

$$\phi_4(x) = 1 + \int_0^x \phi_3(s)\,ds = 1 + \int_0^x 1 + s + \frac{s^2}{2} + \frac{s^3}{2\cdot 3}\,ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4}$$

By induction

If $\phi_{n-1} = 1 + x + \cdots + \frac{x^{n-1}}{(n-1)!}$

$$\phi_n(x) = 1 + \int_0^x 1 + s + \frac{s^2}{2} + \cdots \frac{s^{n-1}}{(n-1)!}\,ds = 1 + x + \frac{x^2}{2} + \cdots \frac{x^n}{n\cdot(n-1)!}$$

$$\vdots \quad \phi_n \to 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$$

$\boxed{5}$ $u = y'$ $\quad u' = \sqrt{1-u^2}$ $\quad \frac{u'}{\sqrt{1-u^2}} = 1$ $\quad \sin^{-1} u = x + c$

$u = \sin(x+c)$

$$u = y' = \sin(x+c) \Rightarrow$$

$$y = -\cos(x+c) + d$$

solves $y'' = \sqrt{1-(y')^2}$

$y = y(x)$

correct problem #5 $\Rightarrow$ last page

$$\begin{cases} y'' = \sqrt{1+(y')^2} \\ y(-1) = y(1) = 0 \end{cases}$$

$$\begin{cases} y'' = \sqrt{1-(y')^2} \\ y(-1) = y(1) = 0 \end{cases}$$

$\boxed{4}$ a) $\begin{cases} y'' + 2y' + y = 0 \\ y(0) = 1, \ y'(0) = 0 \end{cases}$ $\longleftarrow$ constant coeff. homogeneous, lin, second ord.

look for linearly independent solutions in the form

$$y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x}$$
$$y'' = \lambda^2 e^{\lambda x}$$

$$y'' + 2y' + y = e^{\lambda x}(\lambda^2 + 2\lambda + 1) = 0$$
$$\Rightarrow (\lambda^2 + 2\lambda + 1) = (\lambda + 1)^2 = 0$$
$$\Rightarrow \lambda = -1$$
double root

$y_1 = e^{-x}$ is solution, second independent solution can be obtained as

$$y_2 = xe^{-x} = u(x)\, y_1(x)$$

general solution: $y = c_1 y_1 + c_2 y_2$

$y_2 = u y_1$

$(uy_1)'' + 2(uy_1)' + uy_1 = 0$

$u\underbrace{(y_1'' + 2y_1' + y_1)}_{0} + u''y_1 + 2u'y_1' + 2u'y_1 = 0$

$\frac{u''}{u'} = \frac{U'}{U} = -\frac{2y_1'}{y_1} - 2 = 0$ $\quad \left(\frac{y_1'}{y_1} = \frac{-e^{-x}}{e^{-x}} = -1\right)$

$\ln|U| = c$

$U = C = \pm e^c$

$u = \int u = Cx + D$

take $C = 1, D = 0 \Rightarrow u = x$

Initial conditions:

$$\left.\begin{array}{l} 1 = y(0) = c_1 \\ \quad \Downarrow \\ 0 = y'(0) = -e^0 + c_2 e^0 + 0 \\ \Rightarrow \quad c_2 = e^0 = 1 \end{array}\right\} \Leftarrow \begin{cases} y = c_1 e^{-x} + c_2 x e^{-x} \\ y' = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x} \end{cases}$$

thus $\underline{y(x) = e^{x} + xe^{x}}$

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b) $\begin{cases} y'' + 9y = 0 \\ y(0) = 4, \ y'(0) = -6 \end{cases}$ if $y = e^{\lambda x}$ then $y'' + 9y = e^{\lambda x}(\lambda^2 + 9) = 0$

$\lambda^2 + 9 = 0 \quad \lambda = \pm 3i$

general real solution

$c_1 = y(0) = 4 \Leftarrow y(x) = e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$

$3c_2 = y'(0) = -6 \Leftarrow y'(x) = -3c_1 \sin 3x + 3c_2 \cos 3x$

$\Rightarrow c_2 = -2$ thus, $\underline{y(x) = 4\cos 3x - 2\cos 3x}$

5 (4)

$$\begin{cases} y'' = \sqrt{1+(y')^2} \; (>0) \\ y(-1) = y(1) = 0 \end{cases} \qquad u = y'$$

"Second order boundary value problem"

$$\frac{u'}{\sqrt{1+u^2}} = 1$$

$$\int \frac{1}{\sqrt{1+u^2}} \boxed{u' dx}^{du} = x + c$$

integrate change of var

$(\theta \neq x)$ $u = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$du = \sec^2\theta \, d\theta$;

$\sqrt{1+u^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta}$

$= |\sec\theta| = \sec\theta$

$$\int \frac{1}{\sqrt{1+u^2}} du = \int \frac{1}{\sec\theta} \cdot \sec^2\theta \, d\theta =$$

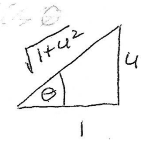
$$= \int \sec\theta \, d\theta = \int \sec\theta \, \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} d\theta =$$

$$= \int \frac{\sec^2\theta + \tan\theta \cdot \sec\theta}{\tan\theta + \sec\theta} d\theta = \int \frac{dv}{v} = \ln v + \tilde{c}$$

$$= \ln|\sec\theta + \tan\theta| + \tilde{c}$$

$v = \tan\theta + \sec\theta$

$dv = \sec^2\theta + \sec\theta \cdot \tan\theta \, d\theta$



$\tan\theta = \frac{u}{1}$

$\sec\theta = \sqrt{1+u^2}$

$$\int \frac{1}{\sqrt{1+u^2}} du = \ln|u + \sqrt{1+u^2}| + \tilde{c} = x + c$$

$$u + \sqrt{1+u^2} = C e^x$$

(*) Note also, that

$u' = y'' = \sqrt{1+(y')^2} > 0$

$u' = \frac{ce^x}{2} + \frac{1}{2c} \cdot e^{-x} > 0$

$\Rightarrow c > 0$

$$\sqrt{1+u^2} = ce^x - u$$

$$1 + \cancel{u^2} = c^2 e^{2x} - 2ce^x u + \cancel{u^2}$$

$$\Rightarrow u = \frac{c^2 e^{2x} - 1}{2ce^x} = \frac{ce^x}{2} - \frac{1}{2c} e^{-x}$$

$$y = \int u \, dx = \frac{ce^x}{2} + \frac{e^{-x}}{2c} + d$$

general sol for y.

$$\left.\begin{aligned} y(1) &= \frac{ce}{2} + \frac{1}{2}\frac{1}{ce} + d = 0 \\ y(-1) &= \frac{c}{2e} + \frac{e}{2c} + d = 0 \end{aligned}\right\} \quad \frac{ce}{2} + \frac{1}{2}\frac{1}{ce} - \frac{c}{2e} - \frac{e}{2c} = 0 \quad \cdot 2e$$

$$e^2c^2 + 1 - c^2 - e^2 = 0$$

$$\Rightarrow c^2(e^2 - 1) = e^2 - 1 \Rightarrow$$

$$\Rightarrow c = \pm 1 \quad (*) \text{ implies } c = 1$$

If $c = 1 \Rightarrow d = -\left(\frac{e + 1/e}{2}\right)$ "$= -\cosh 1$" thus $y(x) := \cosh x - \cosh 1$

2.7 HW 7

2.7.1 problems description

MATH 601 HW #4 HW # 7. 4^{th} on ODE

1 Find the general solution of

a) $16y'' - 8y' + 5y = 0$ b) $y'' + 4y' + (4 - w^2)\,y = 0$

2 Find the solution of the ODE that satisfy the given conditions:

a) $y'' + 0.4y' + 0.29y = 0$ $y(0) = 1$; $y'(0) = -1.2$

b) $y'' + 2y' + 2y = 0$ $y(0) = 1$, $y(\frac{\pi}{2}) = 0$

3 Can you make a harmonic oscillator

$m\,y'' + ky = 0$

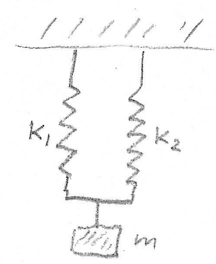
move faster by giving the body greater initial push?

4 What are the frequencies of vibration of a 5 kg mass vibrating in vacuum

a) on a spring with $k_1 = 20\,\frac{N}{m}$ spring constant

b) on a spring with $k_2 = 45\,\frac{N}{m}$ spring constant

c) on the two springs connected parallelly?



2.7.2 Key solution

Math 601 HW #4 Solution

$\boxed{1}$ a) $16y''-8y'+5y=0$ characteristic eq. $16r^2-8r+5=0$

$$r=\frac{8\pm\sqrt{64-5\cdot 64}}{2\cdot 16}=\frac{1}{4}\pm\frac{i}{2}$$

$$y(x)=e^{x/4}\left(C_1\cos\frac{x}{2}+C_2\sin x/2\right)$$

b) $y''+4y'+(4-\omega^2)y=0$ char eq $r^2+4r+4-\omega^2=0$

$\omega>0$ $$r=\frac{-4\pm\sqrt{16-16+4\omega^2}}{2}=-2\pm\omega$$

solution $y(x)=e^{-2x}(C_1\cos\omega x+C_2\sin\omega x)$

or $\omega=0$ $y(x)=C_1e^{2x}+C_2xe^{2x}$

$\boxed{2}$ a) $\begin{cases} y''+0.4y'+0.29y=0 \\ y(0)=1\ ,\ y'(0)=1.2 \end{cases}$ $r^2+\frac{4}{10}r+\frac{29}{100}=0$

$$r=\frac{-0.4\pm\sqrt{0.16-1.16}}{2}=-0.2\pm i/2$$

$$y(x)=e^{-\frac{2x}{10}}\left(C_1\cos\frac{x}{2}+C_2\sin\frac{x}{2}\right)$$

$$y'(x)=-\frac{2}{10}e^{-\frac{2x}{10}}\left(C_1\cos\frac{x}{2}+C_2\sin\frac{x}{2}\right)+\frac{1}{2}e^{-\frac{2x}{10}}\left(-C_1\sin\frac{x}{2}+C_2\cos\frac{x}{2}\right)$$

$y(0)=C_1=1$

$y'(0)=-\frac{2}{10}C_1+\frac{1}{2}C_2=-\frac{2}{10}+\frac{C_2}{2}=-1.2 \Rightarrow C_2=-2$

$$y(x)=e^{-\frac{2x}{10}}\left(\cos\frac{x}{2}-2\cdot\sin x/2\right)$$

b) $\begin{cases} y''+2y'+2y=0 \\ y(0)=1,\ y(\pi/2)=0 \end{cases}$ $r^2+2r+2=0$ $r=\frac{-2\pm\sqrt{4-8}}{2}=-1\pm i$

$y(0)=C_1=1$ $y(x)=e^{-x}(C_1\cos x+C_2\sin x)$

$y(\pi/2)=e^{-\pi/2}(\underbrace{\cos\pi/2}_{0}+C_2\sin\pi/2)=C_2e^{-\pi/2}=0 \Rightarrow C_2=0$

$y(x)=e^{-x}\cdot\cos x$

$\boxed{3}$

$$\begin{cases} my'' + ky = 0 \\ y(0) = y_0,\ y'(0) = v_0 \end{cases}$$

$mr^2 = -k \qquad r = \pm i\sqrt{\frac{k}{m}}$

$$y(x) = c_1 \cos\sqrt{\tfrac{k}{m}}\,x + c_2 \sin\sqrt{\tfrac{k}{m}}\,x$$

$$y'(x) = \sqrt{\tfrac{k}{m}}\left[c_1\left(-\sin\sqrt{\tfrac{k}{m}}\,x\right) + c_2 \cos\sqrt{\tfrac{k}{m}}\,x\right]$$

$$y(0) = c_1 = y_0$$

$$y'(0) = \sqrt{\tfrac{k}{m}}\,c_2 = v_0 \Rightarrow c_2 = v_0\sqrt{\tfrac{m}{k}}$$

$$y(x) = y_0 \cos\sqrt{\tfrac{k}{m}}\,x + v_0\sqrt{\tfrac{m}{k}}\sin\sqrt{\tfrac{k}{m}}\,x$$

$$y'(x) = \underset{A}{-\sqrt{\tfrac{k}{m}}\,y_0}\sin\sqrt{\tfrac{k}{m}}\,x + \underset{B}{v_0}\cos\sqrt{\tfrac{k}{m}}\,x = \sqrt{\tfrac{k}{m}y_0^2 + v_0^2}\,\cos\left(\sqrt{\tfrac{k}{m}}\,x - \delta\right)$$

velocity

$$\delta = \tan^{-1}\left(-\frac{y_0}{v_0}\sqrt{\tfrac{k}{m}}\right)$$

The magnitude of the velocity $\sqrt{\frac{k}{m}y_0^2 + v_0^2}$ increases as v_0 increases.

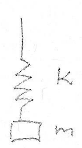
The period of the oscillation is not affected by changing v_0.

$\boxed{4}$

$my'' = -ky - cy' \qquad k$ spring coefficient

$my'' + ky = 0 \qquad c = 0$ damping coefficient

in vacuum no air resistance



Solution as in $\boxed{3}$ is

$$y(x) = c_1 \cos\sqrt{\tfrac{k}{m}}\,x + c_2 \sin\sqrt{\tfrac{k}{m}}\,x$$

a) $k_1 = 20\ \frac{kg}{sec^2}$, $m = 5$, $y(x)$ has period

$$T = 2\pi\sqrt{\frac{m}{k_1}} = \frac{2\pi}{2} = \pi \text{ sec}$$

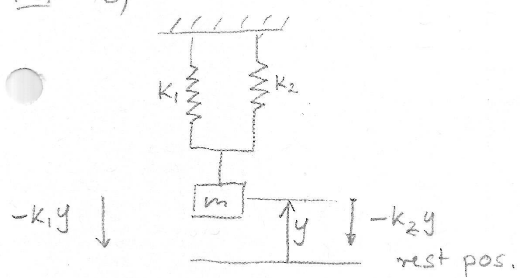
and frequency $\sqrt{\frac{k_1}{m}}\,\frac{1}{2\pi} = \frac{1}{\pi}\,\frac{1}{sec}$

b) $k_2 = 45\ \frac{kg}{sec^2}$, $m = 5$, $y(x)$ has period

$$T = 2\pi\sqrt{\frac{m}{k_2}} = \frac{2\pi}{3} \text{ sec}$$

oscillations per second. $\longrightarrow$ and frequency $\frac{3}{2\pi}\ \frac{1}{sec}$

$\boxed{4}$ c,



Spring force

from spring1 / from spring 2

$$my'' = -k_1 y - k_2 y = -(k_1 + k_2) y$$

$$my'' + ky = 0 \qquad k = k_1 + k_2$$

$$= 65 \ \frac{kg}{sec^2}$$

has period $2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\sqrt{11}}$ sec,

frequency $\frac{\sqrt{13}}{2\pi}$ $\frac{1}{sec}$

2.8 HW 8

2.8.1 problems description

Homework MATH 601 | HW 8 Nov 16, 2018. First on Laplace

1. Find the Laplace transform of $f(t)=\begin{cases} t & \text{if } 0\le t<1 \\ 1 & \text{if } 1\le t<2 \\ 0 & \text{if } t>2\end{cases}$

2. —— " —— $g(t)=e^{-\alpha t}\cos\beta t$.

3. Find the inverse Laplace transform of $4/(s^2-2s-3)$

4. —— " —— $\sum_{k=1}^{5}\frac{a_k}{s^2+k^2}$

Using Laplace transform solve the IVPs

5. (IVP) $\begin{cases} y'+3y=10\sin t \\ y(0)=0\end{cases}$ (Note: solution can be obtained by integrating factor)

6. (IVP) $\begin{cases} y''+y=2\cos t \\ y(0)=6, \quad y'(0)=0\end{cases}$

7. Sketch $f(t)=(t-1)\,u(t-1)$ and find its Laplace transf.

8. $f(t)=\begin{cases} e^t & \text{if } t\in(0,1) \\ 0 & \text{otherwise}\end{cases}$ Use the shifting property to compute $\mathcal{L}[f](s)$

9. Find the inverse Laplace transform of $3\,\frac{1-e^{-\pi s}}{s^2+9}$

10. Solve the IVP using Laplace transform

(IVP) $\begin{cases} y''+3y'+2y=g(t)=\begin{cases} 4t & \text{if } t\in(0,1) \\ 8 & \text{if } t>1\end{cases} \\ y(0)=y'(0)=0\end{cases}$

11. Find a) $\mathcal{L}(te^{-t}\cos t)$ b) $\mathcal{L}^{-1}\left(\frac{s}{(s^2-9)^2}\right)$

12. Compute a) $1*\sin\omega t$ b) e^t*e^{-t}

13. Find the solution by applying convolution (over)

$$\begin{cases} y'' + 4y = \begin{cases} 1 & \text{if } t \in (0,1) \\ 0 & \text{if } t > 1 \end{cases} \\ y(0) = 1, \; y'(0) = 0 \end{cases}$$

$\boxed{14}$ Solve the IVPs by means of Laplace transform:

a) $\begin{cases} y_1' = 6y_1 + 9y_2 \\ y_2' = y_1 + 6y_2 \end{cases}$ $\quad y_1(0) = -3$, $\; y_2(0) = -3$

b) $\begin{cases} y_1'' + y_2 = -5\cos 2t \\ y_2'' + y_1 = 5\cos 2t \end{cases}$ $\quad y_1(0) = y_1'(0) = 1$, $\; y_2(0) = -y_2'(0) = -1$

2.8.2 Key solution

MATH 601 HW #5 Solutions (1)

Laplace tr.

[1] $$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\cdot t\,dt + \int_1^2 e^{-st}\,dt =$$

$$= -\frac{1}{s}e^{-st}\cdot t\Big|_{t=0}^{1} + \frac{1}{s}\int_0^1 e^{-st}\,dt - \frac{1}{s}e^{-st}\Big|_1^2$$

$$-\frac{1}{s}e^{-s} + 0 + \frac{1}{s}\int_0^1 e^{-st}\,dt - \frac{1}{s}e^{-2t} + \cancel{\frac{1}{s}e^{-s}} =$$

$$-\frac{1}{s}e^{-2t} + \frac{1}{s}\left[\frac{-1}{s}e^{-st}\Big|_0^1\right] = -\frac{1}{s^2}e^{-s} + \frac{1}{s^2} - \frac{1}{s}e^{-2t}$$

$$\frac{1}{s^2}(1-e^{-s}) - \frac{1}{s}e^{-2t}$$

[2] $$e^{-\alpha t}\cos\beta t = \mathcal{L}(\cos\beta t)(s+\alpha) = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

[3] $$\mathcal{L}^{-1}\left(\frac{4}{s^2+2s-3}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{t} - e^{-3t}$$

$$\frac{4}{s^2+2s-3} = \frac{4}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3} = \frac{As+3A+Bs-B}{(s-1)(s-3)}$$

$$\parallel$$

$$\frac{1}{s-1} - \frac{1}{s+3}$$

$$\Rightarrow \begin{cases} A+B=0 & \Rightarrow A=-B \\ 3A-B=4 & \Rightarrow 4A=4 \Rightarrow A=1 \end{cases}$$

$$\Rightarrow B=-1$$

[4] $$\mathcal{L}^{-1}\left(\sum_{k=1}^{5} \frac{a_k}{s+k^2}\right) = \sum_{k=1}^{5} \frac{a_k}{k}\mathcal{L}^{-1}\left(\frac{k}{s+k^2}\right) = \sum_{k=1}^{5} \frac{a_k}{k}\sin kt =$$

$$= a_1\sin t + \frac{a_2}{2}\sin 2t + \frac{a_3}{3}\sin 3t + \frac{a_4}{4}\sin 4t + \frac{a_5}{5}\sin 5t$$

2

5 (IVP) $\begin{cases} y' + 3y = 10\sin t \\ y(0) = 0 \end{cases}$ $\qquad \mathcal{L}(y') = sY - y(0) = sY$

Transforming the equation

$$sY + 3Y = 10\frac{1}{s^2+1} \Rightarrow Y(s) = \frac{10}{(s+3)(s^2+1)} =$$

$$\frac{A}{s+3} + \frac{Bs+C}{s^2+1} =$$

$$= \frac{As^2 + A + Bs^2 + Cs + 3Bs + 3C}{(s+3)(s^2+1)}$$

$$\begin{cases} A+B=0 & \Rightarrow A=-B \\ C+3B=0 & C=3A \\ A+3C=10 & \frac{C}{3}+3C=10 \end{cases} \Leftarrow$$

$10C = 30$

$\boxed{C=3,\ A=1,\ B=-1}$

Thus $Y(s) = \frac{1}{s+3} - \frac{s}{s^2+1} + \frac{3}{s^2+1}$

$\mathcal{L}^{-1} \Downarrow$

$$\underline{y(t) = e^{-3t} - \cos t + 3\sin t}$$

Indeed solution

$$\begin{cases} y' + 3y = \left[-\cancel{3e^{-3t}} + \sin t + \cancel{3\cos t}\right] + \left[\cancel{3y^{-3t}} - \cancel{3\cos t} + 9\sin t\right] = 10\sin t \\ y(0) = 1 - 1 + 0 = 0 \end{cases}$$

6 $\begin{cases} y'' + y = 2\cos t \\ y(0) = 6 \\ y'(0) = 0 \end{cases} \Rightarrow \mathcal{L}$ $\qquad s^2Y - s\,y(0) - y'(0) + Y = 2\frac{s}{s^2+1}$

$$Y = \frac{2s}{(s^2+1)(s^2+1)} + \frac{6s}{s^2+1} =$$

$$\Rightarrow y(t) = 2\overbrace{\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\cdot\frac{1}{s^2+1}\right)}^{\cos t * \sin t} + 6\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= 2\cdot\int_0^t \cos\tau \sin(t-\tau)\,d\tau + 6\cos t = \underline{t\sin t + 6\cos t}$$

3

since

$$\cos t * \sin t = \int_0^t \cos\tau \sin(t-\tau)\,d\tau =$$

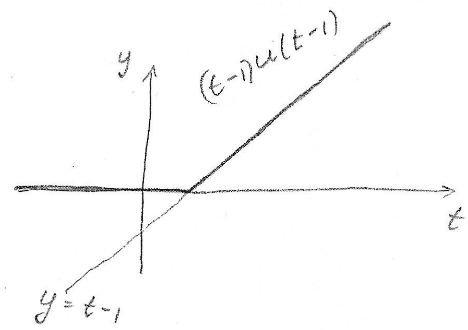
$$= \int_0^t \cos\tau\,(\sin t\cos\tau - \cos t\sin\tau)\,d\tau =$$

$$= \sin t \underbrace{\int_0^t \underbrace{\cos^2\tau}_{\frac{1+\cos 2\tau}{2}}\,d\tau}_{\frac{\tau}{2} + \frac{\sin 2\tau}{4}\Big|_0^t} - \frac{1}{2}\cos t \underbrace{\int_0^t 2\sin\tau\cos\tau\,d\tau}_{\sin^2\tau\Big|_0^t} =$$

$$= \frac{t}{2}\sin t + \cancel{\frac{\sin^2 t\cos t}{2}} - \cancel{\frac{1}{2}\cos t\sin^2 t} = \frac{t}{2}\sin t$$

(4)

$\boxed{7}$ $\mathcal{L}\left[(t-1)\,u(t-1)\right]$

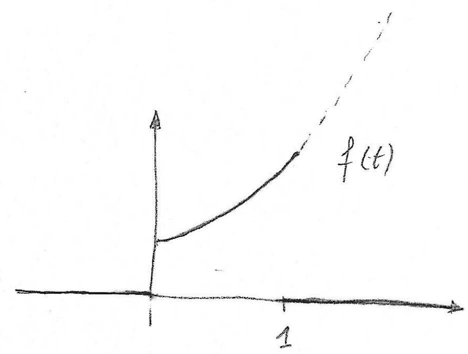


$= e^{-s}\mathcal{L}(t-1) =$

$e^{-s}\left[\mathcal{L}(t)-\mathcal{L}(1)\right] =$

$e^{-s}\left[\frac{1}{s^2}+\frac{1}{s}\right]$

$\boxed{8}$ $f(t)=\begin{cases} e^t & \text{if } t\in(0,1) \\ 0 & \text{otherwise}\end{cases}$



$f(t) = e^t u(t) - e^t u(t-1)$

$\mathcal{L}(f(t)) = \underbrace{\mathcal{L}\left(e^t u(t)\right)}_{\mathcal{L}(e^t)} - \underbrace{\mathcal{L}\left(e^t u(t-1)\right)}_{\mathcal{L}(g(t-1)\,u(t-1)) = e^{-s}\mathcal{L}(g(t))} =$

where $g(t) = e^{(t+1)} = e\cdot e^t$

$\mathcal{L}(f(t)) = \frac{1}{s-1} - e^{-s}\mathcal{L}(e\cdot e^t) = \frac{1}{s-1} - e^{-s}\cdot e\cdot\frac{1}{s-1} = \frac{1-e^{1-s}}{s-1}$

$\boxed{9}$ $\mathcal{L}^{-1}\left(3\,\frac{1-e^{-\pi s}}{s^2+9}\right) = \underbrace{\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)}_{\sin 3t} - \underbrace{\mathcal{L}^{-1}\left[e^{-\pi s}\underbrace{\frac{3}{s^2+9}}_{F(s)}\right]}_{\sin 3(t-\pi)\cdot u(t-\pi)} =$

$f(t) = \sin 3t$

$= \sin 3t - \underbrace{\sin 3(t-\pi)}_{\sin 3t\,\underbrace{\cos(3\pi)}_{-1} - \cos 3t\,\cancel{\sin 3\pi}^{\,0}}\,u(t-\pi) = \sin 3t + \sin 3t\,u(t-\pi)$

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Thus
$$\mathcal{L}^{-1}\left(3\frac{1-e^{-\pi s}}{s^2+9}\right)=\begin{cases}\sin 3t & \text{if } t<\pi\\ 2\sin 3t & \text{if } t\geq\pi\end{cases}$$

$\boxed{10}$
$$\begin{cases} y''+3y'+2y=g(t)=\begin{cases}4t & \text{if } t\in(0,1)\\ 8 & \text{if } t>1\end{cases}\\ y(0)=y'(0)=0\end{cases}$$

$g(t)=4t(1-u(t-1))+8u(t-1)$ if $t\geq 0$

Use Laplace transform:

$$\mathcal{L}(y''+3y'+2y)=\mathcal{L}(g(t))$$

$$[s^2Y-s^2\cancel{y(0)}-s\cancel{y'(0)}]+3[sY-\cancel{y(0)}]+2Y=\mathcal{L}(4t(1-u(t-1)))+\mathcal{L}[8u(t-1)]$$

$$Y(s^2+3s+2)=\mathcal{L}(4t)-\mathcal{L}(4t\,u(t-1))+8\mathcal{L}(u(t-1))$$
$$=\frac{4}{s^2}-4\frac{s+1}{s^2}e^{-s}+\frac{8e^{-s}}{s}$$

$$\mathcal{L}(t\,u(t-1))=\mathcal{L}((t+1-1)u(t-1))$$
$$=e^{-s}\mathcal{L}(t+1)=e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

$$\Rightarrow Y=\frac{4}{s^2(s+1)(s+2)}+e^{-s}\frac{4s-4}{s^2(s+1)(s+2)}$$
$$=\frac{4}{s^2(s+1)(s+2)}+e^{-s}\frac{4}{s(s+1)(s+2)}-e^{-s}\frac{4}{s^2(s+1)(s+2)}$$

Use Partial fraction decomposition.

$$\frac{4}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}=\frac{As^2+3As+2A+Bs^2+2Bs+Cs^2+Cs}{s(s+1)(s+2)}$$

6

$$A+B+C=0$$
$$3A+2B+C=0$$
$$2A=4 \qquad A=2$$

$$B+C=-2$$
$$2B+C=-6$$
$$B=-4 \quad C=2$$

$$\frac{4}{s(s+1)(s+2)} = \frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2}$$

$$\frac{4}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} = \frac{-3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2}$$

$$= \frac{As^3+3As^2+2As + Bs^2+3Bs+2B + Cs^3+2Cs^2 + Ds^3 + Ds^2}{s^2(s+1)(s+2)}$$

$$A + C + D = 0$$
$$3A + B + 2C + D = 0$$
$$2A + 3B = 0$$
$$2B = 4 \Rightarrow B=2$$

$$A+C+D=0$$
$$3A+2C+D=-2$$
$$2A=-6$$
$$A=-3$$

$$C+D=3$$
$$2C+D=-2+9=7$$
$$C=7-3=4$$
$$D=-1$$

$$A=-3, \; B=2, \; C=-4, \; D=1$$

$$Y = -\frac{3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2} + e^{-s}\frac{2}{s} - e^{-s}\frac{4}{s+1} + e^{-s}\frac{2}{s+2}$$
$$+ e^{-s}\frac{3}{s} - e^{-s}\frac{2}{s^2} - e^{-s}\frac{4}{s+1} + e^{-s}\frac{1}{s+2}$$
$$= -\frac{3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2} + e^{-s}\frac{5}{s} - e^{-s}\frac{2}{s^2} - e^{-s}\frac{8}{s+1} + e^{-s}\frac{3}{s+2}$$

$$y(t) = -3 + 2t + 4e^{-t} - e^{-2t} + 5u(t-1) + 2(t-1)u(t-1) - 8e^{(t-1)}u(t-1)$$
$$+3e^{2(t-1)}u(t-1)$$

(8)

$\boxed{11}$

a) $\mathcal{L}(te^{-t}\cos t) = -\frac{d}{ds}\mathcal{L}(e^{-t}\cos t) = -\frac{d}{ds}\mathcal{L}(\cos t)(s+1)$

$$= -\frac{d}{ds}\left[\frac{s+1}{(s+1)^2+1}\right] =$$

$$-\left[\frac{(s+1)^2+1-(s+1)\cdot 2(s+1)}{\left[(s+1)^2+1\right]^2}\right] =$$

$$-\frac{(s+1)^2-2(s+1)^2+1}{\left[(s+1)^2+1\right]^2} = \frac{(s+1)^2-1}{\left[(s+1)^2+1\right]^2}$$

b) $\mathcal{L}^{-1}\left[\frac{s}{(s^2-9)^2}\right] = \mathcal{L}^{-1}\left(\frac{d}{ds}\left[\frac{1}{2(9-s^2)}\right]\right) = -t\,\mathcal{L}^{-1}\left(-\frac{1}{2}\,\frac{1}{(s-3)(s+3)}\right)$

$$\left[\frac{1}{2(9-s^2)}\right]' = \frac{1}{2}\left[-\frac{-2s}{(9-s^2)^2}\right] = \frac{s}{(s^2-9)^2}$$

$$= \frac{t}{2}\mathcal{L}^{-1}\left(\frac{A}{s-3} + \frac{B}{s+3}\right)$$

$$\frac{1}{(s-3)(s+3)} = \frac{As+3A+Bs-3B}{(s-3)(s+3)}$$

$A+B=0$

$3A-3B=1$

$6A=1$ $\quad A=1/6$

$B=-1/6$

$$= \frac{t}{2}\left(\frac{1}{6}e^{3t} - \frac{1}{6}e^{-3t}\right) =$$

$$\frac{t}{12}\left(e^{3t}-e^{-3t}\right)$$

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$\boxed{12}$ compute

a) $$1 * \sin\omega t = \int_0^t 1 \sin\omega(\tau - t)\, d\tau = \int_0^t \sin\omega t \cos\omega\tau - \cos\omega t \sin\omega\tau \, d\tau$$

$$= \sin\omega t \int_0^t \cos\omega\tau \, d\tau - \cos\omega t \int_0^t \sin\omega\tau \, d\tau =$$

$$\sin\omega t \left[\frac{\sin\omega\tau}{\omega} \Big|_0^t \right] - \cos\omega t \left[-\frac{\cos\omega\tau}{\omega} \Big|_0^t \right] =$$

$$\frac{\sin^2\omega t}{\omega} + \frac{\cos^2\omega t}{\omega} - \frac{\cos\omega t}{\omega} = \frac{1-\cos\omega t}{\omega}$$

b) $$e^t * e^{-t} = \int_0^t e^t e^{-(\tau - t)} d\tau = e^{2t} \int_0^t e^{-\tau} d\tau =$$

$$= e^{2t} \left[-e^{-\tau} \Big|_0^t \right] = e^{2t} \left[1 - e^{t} \right] = e^{2t} - e^{3t}$$

$\boxed{13}$ $$\begin{cases} y'' + 4y = \begin{cases} 1 & \text{if } t \in (0,1) \\ 0 & \text{if } t > 0 \end{cases} = (1 - u(t-1)) \quad \text{when } t \geq 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Laplace tr. $$s^2 Y - s\,y(0) - y'(0) - 4Y = \mathcal{L}(1) - \mathcal{L}(u(t-1))$$
$$= \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\Rightarrow Y = \frac{1}{s^2 - 4} \left[s + \frac{1}{s} - \frac{e^{-s}}{s} \right]$$

$$= \frac{s}{(s-2)(s+2)} + \frac{1}{s(s-2)(s+2)} - e^{-s} \frac{1}{s(s-2)(s+2)}$$

$$= \frac{1}{2}\frac{1}{s-2} + \frac{1}{2}\frac{1}{s+2} - \frac{1}{4}\frac{1}{s} + \frac{1}{8}\frac{1}{s-2} + \frac{1}{8}\frac{1}{s+2} + \frac{1}{4} e^{-s} \frac{1}{s} - \frac{1}{8} e^{-s} \frac{1}{s-2} - \frac{1}{8} e^{-s} \frac{1}{s+2} = ^*$$

(11)

$$\frac{s}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2} = \frac{As+2A+Bs-2B}{(s-2)(s+2)} = \frac{1}{2}\left[\frac{1}{s-2} + \frac{1}{s+2}\right]$$

$A+B=1 \qquad 2A=1 \qquad A=B=\frac{1}{2}$

$2(A-B)=0 \Rightarrow A=B$

$$\frac{1}{s(s-2)(s+2)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2} = \frac{As^2-4A+Bs^2+2Bs+Cs^2-2Cs}{s(s-2)(s+2)}$$

$$= -\frac{1}{4}\frac{1}{s} + \frac{1}{8}\frac{1}{s-2} + \frac{1}{8}\frac{1}{s+2}$$

$A+B+C=0 \qquad 2B=\frac{1}{4} \qquad B=C=\frac{1}{8}$

$2(B-C)=0 \qquad B=C$

$-4A=1 \qquad A=-\frac{1}{4}$

$$Y = -\frac{1}{4}\frac{1}{s} + \frac{5}{8}\frac{1}{s-2} + \frac{5}{8}\frac{1}{s+2} + \frac{1}{4}e^{-s}\frac{1}{s} - \frac{1}{8}e^{-s}\frac{1}{s-2} - \frac{1}{8}e^{-s}\frac{1}{s+2}$$

$$y(t) = -\frac{1}{4} + \frac{5}{8}e^{2t} + \frac{5}{8}e^{-2t} + \frac{1}{4}u(t-1) - \frac{1}{8}e^{2(t-1)}u(t-1) - \frac{1}{8}e^{-2(t-1)}u(t-1)$$

(12)

$\boxed{14}$ a)

$$y_1' = 6y_1 + 9y_2 \qquad y_1(0) = -3$$
$$y_2' = y_1 + 6y_2 \qquad y_2(0) = -3$$

$$sY_1 - y_1(0) = 6Y_1 - 9Y_2$$
$$sY_2 - y_2(0) = Y_1 + 6Y_2$$
$$\Rightarrow \begin{bmatrix} 6-s & -9 \\ 1 & 6-s \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$Y_1 = \frac{1}{(6-s)^2+9}\left[3(6-s)+27\right]$$

$$Y_2 = \frac{1}{(6-s)^2+9}\left[3(6-s)-3\right]$$

$$Y_1 = -3\,\frac{s-6}{(s-6)^2+9} + 9\,\frac{3}{(s-6)^2+9}$$

$$\underline{y_1(t)} = -3e^{6t}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) + 9e^{6t}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) =$$

$$\underline{e^{6t}\left(-3\cos 3t + 9\sin 3t\right)}$$

$$Y_2 = -3\,\frac{s-6}{(s-6)^2+9} - \frac{3}{(6-s)^2+9}$$

$$\underline{y_2(t)} = -3e^{6t}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) - e^{6t}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)$$

$$\underline{e^{6t}\left(-3\cos 3t - \sin 3t\right)}$$

b)

$$y_1'' + y_2 = -5\cos 2t \qquad y_1(0) = y_1'(0) = 1$$
$$y_2'' + y_1 = 5\cos 2t \qquad y_2(0) = -y_2'(0) = -1$$

$$\mathcal{L}\ \begin{cases} s^2Y_1 - sy_1(0) - y_1'(0) + Y_2 = -5\,\dfrac{s}{s^2+4} \\ s^2Y_2 - sy_2(0) - y_2'(0) + Y_1 = 5\,\dfrac{s}{s^2+4} \end{cases}$$

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$$\begin{bmatrix} s^2 & 1 \\ 1 & s^2 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} -5\frac{s}{s^2+4} + s + 1 \\ 5\frac{s}{s^2+4} + s - 1 \end{bmatrix}$$

$$Y_1 = \frac{1}{s^4-1}\left[s^2\left[-5\frac{s}{s^2+4} + s+1\right] - \left[5\frac{s}{s^2+4} + s - 1\right]\right] =$$

$$= -5\frac{s^2+1}{s^4-1}\cdot\frac{s}{s^2+4} + \frac{s^2-1}{s^4-1}s + \frac{s^2+1}{s^4-1} = \frac{-5s}{(s-1)(s+1)(s^2+4)}$$

$$+ \frac{s}{s^2+1} + \frac{1}{(s-1)(s+1)} =$$

$$Y_1 = \frac{1}{2}\frac{1}{s-1} + \frac{5}{2}\frac{1}{s+1} - 3\frac{s}{s^2+4} + \frac{2}{s^2+4} + \frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$\frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+4} = \frac{As^3 + As^2 + 4As + 4A + Bs^3 - Bs^2 + 4Bs - 4B + Cs^3 - Cs + Ds^2 - D}{(s-1)(s+1)(s^2+4)}$$

s^3: $A+B+C=0$

s^2: $A-B+D=0$

s: $4A-4B-C=-5$

1: $4A-4B-D=0$

$A+B+D=5$

$A-B+D=0$

$4A-4B-D=-10$

$\Rightarrow D-C=-5$

$C=D-5$

$2B=5$

$A+D=5/2$

$4A-D=0$

$\boxed{B=5/2,\ A=\frac{1}{2},\ D=2,\ C=-3}$

$$= \frac{-5s}{(s-1)(s+1)(s^2+4)}$$

1

$$\frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{As+A+Bs-B}{(s-1)(s+1)}$$

$A+B=0$

$A-B=1$

$\boxed{A=1/2,\ B=-\frac{1}{2}}$

$$y_1 = e^t + 2e^{-t} - 3\cos 2t + \sin 2t + \cos t$$

14

$$Y_2 = \frac{1}{s^4-1}\left[s^2\left[5\frac{s}{s^2+4} + s - 1\right] + \left[5\frac{s}{s^2+4} - s - 1\right]\right]$$

$$= 5\,\frac{s^2+1}{s^4-1}\,\frac{s}{s^2+4} + \frac{s^2-1}{s^4-1}\,s - \frac{s^2+1}{s^4-1} = \frac{5s}{(s-1)(s+1)(s^2+4)}$$

$$+ \frac{s}{s^2+1} - \frac{1}{(s-1)(s+1)}$$

$$Y_2 = -\frac{1}{2}\,\frac{1}{s-1} - \frac{5}{2}\,\frac{1}{s+1} + 3\,\frac{s}{s^2+4} - \frac{2}{s^2+4} + \frac{s}{s^2+1} - \frac{1}{2}\,\frac{1}{s-1} + \frac{1}{2}\,\frac{1}{s+1}$$

$$-\frac{1}{s-1} - 2\,\frac{1}{s+1} + 3\,\frac{s}{s^2+4} - \frac{2}{s^2+4} + \frac{s}{s^2+1}$$

$$y_2(t) = -e^{t} - 2e^{-t} + 3\cos 2t - \sin 2t + \cos t$$

Chapter 3

exams

3.1 first exam

3.1.1 Problem 1

Consider the complex exponential function $f(z) = e^z = e^x\left(\cos y + i\sin y\right)$, where $x = \operatorname{Re}(z), y = \operatorname{Im}(z)$. Use the Cauchy-Riemann equations to show that $f(z)$ is analytic in the whole complex plane $\mathbb{C}$, and using the definition of the derivative, show that $f'(z) = f(z)$.

Solution

$$f(z) = e^x \cos y + ie^x \sin y$$

Comparing the above to $f(z) = u + iv$, shows that

$$\begin{aligned} u &= e^x \cos y \\ v &= e^x \sin y \end{aligned}$$

Cauchy-Riemann equations in Cartesian coordinates are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

Since $\frac{\partial u}{\partial x} = e^x \cos y$ and $\frac{\partial v}{\partial y} = e^x \cos y$, then (1) is satisfied. Looking at (2), since $\frac{\partial u}{\partial y} = -e^x \sin y$ and $\frac{\partial v}{\partial x} = e^x \sin y$, then (2) is also satisfied.

In addition, since all these partial derivatives are continuous everywhere because the elementary $\cos, \sin, \exp$ are all continuous everywhere, then $f(z) = e^z$ is entire, or in other words, analytic everywhere.

To show that $f'(z) = f(z)$, by the definition of derivative, which is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

And since $\Delta z = \Delta x + i\Delta y$ and $f(z) = u\left(x, y\right) + iv\left(x, y\right)$ then the above becomes

$$\begin{aligned} f'(z) &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\left(u\left(x + \Delta x, y + \Delta y\right) + iv\left(x + \Delta x, y + \Delta y\right)\right) - \left(u\left(x, y\right) + iv\left(x, y\right)\right)}{\Delta x + i\Delta y} \\ &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u\left(x + \Delta x, y + \Delta y\right) - u\left(x, y\right)}{\Delta x + i\Delta y} + i\frac{v\left(x + \Delta x, y + \Delta y\right) - v\left(x, y\right)}{\Delta x + i\Delta y} \end{aligned}$$

Since e^z is analytic, then the limit does not depend on the direction, so we can pick any direction to approach z. Let us choose a direction such that the approach is on the x axis only keeping y fixed in order to simplify the above. This implies that now $\Delta y = 0$. The above simplifies to

$$f'(z) = \lim_{\Delta x \to 0} \frac{u\left(x + \Delta x, y\right) - u\left(x, y\right)}{\Delta x} + i\frac{v\left(x + \Delta x, y\right) - v\left(x, y\right)}{\Delta x}$$

But $\lim_{\Delta x \to 0} \frac{u\left(x+\Delta x, y\right) - u\left(x, y\right)}{\Delta x} = \frac{\partial u}{\partial x}$ and $\frac{v\left(x+\Delta x, y\right) - v\left(x, y\right)}{\Delta x} = \frac{\partial v}{\partial x}$, then the above reduces to

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

From the first part we obtained that $\frac{\partial u}{\partial x} = e^x \cos y$ and $\frac{\partial v}{\partial x} = e^x \sin y$. Using these in the above gives

$$\begin{aligned}
f'(z) &= e^x \cos y + ie^x \sin y \\
&= e^x \left(\cos y + i \sin y\right) \\
&= e^x e^{iy} \\
&= e^{x+iy} \\
&= e^z
\end{aligned}$$

Therefore $f'(z) = f(z)$. QED.

3.1.2 Problem 2

Determine the domain D of the z values on the complex plane where the complex function, given by the following series

$$F(z) = z^{\frac{1}{3}} + z^{\frac{1}{7}} + z^{\frac{1}{11}} + z^{\frac{1}{15}} + \cdots$$

is well defined. What is the set of values $z \in \mathbb{C}$, for which it holds that

$$F'(z) = \frac{1}{3} z^{\frac{-2}{3}} + \frac{1}{7} z^{\frac{-6}{7}} + \frac{1}{11} z^{\frac{-10}{11}} + \frac{1}{15} z^{\frac{-14}{15}} + \cdots$$

Solution

z can be either zero or not zero. When $z = 0$, then clearly $F(z)|_{z=0} = 0$ from the expression given for $F(z)$ above. So $F(z)$ is defined at z.

When $z \neq 0$, then each term in the series will now become multivalued since the terms are of the form $z^{\frac{1}{n}}$ for integer n. So we need to first make $F(z)$ single valued before considering the sum. We need to decide on which branch cut to use. Writing

$$\begin{aligned}
z^{\frac{1}{n}} &= \left(re^{i(\theta+2\pi k)}\right)^{\frac{1}{n}} \qquad k = 0, 1, 2, \cdots, n-1 \\
&= r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)} \\
&= r^{\frac{1}{n}} \left(\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right)
\end{aligned}$$

In order to make the multivalued $z^{\frac{1}{n}}$ function single valued, we select $k = 0$ and limit principal argument θ to

$$-\pi < \theta < \pi$$

with $z \neq 0$ for each term. Hence $z^{\frac{1}{n}}$ simplifies to

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(\cos\left(\frac{\theta}{n}\right) + i \sin\left(\frac{\theta}{n}\right)\right)$$

Where $r = |z|$ is the modulus of z. Now that each term is single valued, we can now look at the sum. Writing $F(z)$ as

$$\begin{aligned}
F(z) &= \sum_{n=0}^{\infty} z^{\frac{1}{4n+3}} \\
&= \sum_{n=0}^{\infty} r^{\frac{1}{4n+3}} e^{i\frac{\theta_0}{4n+3}}
\end{aligned}$$

We start with the preliminarily test to check if the above sum could be converging or not. Since the magnitude of the complex exponential is unity, we only need to check the modulus. Hence let

$$a_n = r^{\frac{1}{4n+3}}$$

Now we check if $\lim_{n\to\infty} a_n = 0$ or not. This is a a necessary condition for convergence but not a sufficient condition.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} r^{\frac{1}{4n+3}} \\
&= 1
\end{aligned}$$

We see that the limit is not zero. Therefore when $z \neq 0$, then $F(z)$ does not converge. Which means

$$\boxed{F(z) \text{ is defined only at } z = 0}$$

To answer that last part. Since we showed that $F(z)$ only defined at one point $z = 0$, then its derivative is not defined. Because a derivative requires a small neighborhood region around any point where the derivative to be evaluated due to using the limit as $\Delta z \to 0$ in the definition of derivative. Since there is no such neighborhood around $z = 0$, then it follows immediately that

$$\underline{F'(z) \text{ is not defined anywhere}}$$

3.1.3 Problem 3

Consider the real function defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \left(\frac{x}{6}\right)^n$$

Use the results on complex power series to determine the largest open interval on which $f(x)$ is defined. For what values of $a < b$ does $f(x)$ converges uniformly on $[a, b]$?

Solution

Using the ratio test

$$\begin{aligned} L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{\frac{(4(n+1))!}{((n+1)!)^4}\left(\frac{x}{6}\right)^{n+1}}{\frac{(4n)!}{(n!)^4}\left(\frac{x}{6}\right)^n}\right| \end{aligned}$$

Which simplifies to

$$\begin{aligned} L &= \lim_{n\to\infty} \left|\frac{(4(n+1))!\,(n!)^4 \frac{x}{6}}{(4n)!\,((n+1)!)^4}\right| \\ &= \left|\frac{x}{6}\right| \lim_{n\to\infty} \left|\frac{(4(n+1))!\,(n!)^4}{(4n)!\,((n+1)!)^4}\right| \end{aligned}$$

But $((n+1)!)^4 = ((n+1)\,n!)^4 = (n+1)^4\,(n!)^4$ and the above simplifies to

$$L = \left|\frac{x}{6}\right| \lim_{n\to\infty} \left|\frac{(4(n+1))!}{(4n)!\,(n+1)^4}\right|$$

But $(4(n+1))! = (4n+4)! = ((4n+4)(4n+3)(4n+2)(4n+1)(4n)!)$ and the above simplifies to

$$\begin{aligned} L &= \left|\frac{x}{6}\right| \lim_{n\to\infty} \left|\frac{(4n+4)(4n+3)(4n+2)(4n+1)(4n)!}{(4n)!\,(n+1)^4}\right| \\ &= \left|\frac{x}{6}\right| \lim_{n\to\infty} \left|\frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)^4}\right| \end{aligned}$$

Expanding gives

$$L = \left|\frac{x}{6}\right| \lim_{n\to\infty} \left|\frac{256n^4 + 640n^3 + 560n^2 + 200n + 24}{n^4 + 4n^3 + 6n^2 + 4n + 1}\right|$$

Dividing numerator and denominator by n^4 gives

$$L = \left|\frac{x}{6}\right| \lim_{n\to\infty} \left|\frac{256 + 640\frac{1}{n} + 560\frac{1}{n^2} + 200\frac{1}{n^3} + \frac{24}{n^4}}{1 + 4\frac{1}{n} + 6\frac{1}{n^2} + 4\frac{1}{n^3} + \frac{1}{n^4}}\right|$$

Now we can take the limit which gives 256. Hence

$$L = \frac{256}{6}|x|$$

For convergence, we want $|L| < 1$, which implies

$$\begin{aligned} \frac{256}{6}|x| &< 1 \\ |x| &< \frac{6}{256} \\ |x| &< \frac{3}{128} \end{aligned}$$

Therefore $f(x)$ is defined and absolutely converges for $\frac{-3}{128} < x < \frac{3}{128}$. Therefore by using theorem 1, page 699 in the textbook, we conclude that for uniform convergence we need

$$|x| \leq |r| < \frac{3}{128}$$

Or

$$\underline{x \geq a > -\frac{3}{128} \text{ and } x \leq b < \frac{3}{128}}$$

Hence the series converges uniformly on $[a,b]$ where

$$a > -\frac{3}{128}$$
$$b < \frac{3}{128}$$

3.1.4 Problem 5

Let $f(z)$ be given as

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{z}{4}\right)^{n+1}$$

(a) Find the domain D on which $f\left(\frac{1}{z}\right)$ is analytic. (b) For what z values does $g(z)$ defined by the Laurent series

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{1}{4z}\right)^{n+1} + \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$$

Converge?

Solution

part (a)

First we find where $f(z)$ converges.

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{z}{4}\right)^{n+1} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \left(\frac{z}{4}\right)^{n+2} \\
&= \left(\frac{z}{4}\right)^2 \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \left(\frac{z}{4}\right)^{n} \\
&= \left(\frac{z}{4}\right)^2 \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)4^n} (z-z_0)^{n}
\end{aligned}$$

$f(z)$ converges in a disk centered at $z_0 = 0$ if the series $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)6^n} z^n$ converges there. Using the ratio test to find L gives

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{\frac{1}{(n+2)(n+3)4^{n+1}}}{\frac{1}{(n+1)(n+2)4^n}}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)(n+2)4^n}{(n+2)(n+3)4^{n+1}}\right| \\
&= \frac{1}{4} \lim_{n\to\infty} \left|\frac{n^2+3n+2}{n^2+5n+6}\right| \\
&= \frac{1}{4} \lim_{n\to\infty} \left|\frac{1+3\frac{1}{n}+2\frac{1}{n^2}}{1+5\frac{1}{n}+\frac{6}{n^2}}\right| \\
&= \frac{1}{4}
\end{aligned}$$

Since $L = \frac{1}{4}$ then the radius of convergence $R = \frac{1}{L}$ or $R = 4$. This means $f(z)$ converges inside disk centered at zero of radius $R = 4$. Therefore $f\left(\frac{1}{z}\right)$ converges everywhere outside this disk. Since there are no other singularities in the function given by $f\left(\frac{1}{z}\right) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{1}{4z}\right)^{n+1}$ outside disk of radius 4 then it is analytic there everywhere (it is differentiable everywhere outside this disk). Therefore, we conclude this part by saying that

$$f\left(\frac{1}{z}\right) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{1}{4z}\right)^{n+1}$$

is analytic outside disk of radius 4.

Part (b)

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{1}{4z}\right)^{n+1} + \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$$

The first series in the right side above, we found from part (a) where it converges, which is for $|z| > 4$. Now we need to find where the second series converges.

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)!n^n}{n!(n+1)^{n+1}}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)n!n^n}{n!(n+1)^{n+1}}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)n^n}{(n+1)^{n+1}}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)n^n}{(n+1)(n+1)^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{n^n}{(n+1)^n}\right| \\
&= \frac{1}{e}
\end{aligned}$$

Hence the radius of convergence is $R = \frac{1}{L} = e \approx 2.718$. This means the second series $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$ convergence for $|z| < e$, or <u>inside</u> disk of radius $R = e$. But the first series converges <u>outside</u> disk of radius 4. Therefore, there is <u>no common annulus</u> where <u>both</u> series converge. Therefore

$$\boxed{\text{There are no } z \text{ values where } g(z) \text{ converges}}$$

3.1.5 Problem 6

Determine the MacLaurin series for the following special functions for $z \in \mathbb{R}$. The resulting series defines the functions for complex numbers as well. Give the radius of convergence of the resulting series. Determine whether any of them is even $f(-z) = f(z)$ or odd $f(-z) = f(z)$.

(a) $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ (b) $\operatorname{Si}(z) = \int_0^z \frac{\sin t}{t} dt$

<u>solution</u>

Part (a)

Starting with MacLaurin series for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, hence e^{-t^2} series expansion around zero becomes

$$\begin{aligned}
e^{-t^2} &= 1 + \left(-t^2\right) + \frac{\left(-t^2\right)^2}{2!} + \frac{\left(-t^2\right)^3}{3!} + \frac{\left(-t^2\right)^4}{43!} + \cdots \\
&= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \cdots
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{erf}(z) &= \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^z \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \cdots\right) dt
\end{aligned}$$

Since $\exp(x)$ is analytic everywhere, we can integrate the above term by term, which gives

$$\begin{aligned}
\operatorname{erf}(z) &= \frac{2}{\sqrt{\pi}} \left(t - \frac{1}{3}t^3 + \frac{1}{5}\frac{t^5}{2!} - \frac{1}{7}\frac{t^7}{3!} + \frac{1}{9}\frac{t^9}{4!} - \cdots\right)_0^z \\
&= \frac{2}{\sqrt{\pi}} \left(z - \frac{1}{3}z^3 + \frac{1}{5}\frac{z^5}{2!} - \frac{1}{7}\frac{z^7}{3!} + \frac{1}{9}\frac{z^9}{4!} - \cdots\right)
\end{aligned}$$

To find its radius of convergence, we need to first find closed form for the above. The general term is seen to be

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!\,(2n+1)}$$

Hence

$$\begin{aligned}\operatorname{erf}(z) &= \frac{2}{\sqrt{\pi}} z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!\,(2n+1)} \\ &= \frac{2}{\sqrt{\pi}} z \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} \left(z^2\right)^n\end{aligned}$$

Now let $z^2 = s$. We now find the radius of convergence R for $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} s^n$ and then find $\sqrt{R}$ to find radius of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n}$.

Applying the ratio test to $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} s^n$ to find its L. Since we are using absolute values, the $(-1)^n$ does not affect the result, hence

$$\begin{aligned}L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)!(2n+2)}}{\frac{1}{n!(2n+1)}}\right| \\ &= \lim_{n\to\infty} \left|\frac{n!\,(2n+1)}{(n+1)!\,(2n+2)}\right| \\ &= \lim_{n\to\infty} \left|\frac{n!\,(2n+1)}{(n+1)\,n!\,(2n+2)}\right| \\ &= \lim_{n\to\infty} \left|\frac{(2n+1)}{(n+1)\,(2n+2)}\right| \\ &= \lim_{n\to\infty} \left|\frac{2n+1}{2n^2+4n+2}\right| \\ &= \lim_{n\to\infty} \left|\frac{\frac{2}{n}+\frac{1}{n^2}}{2+\frac{4}{n}+\frac{2}{n^2}}\right| \\ &= \frac{0}{2} \\ &= 0\end{aligned}$$

Therefore $R = \frac{1}{L} = \infty$. But $\sqrt{\infty} = \infty$. Hence

$$\underline{\operatorname{erf}(z) \text{ is analytic on the whole complex plane}}$$

Now, to find if it is even or odd. Using the above series definition

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!\,(2n+1)}$$

Lets check if it odd. i.e. if $f(-z) = -f(z)$. From above

$$\operatorname{erf}(-z) = \frac{2}{\sqrt{\pi}} (-z) \sum_{n=0}^{\infty} \frac{(-1)^n (-z)^{2n}}{n!\,(2n+1)}$$

But $(-z)^{2n} = z^{2n}$ since the exponent is even, and the above simplifies to

$$\operatorname{erf}(-z) = \frac{-2}{\sqrt{\pi}} z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!\,(2n+1)} \tag{1}$$

Now lets find $-f(z)$. From the definition

$$-\operatorname{erf}(z) = \frac{-2}{\sqrt{\pi}} z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!\,(2n+1)} \tag{2}$$

Since (1) and (2) are the same, then

$$\underline{\operatorname{erf}(z) \text{ is odd}}$$

Part (b)

Starting with MacLaurin series for

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Hence $\frac{\sin(x)}{x}$ becomes

$$\begin{aligned}\frac{\sin(x)}{x} &= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) \\ &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots\end{aligned}$$

Hence

$$\operatorname{Si}(x) = \int_0^z \left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \frac{t^8}{9!} - \cdots\right) dt$$

Since $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$ is analytic everywhere, we can integrate the above term by term, which gives

$$\begin{aligned}\operatorname{Si}(z) &= \left(t - \frac{1}{3}\frac{t^3}{3!} + \frac{1}{5}\frac{t^5}{5!} - \frac{1}{7}\frac{t^7}{7!} + \frac{1}{9}\frac{t^9}{9!} - \cdots\right)_0^z \\ &= z - \frac{1}{3}\frac{z^3}{3!} + \frac{1}{5}\frac{z^5}{5!} - \frac{1}{7}\frac{z^7}{7!} + \frac{1}{9}\frac{z^9}{9!} - \cdots\end{aligned}$$

In closed form, this can be written as

$$\begin{aligned}\operatorname{Si}(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)(2n+1)!} z^{2n+1} \\ &= z\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)(2n+1)!} z^{2n} \\ &= z\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)(2n+1)!} \left(z^2\right)^n\end{aligned}$$

So we need to find radius of convergence R for $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)(2n+1)!} s^n$ and then find $\sqrt{R}$ as we did in part (a).

$$\begin{aligned}L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{\frac{1}{(2n+2)(2n+2)!}}{\frac{1}{(2n+1)(2n+1)!}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(2n+1)(2n+1)!}{(2n+2)(2n+2)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{(2n+1)(2n+1)!}{(2n+2)(2n+2)(2n+1)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{(2n+1)}{(2n+2)(2n+2)}\right| \\ &= \lim_{n\to\infty}\left|\frac{2n+1}{4n^2+8n+4}\right| \\ &= \lim_{n\to\infty}\left|\frac{\frac{2}{n}+\frac{1}{n^2}}{4+\frac{8}{n}+\frac{4}{n^2}}\right| \\ &= \frac{0}{4} \\ &= 0\end{aligned}$$

Hence $R = \frac{1}{L} = \infty$. But $\sqrt{\infty} = \infty$. Hence

$$\underline{\operatorname{Si}(z) \text{ is analytic on the whole complex plane}}$$

Now, to find if it is even or odd. Using the above series definition

$$\operatorname{Si}(z) = z\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)(2n+1)!} z^{2n}$$

Lets check if it odd. i.e. $f(-z) = -f(z)$. From above

$$\operatorname{Si}(-z) = -z\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)(2n+1)!} (-z)^{2n}$$

But $(-z)^{2n} = z^{2n}$ since the exponent is even, so the above becomes

$$\operatorname{Si}(-z) = -z\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)(2n+1)!} z^{2n} \tag{1}$$

Now lets find what $-f(z)$ gives

$$-\operatorname{Si}(z) = -z\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)(2n+1)!}z^{2n} \tag{2}$$

Comparing (1,2) we see they are the same. Hence

$$\underline{\operatorname{Si}(z) \text{ is odd}}$$

3.1.6 Problem 7

(a) Determine the Laurent series of the function

$$f(z) = \frac{2z+6}{z^2-6z+5}$$

In the annulus $1 < z < 5$ and in $|z| > 5$.

(b) Determine the Taylor series representation of the function

$$g(z) = e^{-\frac{z^2}{2}}$$

with center $z_0 = 0$. What is the radius of convergence?

Solution

Part (a)

$$\begin{aligned}
f(z) &= \frac{2z+6}{z^2-6z+5}\\
&= \frac{2z+6}{(z-5)(z-1)}\\
&= \frac{A}{(z-5)} + \frac{B}{(z-1)}
\end{aligned}$$

Hence

$$\begin{aligned}
2z+6 &= A(z-1) + B(z-5)\\
&= z(A+B) - A - 5B
\end{aligned}$$

Solving the above two equations for A, B gives

$$\begin{aligned}
2 &= A + B\\
6 &= -A - 5B
\end{aligned}$$

First equations gives $A = 2 - B$. Second equation becomes

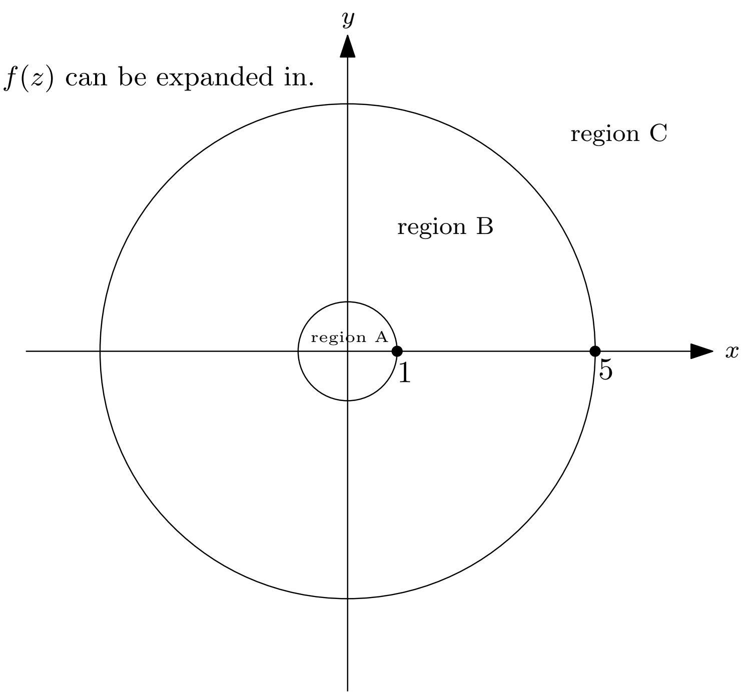
$$\begin{aligned}
6 &= -(2-B) - 5B\\
6 &= -2 - 4B\\
B &= -\frac{8}{4}\\
&= -2
\end{aligned}$$

Hence $A = 4$. Therefore (1) becomes

$$f(z) = \frac{4}{(z-5)} - \frac{2}{(z-1)} \tag{2}$$

We now see there is a pole at $z = 5$ and at $z = 1$. So there are three regions. The following diagram shows these three different regions

Three region $f(z)$ can be expanded in.



<u>For region B</u>, which is annulus $1 < z < 5$, we need to expand $f(z) = \frac{4}{(z-5)} - \frac{2}{(z-1)}$. Looking at first part

$$
\begin{aligned}
\frac{4}{(z-5)} &= \frac{-4}{5-z}\\
&= \frac{-4}{5\left(1-\frac{z}{5}\right)}
\end{aligned}
$$

This can be expanded for $\left|\frac{z}{5}\right| < 1$ or $|z| < 5$. Using Binomial series it gives

$$
\begin{aligned}
\frac{-4}{5\left(1-\frac{z}{5}\right)} &= \frac{-4}{5}\left(1+\left(\frac{z}{5}\right)+\left(\frac{z}{5}\right)^2+\left(\frac{z}{5}\right)^3+\cdots\right)\\
&= \frac{-4}{5}\sum_{n=0}^{\infty}\left(\frac{z}{5}\right)^n \qquad (3)
\end{aligned}
$$

We now consider the second term in (2), which is $\frac{2}{(z-1)}$

$$
\frac{2}{(z-1)} = -\frac{2}{1-z}
$$

This can be expanded only when $|z| < 1$. But we want $|z| > 1$, therefore we need to convert it to negative power. We write

$$
\begin{aligned}
\frac{2}{(z-1)} &= \frac{-2}{z\left(\frac{1}{z}-1\right)}\\
&= \frac{2}{z}\frac{1}{\left(1-\frac{1}{z}\right)}
\end{aligned}
$$

Now $\frac{1}{\left(1-\frac{1}{z}\right)}$ can be expanded for $\left|\frac{1}{z}\right| < 1$ or $z > 1$, which puts in region B. Hence the second term can now be expanded as

$$
\begin{aligned}
\frac{2}{(z-1)} &= \frac{2}{z}\frac{1}{\left(1-\frac{1}{z}\right)}\\
&= \frac{2}{z}\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right)\\
&= \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\\
&= \sum_{n=0}^{\infty}\frac{2}{z^{n+1}} \qquad (4)
\end{aligned}
$$

Therefore (3,4) gives us the expansion of $f(z)$ valid in region B. Substituting results from (3,4) into

(2) gives

$$\begin{aligned}
f(z) &= \frac{4}{(z-5)} - \frac{2}{(z-1)}\\
&= \frac{-4}{5}\sum_{n=0}^{\infty}\left(\frac{z}{5}\right)^n - \sum_{n=0}^{\infty}\frac{2}{z^{n+1}}\\
&= \frac{-4}{5}\left(1+\frac{z}{5}+\frac{z^2}{5^2}+\cdots\right) - 2\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)\\
&= \left(-\frac{4}{5}-\frac{4z}{5^2}-\frac{4z^2}{5^3}+\cdots\right) + \left(-\frac{2}{z}-\frac{2}{z^2}-\frac{2}{z^3}+\cdots\right)
\end{aligned}$$

The above shows that residue is -2, which is the coefficient for the $\frac{1}{z}$ term.

<u>For region C</u>

This is for $|z|>5$ .For the first term in (2), which is $\frac{4}{(z-5)}$ we write it as

$$\frac{4}{(z-5)} = \frac{4}{z}\frac{1}{\left(1-\frac{5}{z}\right)}$$

We can expand this for $\left|\frac{5}{z}\right|<1$ or $|z|>5$ which is what we want. Hence it becomes

$$\begin{aligned}
\frac{4}{z}\frac{1}{\left(1-\frac{5}{z}\right)} &= \frac{4}{z}\left(1+\left(\frac{5}{z}\right)+\left(\frac{5}{z}\right)^2+\left(\frac{5}{z}\right)^3+\cdots\right)\\
&= \frac{4}{z}\sum_{n=0}^{\infty}\left(\frac{5}{z}\right)^n\\
&= 4\sum_{n=0}^{\infty}\frac{5^n}{z^{n+1}}
\end{aligned}$$

For the second in (2), which is $\frac{2}{(z-1)}$, we can use the expansion found earlier since it is valid for $|z|>1$, hence also valid for $|z|>5$ as well. which is $\frac{2}{(z-1)} = \sum_{n=0}^{\infty}\frac{2}{z^{n+1}}$.

Therefore, in region C, the expansion is

$$\begin{aligned}
f(z) &= \frac{4}{(z-5)} - \frac{2}{(z-1)}\\
&= 4\sum_{n=0}^{\infty}\frac{5^n}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2}{z^{n+1}}\\
&= \sum_{n=0}^{\infty}\frac{(4)(5^n)-2}{z^{n+1}}\\
&= \frac{2}{z}+\frac{18}{z^2}+\frac{98}{z^3}+\frac{498}{z^4}+\cdots
\end{aligned}$$

The residue is 2.

Part (b)

$$g(z) = e^{-\frac{z^2}{2}}$$

Taylor series for $g(z)$ expanded around z_0 is given by

$$g(z) = g(z_0) + g'(z_0)z + g''(z_0)\frac{z^2}{2!} + g'''(z_0)\frac{z^3}{3!} + g^{(4)}(z_0)\frac{z^4}{4!} + \cdots$$

But

$$\begin{aligned}
g(z_0) &= g(0) = 1\\
g'(z_0) &= -\left.\frac{2z}{2}e^{-\frac{z^2}{2}}\right|_{z=z_0=0}\\
&= 0
\end{aligned}$$

And

$$
\begin{aligned}
g''(z_0) &= \frac{d}{dz}\left(-ze^{-\frac{z^2}{2}}\right)_{z=z_0} \\
&= \left(-e^{-\frac{z^2}{2}} - z^2 e^{-\frac{z^2}{2}}\right)_{z=z_0=0} \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
g'''(z_0) &= \frac{d}{dz}\left(-e^{-\frac{z^2}{2}} - z^2 e^{-\frac{z^2}{2}}\right)_{z=z_0} \\
&= \left(ze^{-\frac{z^2}{2}} - 2ze^{-\frac{z^2}{2}} - z^3 e^{-\frac{z^2}{2}}\right)_{z=z_0=0} \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
g^{(4)}(z_0) &= \frac{d}{dz}\left(ze^{-\frac{z^2}{2}} - 2ze^{-\frac{z^2}{2}} - z^3 e^{-\frac{z^2}{2}}\right)_{z=z_0} \\
&= \left(e^{-\frac{z^2}{2}} - z^3 e^{-\frac{z^2}{2}} - 2e^{-\frac{z^2}{2}} + 2z^3 e^{-\frac{z^2}{2}} - 3z^2 e^{-\frac{z^2}{2}} + z^4 e^{-\frac{z^2}{2}}\right)_{z=z_0=0} \\
&= 1
\end{aligned}
$$

And so on. We can see the sequence pattern as

$$
\begin{aligned}
g(z) &= g(z_0) + g'(z_0)\, z + g''(z_0)\frac{z^2}{2!} + g'''(z_0)\frac{z^3}{3!} + g^{(4)}(z_0)\frac{z^4}{4!} + \cdots \\
&= 1 + 0 - \frac{z^2}{2} + 0 + \frac{z^4}{4!} + 0 - \frac{z^6}{6!} + \cdots \\
&= 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}
\end{aligned}
$$

To find radius of convergence, we write the above as

$$
\begin{aligned}
g(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}\left(z^2\right)^n \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} s^n \qquad (1)
\end{aligned}
$$

And find R for s then take $\sqrt{R}$. Hence for (1)

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{1}{(2(n+1))!}}{\frac{1}{(2n)!}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n)!}{(2(n+1))!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n)!}{(2n+2)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{(2n+2)(2n+1)}\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{4n^2+6n+2}\right| \\
&= 0
\end{aligned}
$$

Hence $R = \frac{1}{L} = \infty$. Therefore $\sqrt{R} = \infty$. The expansion is valid in the whole complex plane.

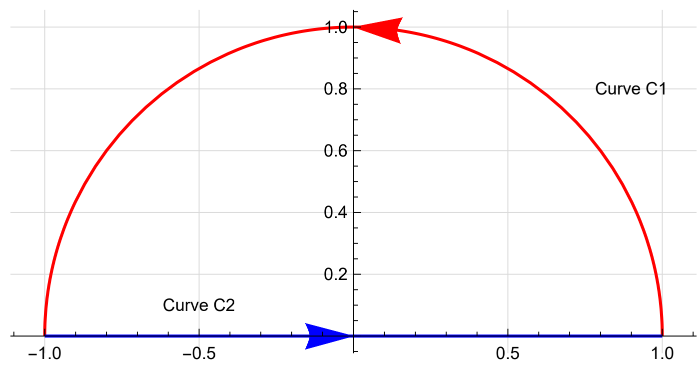
3.1.7 Problem 8

Evaluate the integral below on the curve $C = C_1 \cup C_2$ where $C_1 : z(t) = e^{i\pi t}, 0 \le t \le 1$ and $C_2 : z(t) = 2t - 1, 0 \le t \le 1$.

$$\oint_C \operatorname{Re}(z)\, dz$$

Solution

The diagram below shows the curves



$$\oint_C \operatorname{Re}(z)\, dz = \int_{C_1} \operatorname{Re}(z(t))\, z'(t)\, dt + \int_{C_2} \operatorname{Re}(z(t))\, z'(t)\, dt \tag{1}$$

But on C_1, $z(t) = e^{i\pi t} = \cos(\pi t) + i \sin(\pi t)$, then $\operatorname{Re}(z(t)) = \cos(\pi t)$ and $z'(t) = i\pi e^{i\pi t}$, therefore the integral on C_1 becomes

$$\begin{aligned} \int_0^1 \cos(\pi t)\, i\pi e^{i\pi t} dt &= i\pi \int_0^1 \cos(\pi t)\, e^{i\pi t} dt \\ &= i\pi I \end{aligned} \tag{2}$$

Where $I = \int_0^1 \cos(\pi t)\, e^{i\pi t} dt$. We now evaluate I. Since $e^{i\pi t} = \cos(\pi t) + i \sin(\pi t)$, then

$$I = \int_0^1 \cos(\pi t) \cos(\pi t)\, dt + i \int_0^1 \cos(\pi t) \sin(\pi t)\, dt \tag{3}$$

But first integral in above is

$$\begin{aligned} \int_0^1 \cos(\pi t) \cos(\pi t)\, dt &= \int_0^1 \cos^2(\pi t)\, dt \\ &= \int_0^1 \frac{1}{2} + \frac{1}{2} \cos(2\pi t)\, dt \\ &= \frac{1}{2} + \frac{1}{2} (\sin(2\pi t))_0^1 \\ &= \frac{1}{2} \end{aligned}$$

And for second integral in (3), and using $\sin A \cos A = \frac{1}{2} \sin(2A)$, it becomes

$$\begin{aligned} \int_0^1 \cos(\pi t) \sin(\pi t)\, dt &= \int_0^1 \frac{1}{2} \sin(2\pi t)\, dt \\ &= \frac{1}{2} \left[\frac{\cos(2\pi t)}{2\pi} \right]_0^1 \\ &= \frac{1}{4\pi} [\cos(2\pi t)]_0^1 \\ &= \frac{1}{4\pi} [\cos(2\pi) - 1] \\ &= 0 \end{aligned}$$

Therefore integral on C_1 from (2) becomes

$$\begin{aligned} \int_0^1 \cos(\pi t)\, i\pi e^{i\pi t} dt &= i\pi I \\ &= \frac{i\pi}{2} \end{aligned}$$

Now the second integral on C_2 is found, where $z(t) = 2t - 1$, and $z'(t) = 2$. Hence

$$\begin{aligned}\int_0^1 \mathrm{Re}\,(z(t))\, z'(t)\, dt &= \int_0^1 (2t-1)\, 2\, dt \\ &= 2\left(t^2 - t\right)_0^1 \\ &= 0\end{aligned}$$

Therefore contribution comes only from the integration over C_1 which is

$$\oint_C \mathrm{Re}\,(z)\, dz = \frac{i\pi}{2}$$

3.1.8 Optional choice Problem 4

4. We refer to an open connected set $D \subset \mathbb{C}$ as a domain of the complex plane, and if $F(z)$ is an analytic function on D we call the set

$$F(D) = \{F(z) : z \in D\}$$

the analytic transformation of D by F. (E.g.: if the open unit disk centered at 1 is given as

$$U^1 = \{z \in \mathbb{C} : |z-1| < 1\},$$

and $F(z) = z + i$, $H(z) = iz$, then the analytic transformations $F(U^1)$ and $H(U^1)$ are a shift by i of U^1, and a counterclockwise rotation by 90 degrees – or $\pi/2$ radian – around the origin of U^1, respectively.)

a) Consider

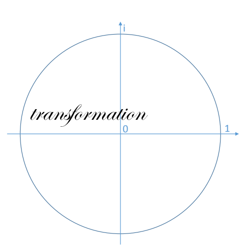
$$U^0 = \{z \in \mathbb{C} : |z| < 1\},$$

and $f_1(z) = \frac{z+1}{1-z}$. Show that $f_1(z)$ is a 1-1 (and analytic) function on U^0, and argue that f_1 , also referred to as a Möbius transformation, transforms U^0 to the right half plane, i.e. $f_1(U^0) = \{z \in \mathbb{C} : Re\, z > 0\}$.

b) Write the function

$$f(z) = e^{-i \ln\left\{\left[i\left(\frac{z+1}{1-z}\right)\right]^{\frac{1}{2}}\right\}}$$

as a composite of 6 analytic functions $f(z) = f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1(z)$ and show that f transforms U^0 into an annulus $f(U^0)$. [Hint: Determine in order $f_1(U^0)$, $f_2 \circ f_1(U^0)$,..., $f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1(U^0)$.]



c) *(Extra credit/fun:)* If you have access to a software performing complex arithmetic (e.g. Matlab), compute the transformation of the *"transformation"* text shaped domain T given inside the unit disk (as black text) on the adjacent image.

<u>Solution</u>

Part (a)

$$f_1(z) = \frac{z+1}{1-z} \tag{1}$$

The domain U^0 is the unit disk centered at origin. To show $f_1(z)$ is 1-1 on U^0 means to show that if

$$f_1(z_1) = f_1(z_2) \tag{2}$$

The this implies that

$$z_1 = z_2$$

Applying (1,2) to both points gives

$$\begin{aligned}
f_1(z_1) &= f_1(z_2)\\
\frac{z_1+1}{1-z_1} &= \frac{z_2+1}{1-z_2}\\
\frac{(x_1+iy_1)+1}{1-(x_1+iy_1)} &= \frac{(x_2+iy_2)+1}{1-(x_2+iy_2)}\\
((x_1+iy_1)+1)(1-(x_2+iy_2)) &= (1-(x_1+iy_1))((x_2+iy_2)+1)\\
(x_1+iy_1+1)(1-x_2-iy_2) &= (1-x_1-iy_1)(x_2+iy_2+1)\\
x_1-x_1x_2-ix_1y_2+iy_1-iy_1x_2+y_1y_2+1-x_2-iy_2 &= x_2+iy_2+1-x_1x_2-iy_2x_1-x_1-iy_1x_2+y_1y_2-iy_1
\end{aligned}$$

Collecting real and imaginary parts

$$(x_1-x_1x_2+y_1y_2+1-x_2)+i(-x_1y_2+y_1-y_1x_2-y_2) = (x_2+1-x_1x_2-x_1+y_1y_2)+i(y_2-y_2x_1-y_1x_2-y_1) \tag{3}$$

If two complex numbers are equal, then the real part and the imaginary part must be equal. Hence in equation (3), equating real parts gives

$$\begin{aligned}
x_1-x_1x_2+y_1y_2+1-x_2 &= x_2+1-x_1x_2-x_1+y_1y_2\\
x_1-x_2 &= x_2-x_1\\
2x_1 &= 2x_2\\
x_1 &= x_2 \qquad (4)
\end{aligned}$$

And equating imaginary parts in (3) gives

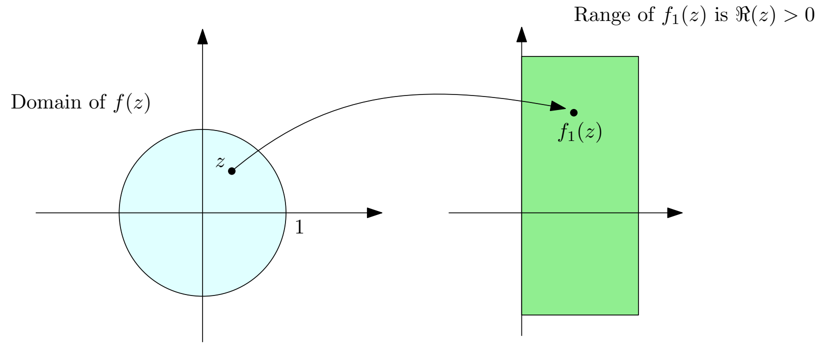
$$\begin{aligned}
-x_1y_2+y_1-y_1x_2-y_2 &= y_2-y_2x_1-y_1x_2-y_1\\
y_1-y_2 &= y_2-y_1\\
2y_1 &= 2y_2\\
y_1 &= y_2 \qquad (5)
\end{aligned}$$

From (4,5) we see that $z_1 = x_1+iy_1$ is the same point as $z_2 = x_2+iy_2$. This shows that

$$\underline{f_1(z) \text{ is 1-1 on } U^0}$$

To show that $f_1(z)$ is analytic, we see that there is a pole at $z=1$. But this is <u>outside</u> the disk $|z|<1$. So there is no singularity inside the disk. And since $\frac{z+1}{z-1}$ is differentiable as many times as we wish, then it is analytic. We can also apply Cauchy Riemann equations also to verify this, but it is not needed for this simple function.

The last part is to show that f_1 is a Mobius transformation.



To show this, we apply $f_1(z)$ to an arbitrary point in the domain $|z|<1$ and see if the real part of $f_1(z)$ comes out to be always positive or not. Let z_0 be any point inside the disk $|z|<1$ where

$z=x+iy$. Hence

$$
\begin{aligned}
f_1(z) &= \frac{z+1}{1-z} \\
&= \frac{\left(x+iy\right)+1}{1-\left(x+iy\right)} \\
&= \frac{(x+1)+iy}{(1-x)-iy}
\end{aligned}
$$

Multiplying the numerator and denominator by the complex conjugate of the denominator gives

$$
\begin{aligned}
f_1(z_0) &= \frac{\left((x+1)+iy\right)\left((1-x)+iy\right)}{\left((1-x)-iy\right)\left((1-x)+iy\right)} \\
&= \frac{(x+1)(1-x)+iy(x+1)+iy(1-x)-y^2}{(1-x)^2+y^2} \\
&= \frac{(x+1)(1-x)+iyx+iy+iy-iyx-y^2}{(1-x)^2+y^2} \\
&= \frac{\left(1-x^2\right)+2iy-y^2}{(1-x)^2+y^2} \\
&= \frac{\left(1-x^2\right)-y^2}{(1-x)^2+y^2}+i\frac{-2yx}{(1-x)^2+y^2} \\
&= u+iv
\end{aligned}
$$

Hence

$$
\begin{aligned}
u\left(x,y\right) &= \frac{\left(1-x^2\right)-y^2}{(1-x)^2+y^2} \\
&= \frac{1-\left(x^2+y^2\right)}{(1-x)^2+y^2} \\
&= \frac{1-|z|^2}{(1-x)^2+y^2}
\end{aligned}
$$

We now need to show that $u\left(x,y\right)$ is always positive. Now, Since $|z|<1$, then $|x|<1$ and also $\left|y\right|<1$. This shows that the denominator is always positive and can not be zero even when $x=0,y=0$ in which case the denominator is 1. The only problem comes when $x=1$ and $y=0$ in which case the mapping goes to infinity. More on this below. But this point is on the boundary itself, and not inside the disk.

Now, for the numerator, since $|z|<1$ then $1-|z|^2$ is always positive. Only when $|z|=1$ (boundary points), then will $u\left(x,y\right)=0$. Therefore we conclude that

$$\underline{\text{Each point inside the disk maps to right side of the complex plane}}$$

For example, the center of the disk, $x=0,y=0$, maps to the right side complex plane, since

$$
\begin{aligned}
f_1\left(x,y\right) &= \frac{\left(1-x^2\right)-y^2}{(1-x)^2+y^2}+i\frac{-2yx}{(1-x)^2+y^2} \\
f_1(0,0) &= 1+0i
\end{aligned}
$$

What about points on the boundary of the disk where $|z|=1$? Lets pick the point $x=1,y=0$, then we see that

$$\lim_{y\to 0}\frac{\left(x+iy\right)+1}{1-\left(x+iy\right)}=\frac{1+x}{1-x}$$

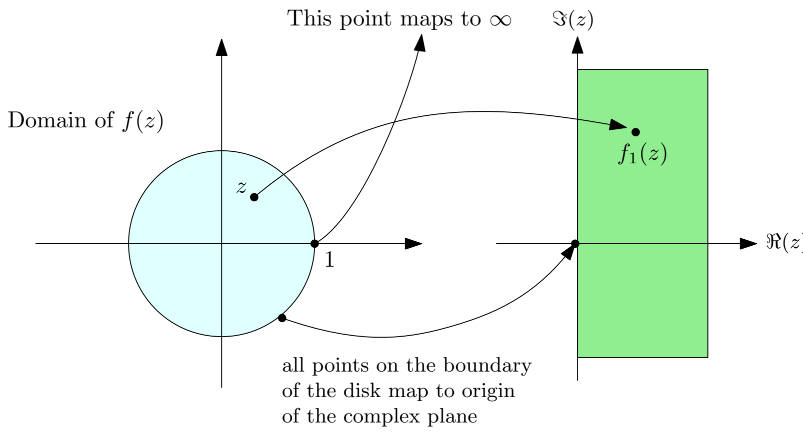
Hence as $x\to 1$ it will blow up and it goes to infinity. How about the point $x=0,y=1$, then this point maps to

$$
\begin{aligned}
f_1(0,1) &= \frac{1-1}{1+1}+i\frac{0}{1+1} \\
&= 0+i0
\end{aligned}
$$

So it maps to origin in the complex plane. The point $x=-1,y=0$ maps to

$$f_1(-1,0)=0+i0$$

All other points on the boundary of disk $|z|=1$ map to the origin of the complex plane, <u>except for the point</u> $x=1,y=0$ which maps to infinity.



Transformation given by $\frac{z+1}{1-z}$

Part (b)

$$f(z)=e^{-i\ln\sqrt{i\frac{z+1}{1-z}}}$$

Since $f_1(z)=\frac{z+1}{1-z}$, where from part(a), we know it maps all points inside disk $|z|<1$ to the right side of the complex plane. Then the above can be written as

$$f(z)=e^{-i\ln\sqrt{if_1(z)}}$$

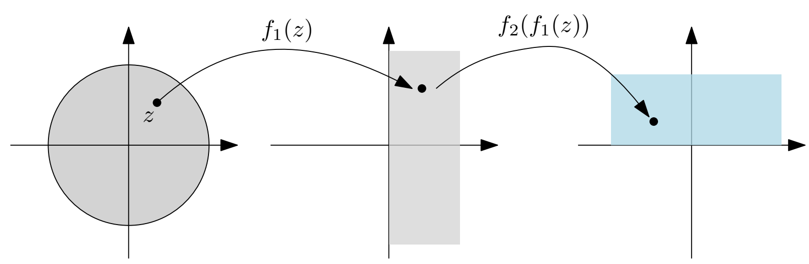
Now let

$$f_2(z)=if_1(z)$$

The effect of this is to rotate each point in the right half plane clockwise by 90^0. This can be seen by considering an arbitrary point $z_0=re^{i\theta}$, then

$$\begin{aligned} iz_0 &= e^{i\frac{\pi}{2}}\left(re^{i\theta}\right)\\ &= re^{i\left(\theta+\frac{\pi}{2}\right)} \end{aligned}$$

Hence the result of applying $f_2(z)=if_1(z)$ is to rotate the right side plane to the upper half plane as shown below



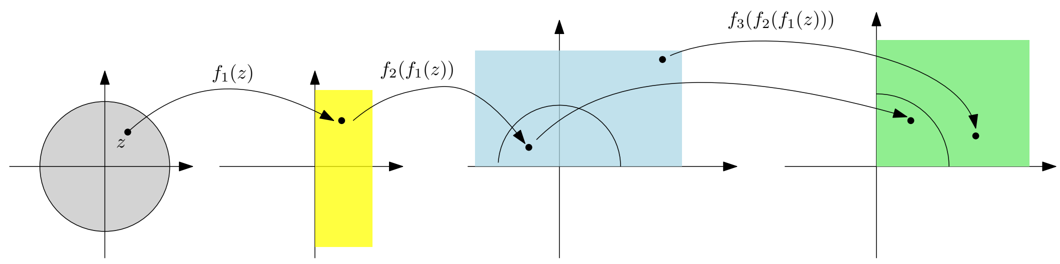
The next step is to apply the square root function. This means

$$f_3(z)=\sqrt{f_2\left(f_1(z)\right)}$$

What does applying a square root to a point z_0 in the complex plane do? Since $z^{\frac{1}{2}}=e^{\frac{1}{2}(\ln|z|+i\operatorname{Arg}(z))}$ where here we used the principal argument of z, therefore

$$\begin{aligned} z^{\frac{1}{2}} &= e^{\frac{1}{2}(\ln r+i\theta)}\\ &= r^{\frac{1}{2}}e^{i\frac{\theta}{2}} \end{aligned}$$

Hence the effect is to take the square root of the module and to reduce the argument by half. Points inside a unit circle will increase their module and move closer to the inner edge of the unit circle, and points outside the unit circle will decrease their modulus and move closer to the outside edge of the unit circle. Points on the unit circle will not change their modulus. But all points will have their argument halved. The result of this is all points will move and end up in the first quadrant of the complex plane



The next step is to apply the $\ln$ function on the resulting points. Hence

$$f_4(z) = \ln\left(f_3(z)\right)$$

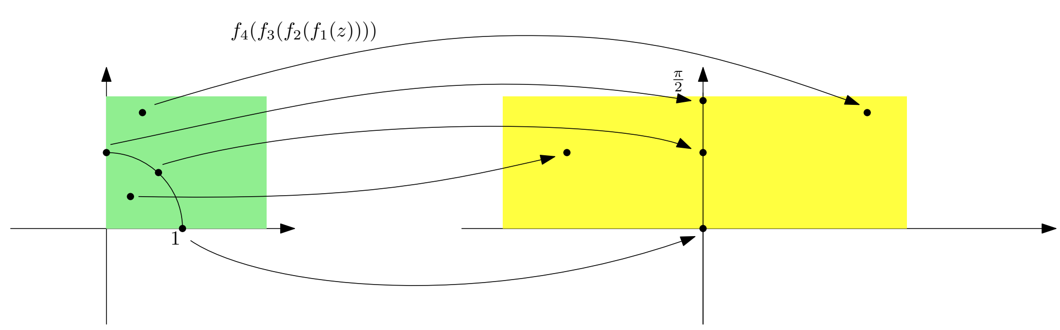
Now we ask, what does $\ln(z)$ do to a point z? Let $z = re^{i\theta}$ then

$$\begin{aligned}\ln(z) &= \ln\left(re^{i\theta}\right)\\ &= \ln r + \ln e^{i\theta}\\ &= \ln r + i\theta\end{aligned}$$

This gives a complex variable whose real part is $\ln|z|$ and whose imaginary part is the argument of z. Since $\ln|z|$ is negative for $|z| < 1$, then all points inside the unit circle will have their real part move to the negative half plane, and all points outside the unit circle will have their real part in the right half plane. And all points on the unit circle will have their real part be zero. So all point on the unit circle will move to the imaginary axis. For example, the point $(1,0)$ will move to $(0,0)$ and the point $(0,1)$ will move to $\left(0,\frac{\pi}{2}\right)$.

As a point is closer to the origin, it will map closer to $-\infty$ in negative half plane, since $\lim_{r\to 0}\ln(r)$ is $-\infty$.

The imaginary part of each point be the argument of the point z. Since all points now reside in the first quadrant as seen in the above diagram, then the imaginary part will extend from $0\cdots\frac{\pi}{2}$. The following diagram just shows the transformation by $f_4(z)$ for selected points



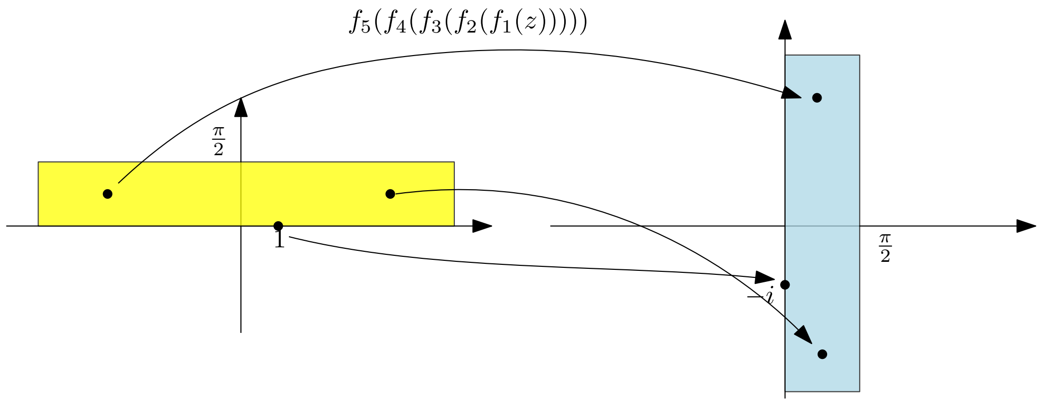
The next step is to apply $-i$ to each point. Hence

$$\begin{aligned}f_5(z) &= -if_4(z)\\ &= e^{-i\frac{\pi}{2}}f_4(z)\end{aligned}$$

Let $z = re^{i\theta} = f_4(z)$ and the above becomes

$$\begin{aligned}f_5(z) &= e^{-i\frac{\pi}{2}}re^{i\theta}\\ &= re^{i\left(\theta-\frac{\pi}{2}\right)}\end{aligned}$$

So the effect of multiplying by $-i$ is rotate each point clockwise by 90^0. Hence the whole strip shown above $(f_4(z))$, will now rotate by 90^0 clockwise. The arguments of each new point location will now be in the range $\frac{\pi}{2}\cdots-\frac{\pi}{2}$ as shown below



The final step is to apply $\exp(z)$ to each point in generated by applying $f_5(z)$. Let a point be $z = x + iy$, then

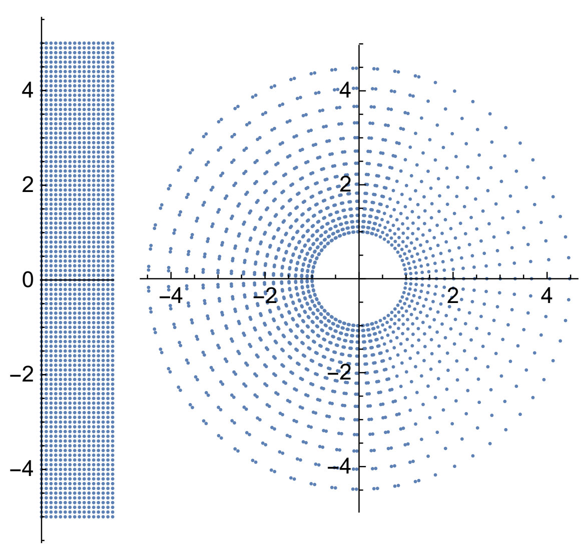
$$\begin{aligned} f_6(z) &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x (\cos y + i \sin y) \\ &= e^x \cos y + i e^x \sin y \end{aligned}$$

Hence the real part of each new point become $e^x \cos y$ and imaginary part become $e^x \sin y$.

All points on imaginary line, with $x = 0$ will map to $\cos y + i \sin y$. All point on the x axis, where $y = 0$ will map to $e^x + 0i$.

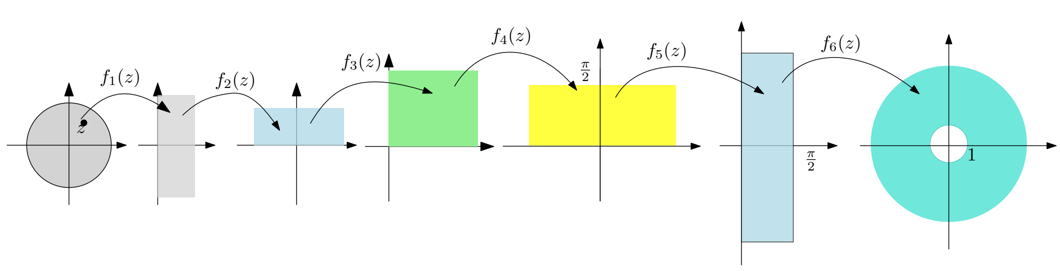
All points on the vertical line $\left(\frac{\pi}{2}, y\right)$ will map to $e^{\frac{\pi}{2}} \cos y + i e^{\frac{\pi}{2}} \sin y$. To better see the mapping, I wrote a small program to plot the above transformation. The function samples points from $x = 0$ to $x = \frac{\pi}{2}$ and samples points from $y = -5$ to $y = 5$. For each such point (x, y) it transforms it to $\left(e^x \cos y, e^x \sin y\right)$.

The result shows that all points map to concentric rings outside the unit circle as shown in the plot below



Result of applying $f_6(z)$. annulus outside unit circle.

Hence the final mapping is to an annulus outside disk on radius 1. The following shows all the transformation applied on the same diagram.

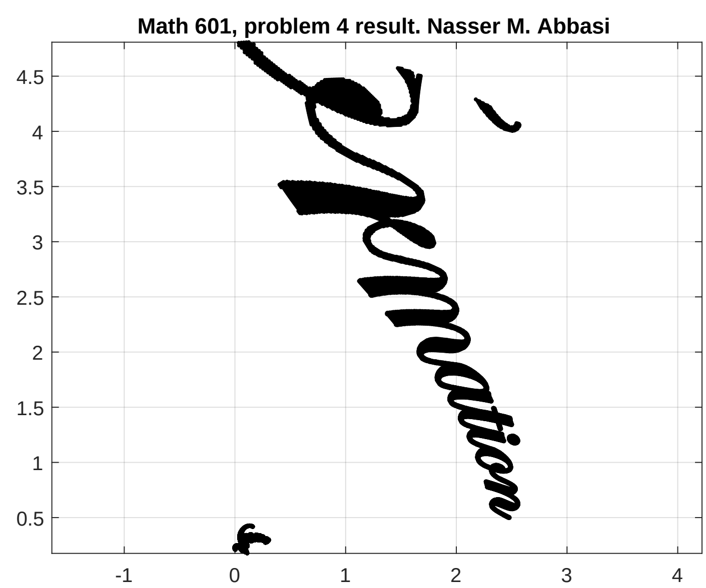


Part (c)

Using Matlab, the code provided was run after applying the function $f_6\left(f_5\left(f_4\left(f_3\left(f_2\left(f_1(z)\right)\right)\right)\right)\right)$ on it.

```
close all
load 'TransPoints.mat';
TR=exp(-1i.*log( sqrt( 1i*((COMPLD+1)./(1-COMPLD) ))));
IMtr=imag(TR);
REtr=real(TR);
plot(REtr,IMtr,'k.')
axis equal
title('Math 601, problem 4 result. Nasser M. Abbasi')
grid
```

The following shows the original image, and the transformed image below it.



3.2 second exam

3.2.1 Problem 1

Find the equilibria of the following differential equation $y' = 1 - y^2$ and determine their stability. Derive the explicit solution for the initial value problem

$$y'(t) = 1 - y^2$$
$$y(0) = -2$$

Find the finite time interval for which the solution exists.

solution

Before solving the problem, the domain of the solution is determined. The RHS of the ODE is $f\left(t,y\right) = 1 - y^2$. This is a continuous and real function for all y. Now $\frac{\partial f}{\partial y} = -2y$ shows it is also continuous and real for all y. Combining these results shows that there exists a solution and is unique in some subset of the domain

$$-\infty < y < \infty$$

The problem is now solved. Since

$$y'(t) = f\left(y\right)$$

Then the equilibrium points are the solution to $f\left(y\right) = 0$ or $1 - y^2 = 0$. Therefore there are two equilibrium points given by

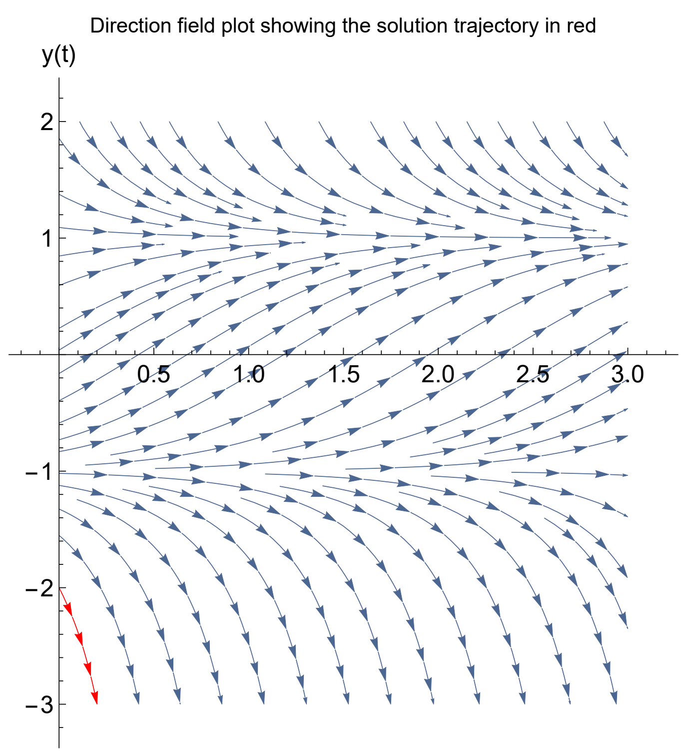
$$y = \pm 1$$

The stability type is determined by taking the second derivative and evaluating it at at each equilibrium point. If the second derivative is negative, then the point is stable equilibrium. If the second derivative is positive then the point is unstable equilibrium. If the second derivative is zero, it is a saddle point. Since

$$y'' = -2y$$

Then at $y = 1$, $y'' < 0$ which implies $\underline{y = 1 \text{ is stable}}$. At $y = -1$, $y'' > 0$ which implies $\underline{y = -1 \text{ is unstable}}$ equilibrium.

The above result was verified by generating the direction field plot for the ODE. It shows that solution lines are moving away from line $y = -1$, which means it is unstable (A solution that starts near $y = -1$ will move away from its initial position). The plot also shows solutions that start near $y = 1$ moving towards $y = 1$. Hence $y = 1$ is stable equilibrium. The line in red is the particular solution trajectory for the initial condition given in the problem.




```
f[t_, y_] := 1 - y^2;
p = StreamPlot[{1, f[t, y]}, {t, 0, 3}, {y, -3, 2},
Frame -> False,
Axes -> True,
AxesLabel -> {"t", "y(t)"},
BaseStyle -> 14,
StreamPoints -> {{{{0, -2}, Red}, Automatic}},
ImageSize -> 400,
PlotLabel -> Style[Text[ "Direction field plot showing the solution trajectory in red"], 12]
]
```

The ODE is now solved.

$$\frac{dy}{dt} = 1 - y^2$$
$$\frac{dy}{1 - y^2} = dt$$

Since it is separable, then Integrating both sides results in

$$\int \frac{dy}{1 - y^2} = \int dt$$
$$\operatorname{arctanh}(y) = t + c$$

Hence the solution is

$$y(t) = \tanh(t + c)$$

But

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

Therefore the solution can be written as

$$
\begin{aligned}
y(t) &= \frac{e^{t+c} - e^{-t-c}}{e^{t+c} + e^{-t-c}} \\
&= \frac{ce^t - \frac{1}{c}e^{-t}}{ce^t + \frac{1}{c}e^{-t}} \\
&= \frac{c^2e^t - e^{-t}}{c^2e^t + e^{-t}} \\
&= \frac{Ce^t - e^{-t}}{Ce^t + e^{-t}} \qquad (1)
\end{aligned}
$$

Using the initial conditions $y(0) = -2$ the above gives the value of C

$$
\begin{aligned}
-2 &= \frac{C-1}{C+1} \\
-2C - 2 &= C - 1 \\
-3C &= 1 \\
C &= \frac{-1}{3}
\end{aligned}
$$

Substituting the constant C value found above into solution (1) gives

$$
\begin{aligned}
y(t) &= \frac{\frac{-1}{3}e^t - e^{-t}}{\frac{-1}{3}e^t + e^{-t}} \\
&= \frac{-e^t - 3e^{-t}}{-e^t + 3e^{-t}} \\
&= \frac{e^t + 3e^{-t}}{e^t - 3e^{-t}}
\end{aligned}
$$

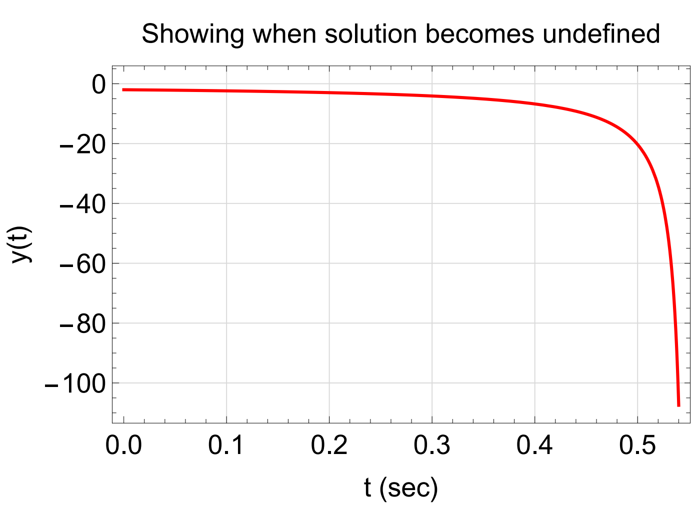
By factoring e^{-t} the above becomes

$$
y(t) = \frac{3 + e^{2t}}{-3 + e^{2t}}
$$

To find when the solution stops, means to find the time when solution becomes undefined. This occurs when the denominator becomes zero (the solution reaches a pole). The denominator of the solution above becomes zero when

$$
\begin{aligned}
-3 + e^{2t} &= 0 \\
2t &= \ln 3 \\
t &= \frac{1}{2}\ln 3
\end{aligned}
$$

Numerically, this is approximately $\underline{t = 0.549 \text{ seconds}}$. Here is a plot of the solution showing what happens when it reaches close to the above t value starting from $t = 0$. The plot shows that the solution diverges to $-\infty$ as the pole is approached from the left and the solution becomes undefined.



```
sol = (3 + Exp[2 t])/(-3 + Exp[2 t]);
p = Plot[sol, {t, 0, 0.54}, PlotRange -> All,
Frame -> True,
GridLines -> Automatic, GridLinesStyle -> LightGray,
PlotStyle -> Red,
FrameLabel -> {{"y(t)", None}%
, {"t (sec)", "Showing when solution becomes undefined"}},
BaseStyle -> 14]
```

3.2.2 Problem 2

2. Consider the initial value problem

$$y'(t) = t\sqrt{y(t)}$$
$$y(1) = k.$$

Determine the k-values for which the above equation has

(a) two real solutions,

(b) infinitely many real solutions,

(c) no real solutions.

(d) Is there a k-value for which $f(t,y) = t\sqrt{y}$ is Lipschitz continuous on the rectangular domain $0.5 \le t \le 1.5, \quad 0.9 \le y \le 1.1$?

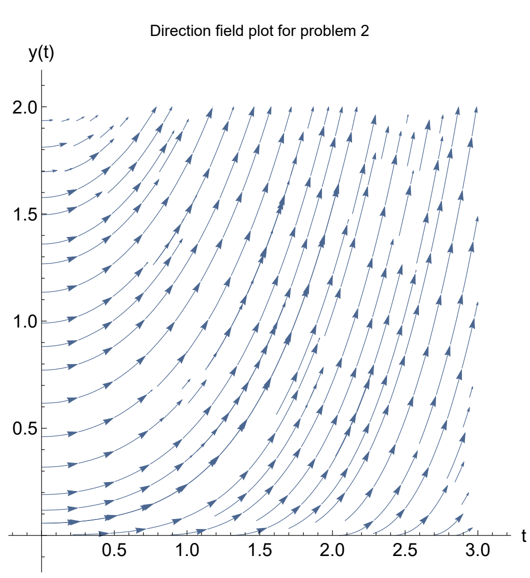
solution

$$\frac{dy}{dt} = t\sqrt{y}$$

The domain of the solution is first found. Since $f\left(t,y\right) = t\sqrt{y}$ then this function is real and continuous for all t and for $y \ge 0$. Since $\frac{\partial f}{\partial y} = \frac{t}{2\sqrt{y}}$ then this is continuous for all t and for $y \neq 0$ (to avoid a pole). Combining these two results shows a solution exists and unique in some subset of the domain

$$-\infty < t < \infty$$
$$y > 0$$

The direction field for the above ode is given in the plot below



```
f[t_, y_] := t Sqrt[y];
p = StreamPlot[{1, f[t, y]}, {t, 0, 3}, {y, 0, 2},
Frame -> False,
Axes -> True,
AxesLabel -> {"t", "y(t)"},
BaseStyle -> 14,
ImageSize -> 400,
PlotLabel -> Style[Text["Direction field plot for problem 2"], 12]
]
```

The ODE is now solved.

$$\begin{aligned}\frac{dy}{dt} &= t\sqrt{y}\\ \frac{dy}{\sqrt{y}} &= tdt\end{aligned}$$

This is separable. Integrating both sides gives

$$\begin{aligned}\int y^{-\frac{1}{2}}dy &= \int tdt\\ 2\sqrt{y} &= \frac{t^2}{2}+c\\ \sqrt{y} &= \frac{t^2}{4}+\frac{c}{2}\end{aligned}$$

Applying initial conditions $y(1)=k$ the above becomes

$$\begin{aligned}\sqrt{k} &= \frac{1}{4}+\frac{c}{2}\\ c &= 2\sqrt{k}-\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\sqrt{y} &= \frac{t^2}{4}+\frac{\left(2\sqrt{k}-\frac{1}{2}\right)}{2}\\ &= \frac{t^2}{4}+\sqrt{k}-\frac{1}{4} \qquad (2)\end{aligned}$$

Or

$$\begin{aligned}y(t) &= \left(\frac{t^2}{4}+\sqrt{k}-\frac{1}{4}\right)^2\\ &= k+\frac{1}{2}\sqrt{k}t^2-\frac{1}{8}t^2+\frac{1}{16}t^4-\frac{1}{2}\sqrt{k}+\frac{1}{16} \qquad (3)\end{aligned}$$

part (a)

Looking at solutions in (3) shows that $k>0$ is needed to obtain two real solutions.

part (b)

When $k=0$ then $y(1)=0$. But from earlier the domain of the unique solution was found to be

$$\begin{aligned}-\infty &< t < \infty\\ y &> 0\end{aligned}$$

Therefore the initial condition point where $y=0$ is outside the above domain. Therefore $k=0$ will generate infinite number of solutions because it the initial condition is outside the domain where the solution have to satisfy in order to be unique.

part (c)

No real solution can be obtained when $k<0$. This is because when k is negative then $\sqrt{k}=i\sqrt{|k|}$ and the solution becomes complex.

part (d)

$$f(t,y) = t\sqrt{y}$$

Let $k=1$. This implies the initial conditions is $y(1)=1$. This means the initial conditions point is inside the domain given. Therefore when $k=1$ then $f(t,y_1)$ becomes, using $y_1(t)$ solution from above, the following

$$
\begin{aligned}
f\left(t,y_1\right) &= t\sqrt{y_1} \\
&= \frac{t}{4}\sqrt{t^4 - t^2 + 8t^2 + \frac{1}{4} + 16 - 4} \\
&= \frac{t}{4}\sqrt{t^4 + 7t^2 + \frac{49}{4}}
\end{aligned}
$$

The above shows that $f\left(t,y_1\right)$ is continuous and real over the range $0.5 \leq t \leq 1.5$. And $\frac{\partial f\left(t,y_1\right)}{\partial y_1}$ becomes

$$
\frac{\partial f\left(t,y_1\right)}{\partial y_1} = \frac{1}{2}\frac{t}{\sqrt{y_1}}
$$

Using $k=1$ in the solution $y_1(t)$ the above becomes

$$
\begin{aligned}
\frac{\partial f\left(t,y_1\right)}{\partial y_1} &= \frac{1}{2}\frac{t}{\frac{1}{4}\sqrt{t^4 + 7t^2 + \frac{49}{4}}} \\
&= \frac{2t}{\sqrt{t^4 + 7t^2 + \frac{49}{4}}}
\end{aligned}
$$

Over the range $0.5 \leq t \leq 1.5$ the denominator above is never zero. Hence there is no pole and therefore $\frac{\partial f\left(t,y_1\right)}{\partial y_1}$ is also continuous and real in the range given. This shows that $f\left(t,y\right)$ is Lipschitz continuous inside a rectangular around initial conditions given for the value $k=1$.

This is not the only k value that could be selected. However the problem is asking for one such k value.

3.2.3 Problem 3

3. If a, b and c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$. If b is set to zero is there any solution with this property; are there any solutions that are not bounded?

Solution

$$
ay'' + by' + cy = 0
$$

Because the coefficients of the ODE are constants, the solution is found by solving for the roots of the characteristic equation

$$
a\lambda^2 + b\lambda + c = 0
$$

The roots are

$$
\lambda = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}
$$

Hence the solution is given by linear combination of each solution $e^{\lambda_1 t}, e^{\lambda_2 t}$ as

$$
\begin{aligned}
y(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
&= c_1 e^{\frac{-b}{2a}t} e^{\frac{\sqrt{b^2-4ac}}{2a}t} + c_1 e^{\frac{-b}{2a}t} e^{\frac{-\sqrt{b^2-4ac}}{2a}t} \\
&= e^{\frac{-b}{2a}t}\left(c_1 e^{\frac{\sqrt{b^2-4ac}}{2a}t} + c_2 e^{\frac{-\sqrt{b^2-4ac}}{2a}t}\right)
\end{aligned}
\tag{1}
$$

The above shows that since $b>0$ and $a>0$ then $e^{\frac{-b}{2a}t}$ will go to zero as $t \to \infty$. This shows that all solutions will eventually go to zero.

When $b=0$, the solution given by (1) reduces to

$$
y(t) = c_1 e^{\frac{\sqrt{-4ac}}{2a}t} + c_2 e^{\frac{-\sqrt{-4ac}}{2a}t}
$$

But because $a>0$ and $c>0$ then $-4ac$ is negative and the discriminant $\sqrt{-4ac}$ becomes complex and the above solution becomes

$$
\begin{aligned}
y(t) &= c_1 e^{\frac{2i\sqrt{ac}}{2a}t} + c_2 e^{\frac{-2i\sqrt{ac}}{2a}t} \\
&= c_1 e^{i\sqrt{\frac{c}{a}}t} + c_2 e^{-i\sqrt{\frac{c}{a}}t} \\
&= C_1 \cos\left(\sqrt{\frac{c}{a}}t\right) + C_2 \sin\left(\sqrt{\frac{c}{a}}t\right)
\end{aligned}
$$

The above shows that the solution never goes to zero as $t \to \infty$ as the solution continues to oscillate. This happened because the damping term b was set to zero, so there is no loss of energy in the system as it moves and therefore once the system is set in motion (by some initial condition away from rest), the system will continue to vibrate for all time.

To obtain unbounded solution, b must be negative while keeping $a > 0$. In this case the solution in (1) becomes

$$
y(t) = e^{\frac{|b|}{2a}t}\left(c_1 e^{\frac{\sqrt{b^2-4ac}}{2a}t} + c_2 e^{\frac{-\sqrt{b^2-4ac}}{2a}t}\right)
$$

The above shows that since $b < 0$ then $e^{\frac{-b}{2a}t} = e^{\frac{|b|}{2a}t}$ and this will cause the solution to blow up as t increases. Negative damping means there is energy being added to the system as it time increases instead of the normal case where damping causes energy to be lost from the system with time. This is why the solution becomes unbounded when $b < 0$. In Physical systems the damping term is always positive.

3.2.4 Problem 4

4. Assume that four bugs moving around the floor with their positions given as

$$
\mathbf{w}_i(t) = \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}, \qquad i = 1, 2, 3, 4.
$$

They are chasing each other in the following fashion: bug No.1. chasing bug No. 2., No. 2. chasing No. 3., No. 3. chasing No. 4., No. 4. chasing No. 1. At any instant each bug travels at its top speed heading staight towards its target (for simplicity assume that the top speed of each is the same: unit speed). Write a vector equation for each bug describing its (vector) velocity in terms of the position of the four bugs. Give an initial value problem for an 8-dimensional nonlinear system of ODEs that describes the dynamics of the chase when the initial positions are

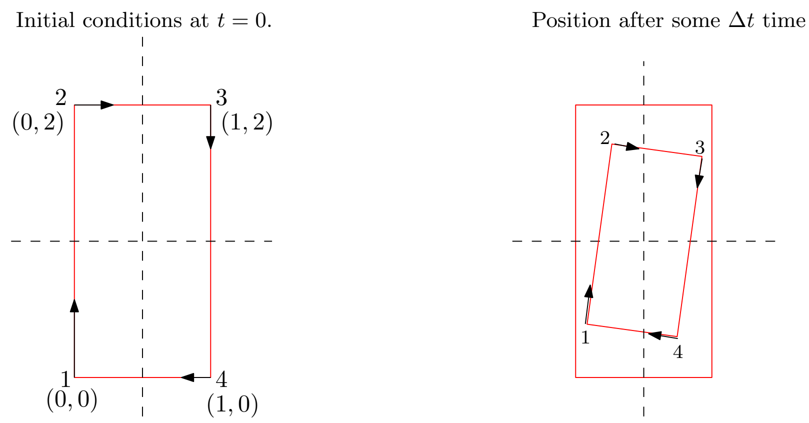
$$
\mathbf{w}_1(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{w}_4(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
$$

respectively

solution

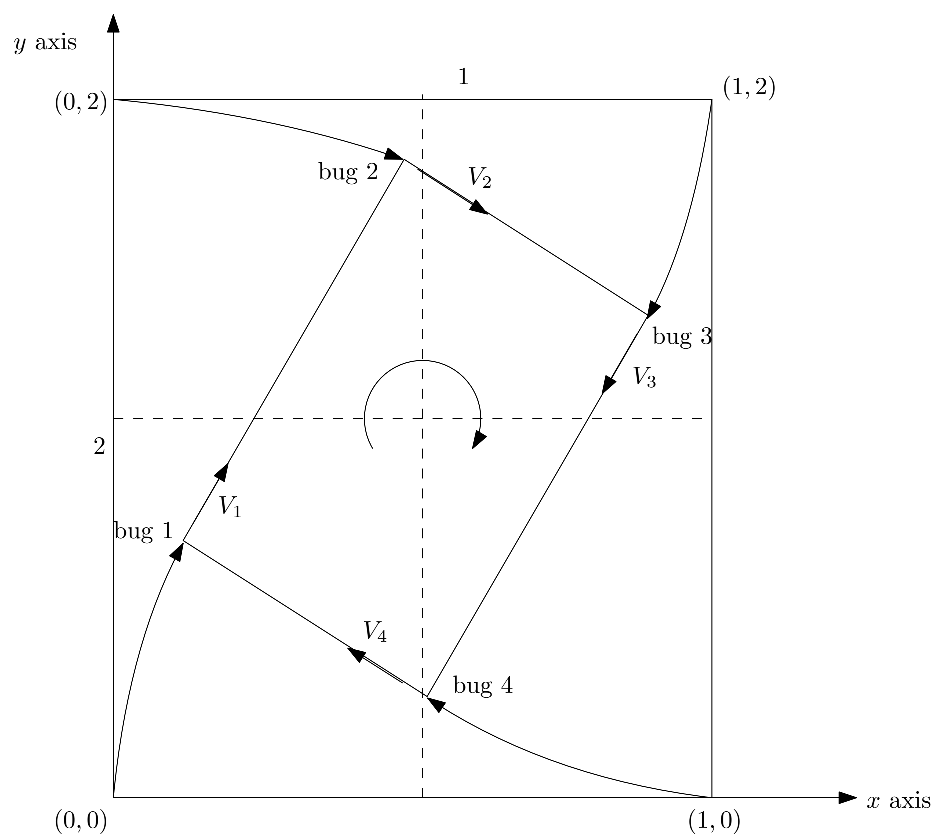
Analysis of motion

The following diagram shows the initial positions of the four bugs and what happens after Δt has elapsed.



Nasser M. Abbasi. ant_1.ipe. 11/11/2018

The four bugs initially are located at the corners of the rectangle. The width is $h = 1$ and the height is $L = 2$. Because each bug moves with the same speed toward the bug adjacent to it (in clockwise direction), then by symmetry, the four bugs will remain on the corners of a rectangle as time increases, but the rectangle shrinks and rotates clockwise in time as the bugs spiral towards the center of the original rectangle where they collide. The following diagram illustrates such motion after some Δt has elapsed.



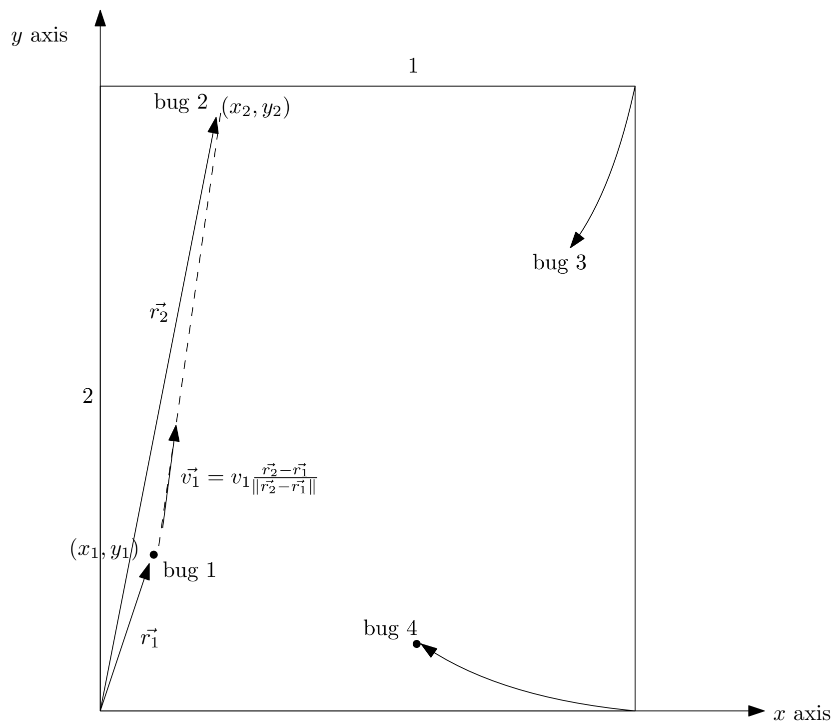
Showing locations of bugs after some Δt.
Rectangle is rotating clockwise and rotating in time. (drawing not to scale)

Nasser M. Abbasi. ant_0.ipe. 11/12/2018

The above shows that at each instance of time, each bug remains at the corner of a scaled down version of the original rectangle that is rotating. Each bug's velocity vector is always pointing straight towards the bug it is chasing. This means that bug's 1 motion is always at 90^0 to the path of bug 2. And bug's 2 motion is at 90^0 to the path of bug 3 and so on.

Equations of motion

To obtain the equation of motion for each bug, each bug's position is considered relative to the bug it is chasing. Starting with bug's 1 relative position to bug 2. This is done with the help of the following diagram



Showing relative locations of bug 1 and 2 after some Δt

Nasser M. Abbasi. ant_2.ipe. 11/11/2018

The position vector of bug 1 is $\vec{r}_1(t)$ and the position vector of bug 2 is $\vec{r}_2(t)$. Therefore

$$
\begin{aligned}
\vec{v}_1 &= \frac{d\vec{r}_1(t)}{dt} \\
&= |\vec{v}_1|\,\hat{r}
\end{aligned}
$$

Where $\hat{r}$ is unit vector in the direction from bug 1 to bug 2. Hence the above can be written as

$$
\frac{d\vec{r}_1(t)}{dt} = |\vec{v}_1| \frac{\vec{r}_2(t) - \vec{r}_1(t)}{\| \vec{r}_2(t) - \vec{r}_1(t) \|}
$$

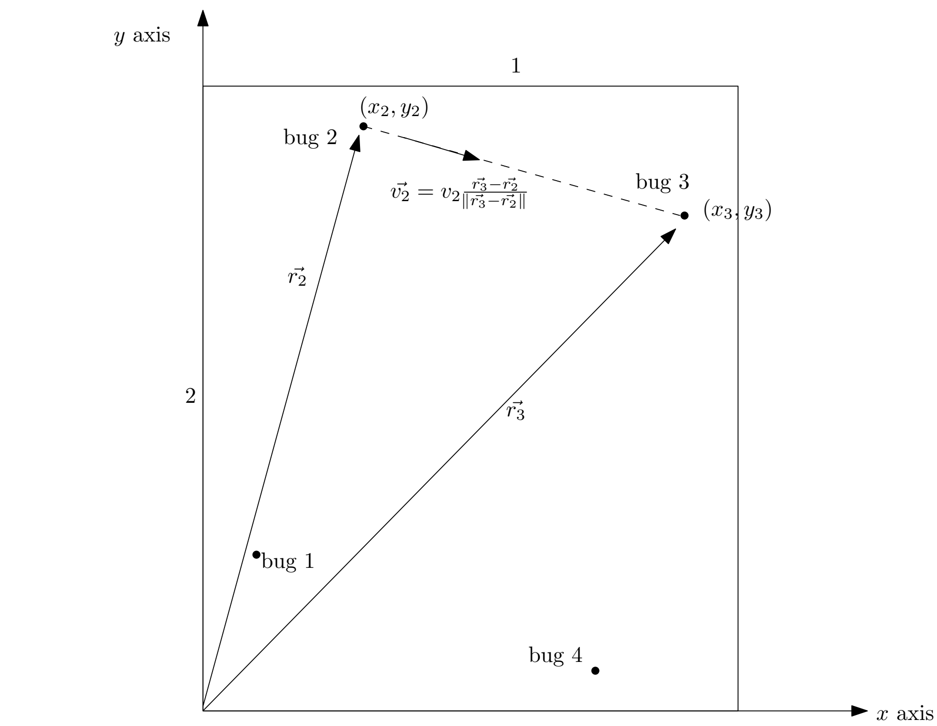
Because $|\vec{v}_1| = 1$ meter per seconds, then the above simplifies to

$$
\begin{aligned}
\frac{d\vec{r}_1(t)}{dt} &= \frac{\left(x_2\hat{\imath} + y_2\hat{\jmath}\right) - \left(x_1\hat{\imath} + y_1\hat{\jmath}\right)}{\| \left(x_2\hat{\imath} + y_2\hat{\jmath}\right) - \left(x_1\hat{\imath} + y_1\hat{\jmath}\right) \|} \\
\left(\frac{dx_1}{dt}\hat{\imath} + \frac{dy_1}{dt}\hat{\jmath}\right) &= \frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + \left(y_2 - y_1\right)^2}}\hat{\imath} + \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + \left(y_2 - y_1\right)^2}}\hat{\jmath}
\end{aligned}
$$

Where x_1, y_1 are the coordinates of bug 1 and x_2, y_2 are the coordinates of bug 2. The above gives the equation of motion for bug 1. Let $x_1' = \frac{dx_1}{dt}$ and $y_1' = \frac{dy_1}{dt}$ for bug 1 then the following are the two equations of motion for bug 1 as function of its position and the position of bug 2

$$
\begin{aligned}
x_1' &= \frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + \left(y_2 - y_1\right)^2}} \\
y_1' &= \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + \left(y_2 - y_1\right)^2}}
\end{aligned} \tag{3}
$$

The same analysis is now carried out to obtain $x_2'(t)$ and $y_2'(t)$ expressions similar to (3) above for bug 2.



Showing relative location of bugs 2 and 3 after some Δt

Nasser M. Abbasi. ant_3.ipe. 11/11/2018

The position vector of bug 2 is $\vec{r}_2(t)$ and the position vector of bug 3 is $\vec{r}_3(t)$. Therefore $\vec{v}_2 = \frac{d\vec{r}_2(t)}{dt} = \left|\vec{v}_2\right| \hat{r}$ where $\hat{r}$ is unit vector in the direction from bug 2 to bug 3. Hence

$$\frac{d\vec{r}_2(t)}{dt} = \left|\vec{v}_2\right| \frac{\vec{r}_3(t) - \vec{r}_2(t)}{\| \vec{r}_3(t) - \vec{r}_2(t) \|}$$

Since $\left|\vec{v}_2\right| = 1$ meter per seconds then

$$\begin{aligned}
\frac{d\vec{r}_2(t)}{dt} &= \frac{\left(x_3\hat{\imath} + y_2\hat{\jmath}\right) - \left(x_3\hat{\imath} + y_2\hat{\jmath}\right)}{\| \left(x_3\hat{\imath} + y_2\hat{\jmath}\right) - \left(x_3\hat{\imath} + y_2\hat{\jmath}\right) \|} \\
\left(\frac{dx_2}{dt}\hat{\imath} + \frac{dy_2}{dt}\hat{\jmath}\right) &= \frac{x_3 - x_2}{\sqrt{(x_3 - x_2)^2 + \left(y_3 - y_2\right)^2}}\hat{\imath} + \frac{y_3 - y_2}{\sqrt{(x_3 - x_2)^2 + \left(y_3 - y_2\right)^2}}\hat{\jmath}
\end{aligned}$$

Where x_2, y_2 are the coordinates of bug 2 and x_3, y_3 are the coordinates of bug 3. The above gives the two equations of motion for bug 2. Using $x_2' = \frac{dx_2}{dt}$ and $y_2' = \frac{dy_2}{dt}$ for bug 2, then the following gives the two equations of motion for bug 2 as function of its position and the position of bug 3

$$\begin{aligned}
x_2' &= \frac{x_3 - x_2}{\sqrt{(x_3 - x_2)^2 + \left(y_3 - y_2\right)^2}} \\
y_2' &= \frac{y_3 - y_2}{\sqrt{(x_3 - x_2)^2 + \left(y_3 - y_2\right)^2}}
\end{aligned} \tag{3}$$

The same analysis is carried out for bug 3 and bug 4, which results in similar equations. Therefore the final equations of motions in vector form are

$$\mathrm{x}' = f(\mathrm{x})$$

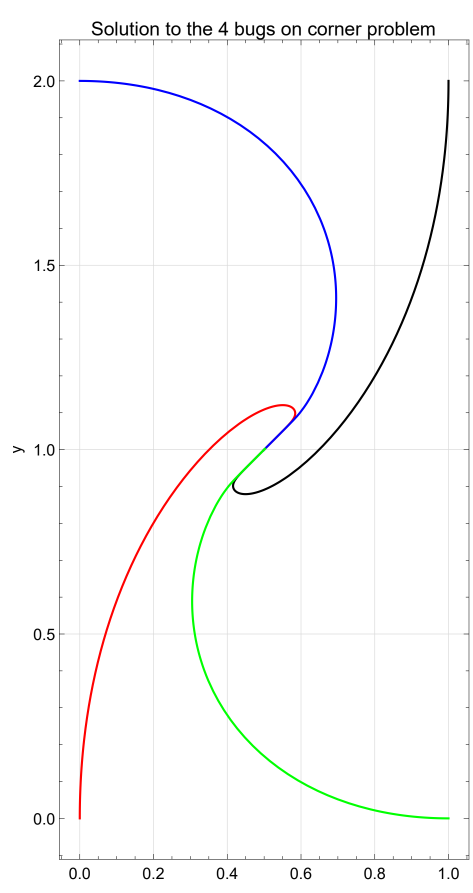
Or

$$\begin{pmatrix} x_1'(t) = \frac{x_2-x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} \\ y_1'(t) = \frac{y_2-y_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} \\ x_2'(t) = \frac{x_3-x_2}{\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}} \\ y_2'(t) = \frac{y_3-y_2}{\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}} \\ x_3'(t) = \frac{x_4-x_3}{\sqrt{(x_4-x_3)^2+(y_4-y_3)^2}} \\ y_3'(t) = \frac{y_4-y_3}{\sqrt{(x_4-x_3)^2+(y_4-y_3)^2}} \\ x_4'(t) = \frac{x_1-x_4}{\sqrt{(x_1-x_4)^2+(y_1-y_4)^2}} \\ y_4'(t) = \frac{y_1-y_4}{\sqrt{(x_1-x_4)^2+(y_1-y_4)^2}} \end{pmatrix}$$

With the initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ y_1(0) \\ x_2(0) \\ y_2(0) \\ x_3(0) \\ y_3(0) \\ x_4(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

The above system of equation can not written as $\mathbf{x}' = A\mathbf{x}$ because the equations of motion are not linear. These ODE's have to solved numerically. The following is the result of running the numerical solution for 1.5 seconds. The code used is listed below. This shows the bugs spiraling down to the center of the original rectangle as expected.

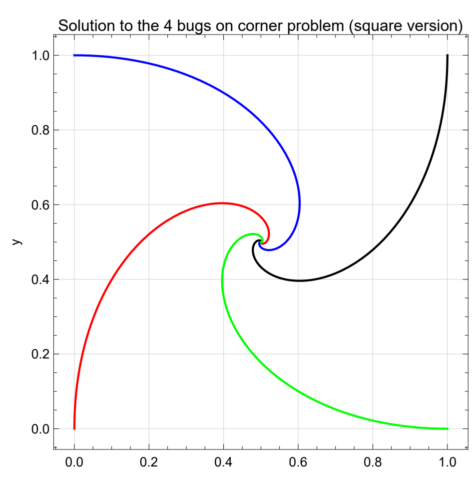


```
ode1 =x1'[t] == (x2[t] - x1[t])/Sqrt[(x2[t] - x1[t])^2 + (y2[t] - y1[t])^2];
ode2 =y1'[t]== (y2[t] - y1[t])/Sqrt[(x2[t] - x1[t])^2 + (y2[t] - y1[t])^2];
ode3 =x2'[t] == (x3[t] - x2[t])/Sqrt[(x3[t] - x2[t])^2 + (y3[t] - y2[t])^2];
ode4 =y2'[t] == (y3[t] - y2[t])/Sqrt[(x3[t] - x2[t])^2 + (y3[t] - y2[t])^2];
ode5 =x3'[t]== (x4[t] - x3[t])/Sqrt[(x4[t] - x3[t])^2 + (y4[t] - y3[t])^2];
ode6 =y3'[t]== (y4[t] - y3[t])/Sqrt[(x4[t] - x3[t])^2 + (y4[t] - y3[t])^2];
ode7 =x4'[t]== (x1[t] - x4[t])/Sqrt[(x1[t] - x4[t])^2 + (y1[t] - y4[t])^2];
ode8 =y4'[t] == (y1[t] - y4[t])/Sqrt[(x1[t] - x4[t])^2 + (y1[t] - y4[t])^2];
sol = NDSolve[{ode1, ode2, ode3, ode4, ode5, ode6, ode7, ode8, x1[0] == 0,
y1[0] == 0, x2[0] == 0, y2[0] == 2, x3[0] == 1, y3[0] == 2, x4[0] == 1,
y4[0] == 0},
{x1[t], y1[t], x2[t], y2[t], x3[t], y3[t], x4[t], y4[t]}, {t, 0, 1.5}];
p = ParametricPlot[{x1[t], y1[t], x2[t], y2[t], x3[t], y3[t], x4[t], y4[t]}
/. sol,
{t, 0, 1.5}, AxesOrigin -> {0, 0},
GridLines -> Automatic, GridLinesStyle -> LightGray, Frame -> True,
FrameLabel -> {{"y", None}, {"x", "Solution to problem 4"}},
ImageSize -> 350]
```

This problem was also solved for a square instead of a rectangle. The only change needed was to modify the initial conditions so as to locate the bugs at corners of unit square as shown below. No changes are needed in the equations of motion.

$$\mathrm{x}(0)=\begin{pmatrix} x_1(0)\\ y_1(0)\\ x_2(0)\\ y_2(0)\\ x_3(0)\\ y_3(0)\\ x_4(0)\\ y_4(0)\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\\1\\1\\1\\0\end{pmatrix}$$

The time needed to reach the center in this case is one second. The following plot shows the path generated for the bugs at the corners of the square.



3.2.5 Problem 5

5. Determine the long term behavior of the solution $(x(t), y(t))$ of the following initial value problem

$$\begin{cases} x' = -2x - y & x(1) = 2 \\ y' = 2x - y & y(1) = 4 \end{cases}$$

by determining the limits $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$

solution

The system can be written using $\mathbf{x}' = A\mathbf{x}$ as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Where $A = \begin{pmatrix} -2 & -1 \\ 2 & -1 \end{pmatrix}$, The eigenvalues of A are found using $\det(A - \lambda I) = 0$ which gives

$$\begin{vmatrix} -2-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)(-1-\lambda) + 2 = 0$$

$$\lambda^2 + 3\lambda + 4 = 0$$

The roots of the above characteristic equation are

$$\begin{aligned} \lambda &= \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \\ &= \frac{-3}{2} \pm \frac{1}{2}\sqrt{9 - 4(4)} \\ &= \frac{-3}{2} \pm \frac{1}{2}\sqrt{-7} \\ &= \frac{-3}{2} \pm \frac{i}{2}\sqrt{7} \end{aligned}$$

Therefore the roots are

$$\lambda_1 = -\frac{3}{2} - i\frac{\sqrt{7}}{2}$$

$$\lambda_2 = -\frac{3}{2} + i\frac{\sqrt{7}}{2}$$

The above shows that the solution will go to zero for large t since the eigenvalues have negative real part. The system is asymptotically stable. The complex conjugate parts of the eigenvalues give solutions that will oscillate with frequency $\frac{\sqrt{7}}{2}$ rad/sec. To obtain the actual solution the eigenvectors are now found for each eigenvalue. Since the eigenvalues are unique, then there is one eigenvector for each eigenvalue.

For $\lambda_1 = -\frac{3}{2} - i\frac{\sqrt{7}}{2}$

$$(A - \lambda_1 I)\,\mathbf{v}_1 = 0$$

$$\begin{pmatrix} -2 - \left(-\frac{3}{2} - i\frac{\sqrt{7}}{2}\right) & -1 \\ 2 & -1 - \left(-\frac{3}{2} - i\frac{\sqrt{7}}{2}\right) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let $v_2 = 1$. The first equation gives $-2 - \left(-\frac{3}{2} - i\frac{\sqrt{7}}{2}\right)v_1 - 1 = 0$ or $v_1 = \frac{1}{-2-\left(-\frac{3}{2}-i\frac{\sqrt{7}}{2}\right)} = \frac{1}{\frac{1}{2}i\sqrt{7}-\frac{1}{2}} = \frac{\frac{-1}{2}i\sqrt{7}-\frac{1}{2}}{\left(\frac{1}{2}i\sqrt{7}-\frac{1}{2}\right)\left(\frac{1}{2}i\sqrt{7}-\frac{1}{2}\right)} = \frac{\frac{-1}{2}i\sqrt{7}-\frac{1}{2}}{2} = -i\frac{\sqrt{7}}{4} - \frac{1}{4}$. Hence the first eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} -i\frac{\sqrt{7}}{4} - \frac{1}{4} \\ 1 \end{pmatrix}$$

For $\lambda_2 = -\frac{3}{2} + i\frac{\sqrt{7}}{2}$

$$(A - \lambda_2 I)\,\mathbf{v}_1 = 0$$

$$\begin{pmatrix} -2 - \left(-\frac{3}{2} + i\frac{\sqrt{7}}{2}\right) & -1 \\ 2 & -1 - \left(-\frac{3}{2} + i\frac{\sqrt{7}}{2}\right) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let $v_2 = 1$. The first equation gives $-2 - \left(-\frac{3}{2} + i\frac{\sqrt{7}}{2}\right)v_1 - 1 = 0$ or $v_1 = \frac{1}{-2-\left(-\frac{3}{2}+i\frac{\sqrt{7}}{2}\right)} = \frac{1}{-\frac{1}{2}i\sqrt{7}-\frac{1}{2}} = \frac{\frac{1}{2}i\sqrt{7}-\frac{1}{2}}{2} =$

$\frac{1}{4}i\sqrt{7}-\frac{1}{4}$. Hence the second eigenvector is

$$\mathrm{v}_2=\begin{pmatrix} i\frac{\sqrt{7}}{4}-\frac{1}{4} \\ 1 \end{pmatrix}$$

Using the above two linearly independent eigenvectors, the two basis solutions are

$$\mathrm{x}_1=\mathrm{v}_1 e^{\lambda_1 t}$$
$$\mathrm{x}_2=\mathrm{v}_2 e^{\lambda_1 t}$$

The solution is a linear combination of the above solutions

$$\mathrm{x}=c_1\mathrm{x}_1+c_1\mathrm{x}_2$$

The solution is converted to real solution by taking the real and imaginary part of one of the basis solution above. Therefore

$$\mathrm{x}_3=\mathrm{Re}\,(\mathrm{x}_1)$$
$$\mathrm{x}_4=\mathrm{Im}\,(\mathrm{x}_1)$$

The solution becomes

$$\mathrm{x}=c_3\mathrm{x}_3+c_4\mathrm{x}_4 \tag{1}$$

But

$$\begin{aligned}
\mathrm{Re}\,(\mathrm{x}_1) &= \mathrm{Re}\left[\begin{pmatrix} -i\frac{\sqrt{7}}{4}-\frac{1}{4} \\ 1 \end{pmatrix} e^{\lambda_1 t}\right] \\
&= \mathrm{Re}\begin{pmatrix} -i\frac{\sqrt{7}}{4}e^{\lambda_1 t}-\frac{1}{4}e^{\lambda_1 t} \\ e^{\lambda_1 t} \end{pmatrix} \\
&= \mathrm{Re}\begin{pmatrix} -i\frac{\sqrt{7}}{4}e^{\left(-\frac{3}{2}-i\sqrt{\frac{7}{4}}\right)t}-\frac{1}{4}e^{\left(-\frac{3}{2}-i\sqrt{\frac{7}{4}}\right)t} \\ e^{\left(-\frac{3}{2}-i\sqrt{\frac{7}{4}}\right)t} \end{pmatrix} \\
&= \mathrm{Re}\begin{pmatrix} -i\frac{\sqrt{7}}{4}e^{\frac{-3}{2}t}\left(\cos\sqrt{\frac{7}{4}}t-i\sin\sqrt{\frac{7}{4}}t\right)-\frac{1}{4}e^{\frac{-3}{2}t}\left(\cos\sqrt{\frac{7}{4}}t-i\sin\sqrt{\frac{7}{4}}t\right) \\ e^{\frac{-3}{2}t}\left(\cos\sqrt{\frac{7}{4}}t-i\sin\sqrt{\frac{7}{4}}t\right) \end{pmatrix} \\
&= \mathrm{Re}\begin{pmatrix} \sqrt{\frac{7}{16}}e^{\frac{-3}{2}t}\left(-i\cos\sqrt{\frac{7}{4}}t-\sin\sqrt{\frac{7}{4}}t\right)-\frac{1}{4}e^{\frac{-3}{2}t}\left(\cos\sqrt{\frac{7}{4}}t-i\sin\sqrt{\frac{7}{4}}t\right) \\ e^{\frac{-3}{2}t}\left(\cos\sqrt{\frac{7}{4}}t-i\sin\sqrt{\frac{7}{4}}t\right) \end{pmatrix} \\
&= \mathrm{Re}\begin{pmatrix} e^{\frac{-3}{2}t}\left(-\sqrt{\frac{7}{16}}\sin\sqrt{\frac{7}{4}}t-\frac{1}{4}\cos\sqrt{\frac{7}{4}}t\right)+ie^{\frac{-3}{2}t}\left(-\sqrt{\frac{7}{16}}\cos\sqrt{\frac{7}{4}}t+\frac{1}{4}\sin\sqrt{\frac{7}{4}}t\right) \\ e^{\frac{-3}{2}t}\cos\sqrt{\frac{7}{4}}t-ie^{\frac{-3}{2}t}\sin\sqrt{\frac{7}{4}}t \end{pmatrix} \\
&= \begin{pmatrix} e^{\frac{-3}{2}t}\left(-\sqrt{\frac{7}{16}}\sin\sqrt{\frac{7}{4}}t-\frac{1}{4}\cos\sqrt{\frac{7}{4}}t\right) \\ e^{\frac{-3}{2}t}\cos\sqrt{\frac{7}{4}}t \end{pmatrix}
\end{aligned} \tag{2}$$

And

$$\begin{aligned}
\mathrm{Im}\,(\mathrm{x}_1) &= \mathrm{Im}\begin{pmatrix} e^{\frac{-3}{2}t}\left(-\sqrt{\frac{7}{16}}\sin\sqrt{\frac{7}{4}}t-\frac{1}{4}\cos\sqrt{\frac{7}{4}}t\right)+ie^{\frac{-3}{2}t}\left(-\sqrt{\frac{7}{16}}\cos\sqrt{\frac{7}{4}}t+\frac{1}{4}\sin\sqrt{\frac{7}{4}}t\right) \\ e^{\frac{-3}{2}t}\cos\sqrt{\frac{7}{4}}t-ie^{\frac{-3}{2}t}\sin\sqrt{\frac{7}{4}}t \end{pmatrix} \\
&= \begin{pmatrix} e^{\frac{-3}{2}t}\left(-\sqrt{\frac{7}{16}}\cos\sqrt{\frac{7}{4}}t+\frac{1}{4}\sin\sqrt{\frac{7}{4}}t\right) \\ -e^{\frac{-3}{2}t}\sin\sqrt{\frac{7}{4}}t \end{pmatrix}
\end{aligned} \tag{3}$$

Using (2,3) in (1) gives the solution

$$\mathrm{x}=c_3\,\mathrm{Re}\,(\mathrm{x}_1)+c_4\,\mathrm{Im}\,(\mathrm{x}_1)$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_3\begin{pmatrix} e^{\frac{-3}{2}t}\left(-\sqrt{\frac{7}{16}}\sin\sqrt{\frac{7}{4}}t-\frac{1}{4}\cos\sqrt{\frac{7}{4}}t\right) \\ e^{\frac{-3}{2}t}\cos\sqrt{\frac{7}{4}}t \end{pmatrix} + c_4\begin{pmatrix} e^{\frac{-3}{2}t}\left(-\sqrt{\frac{7}{16}}\cos\sqrt{\frac{7}{4}}t+\frac{1}{4}\sin\sqrt{\frac{7}{4}}t\right) \\ -e^{\frac{-3}{2}t}\sin\sqrt{\frac{7}{4}}t \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_3e^{\frac{-3}{2}t}\left(-\sqrt{\frac{7}{16}}\sin\sqrt{\frac{7}{4}}t-\frac{1}{4}\cos\left(\frac{\sqrt{7}t}{2}\right)\right)+c_4e^{\frac{-3}{2}t}\left(-\frac{\sqrt{7}}{4}\cos\left(\frac{\sqrt{7}t}{2}\right)+\frac{1}{4}\sin\left(\frac{\sqrt{7}t}{2}\right)\right) \\ c_3e^{\frac{-3}{2}t}\cos\left(\frac{\sqrt{7}t}{2}\right)-c_4e^{\frac{-3}{2}t}\sin\left(\frac{\sqrt{7}t}{2}\right) \end{pmatrix}$$

Therefore

$$
\begin{aligned}
x(t) &= c_3 e^{\frac{-3}{2}t}\left(-\frac{\sqrt{7}}{4}\sin\left(\frac{\sqrt{7}t}{2}\right) - \frac{1}{4}\cos\left(\frac{\sqrt{7}t}{2}\right)\right) + c_4 e^{\frac{-3}{2}t}\left(-\frac{\sqrt{7}}{4}\cos\left(\frac{\sqrt{7}t}{2}\right) + \frac{1}{4}\sin\left(\frac{\sqrt{7}t}{2}\right)\right) \\
y(t) &= c_3 e^{\frac{-3}{2}t}\cos\left(\frac{\sqrt{7}t}{2}\right) - c_4 e^{\frac{-3}{2}t}\sin\left(\frac{\sqrt{7}t}{2}\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
x(t) &= e^{\frac{-3}{2}t}\left(c_3\left(-\frac{\sqrt{7}}{4}\sin\left(\frac{\sqrt{7}t}{2}\right) - \frac{1}{4}\cos\left(\frac{\sqrt{7}t}{2}\right)\right) + c_4\left(-\frac{\sqrt{7}}{4}\cos\left(\frac{\sqrt{7}t}{2}\right) + \frac{1}{4}\sin\left(\frac{\sqrt{7}t}{2}\right)\right)\right) \\
y(t) &= e^{\frac{-3}{2}t}\left(c_3\cos\left(\frac{\sqrt{7}t}{2}\right) - c_4\sin\left(\frac{\sqrt{7}t}{2}\right)\right)
\end{aligned}
$$

Let $C_1 = c_3$ and $C_2 = -c_3$, and the above becomes

$$
\begin{aligned}
x(t) &= -\frac{1}{4}e^{\frac{-3}{2}t}\left(C_1\sqrt{7}\sin\left(\frac{\sqrt{7}t}{2}\right) + C_1\cos\left(\frac{\sqrt{7}t}{2}\right) - \sqrt{7}C_2\cos\left(\frac{\sqrt{7}t}{2}\right) + C_2\sin\left(\frac{\sqrt{7}t}{2}\right)\right) \\
y(t) &= e^{\frac{-3}{2}t}\left(C_1\cos\left(\frac{\sqrt{7}t}{2}\right) + C_2\sin\left(\frac{\sqrt{7}t}{2}\right)\right)
\end{aligned} \tag{4}
$$

Initial conditions are now used to find C_1, C_2. At $t = 1$ the above becomes

$$
\begin{aligned}
2 &= -\frac{1}{4}e^{\frac{-3}{2}}\left(C_1\sqrt{7}\sin\left(\frac{\sqrt{7}}{2}\right) + C_1\cos\left(\frac{\sqrt{7}}{2}\right) - \sqrt{7}C_2\cos\left(\frac{\sqrt{7}}{2}\right) + C_2\sin\left(\frac{\sqrt{7}}{2}\right)\right) \\
4 &= e^{\frac{-3}{2}}\left(C_1\cos\left(\frac{\sqrt{7}}{2}\right) + C_2\sin\left(\frac{\sqrt{7}}{2}\right)\right)
\end{aligned}
$$

In system form the above becomes

$$
\begin{aligned}
\begin{pmatrix} 2 \\ 4 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{4}e^{\frac{-3}{2}}\sqrt{7}\sin\left(\frac{\sqrt{7}}{2}\right) - \frac{1}{4}e^{\frac{-3}{2}}\cos\left(\frac{\sqrt{7}}{2}\right) & \frac{1}{4}e^{\frac{-3}{2}}\sqrt{7}\cos\left(\frac{\sqrt{7}}{2}\right) - \frac{1}{4}e^{\frac{-3}{2}}\sin\left(\frac{\sqrt{7}}{2}\right) \\ e^{\frac{-3}{2}}\cos\left(\frac{\sqrt{7}}{2}\right) & e^{\frac{-3}{2}}\sin\left(\frac{\sqrt{7}}{2}\right) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\
&= \begin{pmatrix} -0.15676 & -0.01786 \\ 0.05475 & 0.21631 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}
\end{aligned}
$$

Solving for $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ by elimination gives

$$
\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} -15.307 \\ 22.367 \end{pmatrix}
$$

Using these constants in the the solution (4) results in

$$
\begin{aligned}
x(t) &= -\frac{1}{4}e^{\frac{-3}{2}t}\left((-15.307)\sqrt{7}\sin\left(\frac{\sqrt{7}t}{2}\right) - 15.307\cos\left(\frac{\sqrt{7}t}{2}\right) - \sqrt{7}(22.367)\cos\left(\frac{\sqrt{7}t}{2}\right) + 22.367\sin\left(\frac{\sqrt{7}t}{2}\right)\right) \\
y(t) &= e^{\frac{-3}{2}t}\left(-15.307\cos\left(\frac{\sqrt{7}t}{2}\right) + 22.367\sin\left(\frac{\sqrt{7}t}{2}\right)\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
x(t) &= -\frac{1}{4}e^{\frac{-3}{2}t}\left(-40.499\sin\left(\frac{\sqrt{7}t}{2}\right) - 15.307\cos\left(\frac{\sqrt{7}t}{2}\right) - 59.178\cos\left(\frac{\sqrt{7}t}{2}\right) + 22.367\sin\left(\frac{\sqrt{7}t}{2}\right)\right) \\
y(t) &= e^{\frac{-3}{2}t}\left(-15.307\cos\left(\frac{\sqrt{7}t}{2}\right) + 22.367\sin\left(\frac{\sqrt{7}t}{2}\right)\right)
\end{aligned}
$$

Simplifying the above using trigonometric relations gives

$$
\begin{aligned}
x(t) &= -\frac{1}{4}e^{\frac{-3}{2}t}\left(-74.485\cos\left(\frac{\sqrt{7}t}{2}\right) - 18.132\sin\left(\frac{\sqrt{7}t}{2}\right)\right) \\
y(t) &= e^{\frac{-3}{2}t}\left(-15.307\cos\left(\frac{\sqrt{7}t}{2}\right) + 22.367\sin\left(\frac{\sqrt{7}t}{2}\right)\right)
\end{aligned}
$$

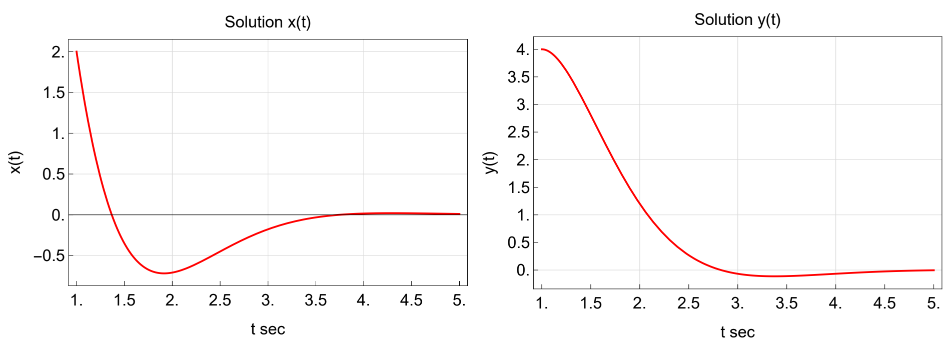
Or

$$x(t) = e^{\frac{-3}{2}t}\left(18.621\cos\left(\frac{\sqrt{7}t}{2}\right) + 4.533\sin\left(\frac{\sqrt{7}t}{2}\right)\right)$$
$$y(t) = e^{\frac{-3}{2}t}\left(-15.307\cos\left(\frac{\sqrt{7}t}{2}\right) + 22.367\sin\left(\frac{\sqrt{7}t}{2}\right)\right) \tag{5}$$

The above shows that due to the exponentially decaying term in the solution, then

$$\lim_{t\to\infty}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The following is a plot of $x(t)$ and $y(t)$ for t from 1 to 5 seconds showing both solutions go to zero quickly due to the $e^{\frac{-3}{2}t}$ term.




```
ClearAll[t];
myXSol=Exp[-3/2 t](18.621 Cos[Sqrt[7] t/2]+4.533 Sin[Sqrt[7] t/2]);
myYSol=Exp[-3/2 t](-15.307 Cos[Sqrt[7] t/2]+22.367 Sin[Sqrt[7] t/2]);
p1=Plot[myXSol,{t,1,5},PlotRange->All,Frame->True,
FrameLabel->{{"x(t)",None},{"t sec","Solution x(t)"}},
PlotStyle->Red,
GridLines->Automatic,GridLinesStyle->LightGray,
BaseStyle->14,ImageSize->400,
FrameTicks->{{Range[-1,2,.5],None},{Range[0,5,.5],None}}];
p2=Plot[myYSol,{t,1,5},PlotRange->All,Frame->True,
FrameLabel->{{"y(t)",None},{"t sec","Solution y(t)"}},
PlotStyle->Red,GridLines->Automatic,GridLinesStyle->LightGray,
BaseStyle->14,ImageSize->400,
FrameTicks->{{Range[-1,4,.5],None},{Range[0,5,.5],None}}];
p=Grid[{{p1,p2}}]
```

3.2.6 Problem 6

6. Find the general solution of the homogeneous linear second order differential equation

$$3t^2y'' + ty' + y = 0$$

(Hint: look for solution as a $t-$power).

<u>Solution</u>

$$3t^2y'' + ty' + y = 0$$

Since the powers on the t coefficients match the order of the derivatives in each term of the ODE, then this is called the Euler ODE. Its solution can be found by assuming solution has this form (Using the hint given)

$$y(t) = t^{\alpha} \tag{1}$$

Therefore

$$y' = \alpha t^{\alpha-1}$$
$$y'' = \alpha(\alpha-1)t^{\alpha-2}$$

Substituting these in the original ODE gives the characteristic equation to solve for α

$$\begin{aligned} 3t^2\alpha(\alpha-1)t^{\alpha-2} + t\alpha t^{\alpha-1} + t^\alpha &= 0 \\ 3\alpha(\alpha-1)t^\alpha + \alpha t^\alpha + t^\alpha &= 0 \\ t^\alpha(3\alpha(\alpha-1)+\alpha+1) &= 0 \end{aligned}$$

Since $t^\alpha \neq 0$ (else this will result in a trivial solution), the characteristic equation is $3\alpha(\alpha-1)+\alpha+1 = 0$ or

$$3\alpha^2 - 2\alpha + 1 = 0$$

Using the quadratic formula, the roots of the above characteristic equation are

$$\begin{aligned} \alpha_1 &= \frac{1}{3} + \frac{1}{3}i\sqrt{2} \\ \alpha_2 &= \frac{1}{3} - \frac{1}{3}i\sqrt{2} \end{aligned}$$

The solution is a linear combination of the basis solutions $t^{\alpha_1}, t^{\alpha_2}$. Hence

$$\begin{aligned} y(t) &= c_1 t^{\alpha_1} + c_2 t^{\alpha_2} \\ &= c_1 t^{\left(\frac{1}{3}+\frac{1}{3}i\sqrt{2}\right)} + c_2 t^{\left(\frac{1}{3}-\frac{1}{3}i\sqrt{2}\right)} \\ &= c_1 t^{\frac{1}{3}} t^{\frac{1}{3}i\sqrt{2}} + c_2 t^{\frac{1}{3}} t^{-\frac{1}{3}i\sqrt{2}} \\ &= t^{\frac{1}{3}}\left(c_1 t^{\frac{1}{3}i\sqrt{2}} + c_2 t^{-\frac{1}{3}i\sqrt{2}}\right) \end{aligned} \tag{2}$$

But

$$\begin{aligned} t^{\frac{1}{3}i\sqrt{2}} &= e^{\ln\left(t^{\frac{1}{3}i\sqrt{2}}\right)} \\ &= e^{\frac{1}{3}i\sqrt{2}\ln t} \end{aligned}$$

And

$$\begin{aligned} t^{\frac{-1}{3}i\sqrt{2}} &= e^{\ln\left(t^{\frac{-1}{3}i\sqrt{2}}\right)} \\ &= e^{\frac{-1}{3}i\sqrt{2}\ln t} \end{aligned}$$

Using the above two equations in (2) then the solution (2) becomes

$$y(t) = t^{\frac{1}{3}}\left(c_1 e^{\frac{1}{3}i\sqrt{2}\ln t} + c_2 e^{\frac{-1}{3}i\sqrt{2}\ln t}\right)$$

Using Euler relation the above solution is written using sin and cos to become

$$\boxed{y(t) = t^{\frac{1}{3}}\left(C_1\cos\left(\frac{\sqrt{2}\ln t}{3}\right) + C_2\sin\left(\frac{\sqrt{2}\ln t}{3}\right)\right)}$$

3.2.7 Problem 7

7. Compute the general solution of the following linear constant coefficient system of ODEs

$$\begin{aligned} y_1' &= 3y_1 + 2y_2 + y_3 \\ y_2' &= -y_1 + 3y_2 + 2y_3 \\ y_3' &= y_1 - 3y_2 - 2y_3 \end{aligned}$$

Solution

$$\begin{aligned} y_1' &= 3y_1 + 2y_2 + y_3 \\ y_2' &= -y_1 + 3y_2 + 2y_3 \\ y_3' &= y_1 - 3y_2 - 2y_3 \end{aligned}$$

The system is written using $\mathbf{y}' = A\mathbf{y}$ as

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & -3 & -2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

Where $A = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & -3 & -2 \end{pmatrix}$. The eigenvalues are found by solving $\det(A - I\lambda) = 0$ which gives

$$\begin{vmatrix} 3-\lambda & 2 & 1 \\ -1 & 3-\lambda & 2 \\ 1 & -3 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 4\lambda^2 + 4\lambda = 0$$

$$\left(\lambda^2 - 4\lambda + 4\right)\lambda = 0$$

$$(\lambda - 2)(\lambda - 2)\lambda = 0$$

Hence the eigenvalues are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$

Where λ_2 has algebraic multiplicity 2. The eigenvector associated with $\lambda_1 = 0$ is now found and then an additional two two linearly independent eigenvectors are needed that are associated with the second eigenvalue λ_2. The eigenvector $\mathbf{v}_1$ is found as normally done by solving

$$(A - \lambda_1 I)\mathbf{v} = 0$$

$$\begin{pmatrix} 3-\lambda_1 & 2 & 1 \\ -1 & 3-\lambda_1 & 2 \\ 1 & -3 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & -3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives three equations

$$3v_1 + 2v_2 + v_3 = 0$$
$$-v_1 + 3v_2 + 2v_3 = 0$$
$$v_1 - 3v_2 - 2v_3 = 0$$

Let $v_1 = 1$, then the above becomes

$$2v_2 + v_3 = -3$$
$$3v_2 + 2v_3 = 1$$
$$-3v_2 - 2v_3 = -1$$

The first equation above gives $v_2 = \frac{-3-v_3}{2}$. Substituting this in the second equation gives $3\left(\frac{-3-v_3}{2}\right) + 2v_3 = 1$, or $v_3 = 11$. Hence $v_2 = \frac{-3-11}{2} = -7$.

Therefore the eigenvector associated with $\lambda_1 = 0$ is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -7 \\ 11 \end{pmatrix}$$

For the eigenvalue $\lambda_2 = 2$, which has algebraic multiplicity 2, it is first checked if it is defective eigenvalue or a complete one. A complete eigenvalue is one with an algebraic multiplicity m and an geometric multiplicity m as well. When this is the case, then m linearly independent eigenvectors associated with the eigenvalue can be found.

However, if the eigenvalue is defective, which means its geometric multiplicity is less than m, then it is not possible to find m linearly independent eigenvectors from the eigenvalue. In this case the defective eigenvalue algorithm is used to find the remaining linearly independent eigenvectors. Note that geometric multiplicity can not be larger than the algebraic multiplicity.

Now a check is made to determine if the eigenvalue $\lambda_2 = 2$ is defective or complete. The geometric multiplicity of an eigenvalue is the dimension of the null-space of the matrix $A - \lambda_2 I$ given by

$$
\begin{aligned}
(A - \lambda_2 I) &= \begin{pmatrix} 3-\lambda_2 & 2 & 1 \\ -1 & 3-\lambda_2 & 2 \\ 1 & -3 & -2-\lambda_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -3 & -4 \end{pmatrix}
\end{aligned}
$$

The null space of the above matrix is now found. By the Rank nullity theorem of linear algebra, which says

$$
\text{column rank}(A) + \text{nullity}(A) = \text{dimension } (A)
$$

Then the column rank needs to be found as well. This is done by converting the matrix to reduced row echelon form as follows

$$
\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -3 & -4 \end{pmatrix} \xrightarrow{R_2=R_2+R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 1 & -3 & -4 \end{pmatrix} \xrightarrow{R_3=R_3-R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & -5 & -5 \end{pmatrix}
$$

$$
\xrightarrow{R_3=R_3+\frac{5}{3}R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2=\frac{R_2}{3}\frac{3}{2}R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}
$$

The above is in reduced row echelon form. The number of columns with 1 on the diagonal is the column rank. The above shows the column rank is 2. Using the rank nullity the dimension of the null space is now found as follows

$$
\begin{aligned}
\text{nullity}(A) &= \text{dimension } (A) - \text{column rank}(A) \\
&= 3 - 2 \\
&= 1
\end{aligned}
$$

Therefore the geometric multiplicity is 1 which is less than the algebraic multiplicity 2. This means only one eigenvector can be obtained directly from λ_2 since this eigenvalue is defective.

The defective eigenvalue method is used next to find the second eigenvector associated with λ_2. In this method the first eigenvector from λ_2 is first found as is done normally by solving

$$
(A - \lambda_2 I)\,\mathrm{v}_2 = 0
$$

$$
\begin{pmatrix} 3-\lambda_2 & 2 & 1 \\ -1 & 3-\lambda_2 & 2 \\ 1 & -3 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -3 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

This gives the three equations

$$
\begin{aligned}
v_1 + 2v_2 + v_3 &= 0 \\
-v_1 + v_2 + 2v_3 &= 0 \\
v_1 - 3v_2 - 4v_3 &= 0
\end{aligned}
$$

Let $v_1 = 1$, then the above becomes

$$
\begin{aligned}
2v_2 + v_3 &= -1 \\
v_2 + 2v_3 &= 1 \\
-3v_2 - 4v_3 &= -1
\end{aligned}
$$

From the first equation $v_2 = \frac{-1-v_3}{2}$ and from the second equation $\frac{-1-v_3}{2} + 2v_3 = 1$, or $v_3 = 1$. Hence $v_2 = \frac{-1-1}{2} = -1$. Therefore the first eigenvector associated with λ_2 is

$$
\mathrm{v}_2 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}
$$

The second eigenvector associated with λ_2 is given by

$$
\mathrm{v}_3 = t\,\mathrm{v}_2 + \mathrm{p}
$$

Where p is the solution to

$$(A - \lambda_2 I)\,\mathrm{p} = \mathrm{v}_2$$

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -3 & -4 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

The above gives the equations

$$\begin{aligned} p_1 + 2p_2 + p_3 &= 1 \\ -p_1 + p_2 + 2p_3 &= -1 \\ p_1 - 3p_2 - 4p_3 &= 1 \end{aligned}$$

Let $p_1 = 1$, and the above becomes

$$\begin{aligned} 2p_2 + p_3 &= 0 \\ p_2 + 2p_3 &= -2 \\ -3p_2 - 4p_3 &= 0 \end{aligned}$$

The first equation gives $p_2 = \frac{p_3}{2}$. Hence the second equation becomes $\frac{p_3}{2} + 2p_3 = 0$. Therefore $p_3 = 0$ and therefore $p_2 = 0$. Which results in

$$\mathrm{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore the third eigenvector is found from

$$\begin{aligned} \mathrm{v}_3 &= t\mathrm{v}_2 + \mathrm{p} \\ &= t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

The three eigenvectors are the following

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ -7 \\ 11 \end{pmatrix}, \mathrm{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathrm{v}_3 = t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The solution can now be written as

$$\mathrm{y}(t) = c_1 e^{\lambda_1 t}\mathrm{v}_1 + c_2 e^{\lambda_2 t}\mathrm{v}_2 + c_3 e^{\lambda_2 t}\mathrm{v}_3$$

Since $\lambda_1 = 0$ and $\lambda_2 = 2$ then the above becomes

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -7 \\ 11 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{2t}\left[t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]$$

Which can be simplified to

$$\begin{aligned} y_1(t) &= c_1 + c_2 e^{2t} + c_3 e^{2t}(t+1) \\ y_2(t) &= -7c_1 - c_2 e^{2t} - c_3 t e^{2t} \\ y_3(t) &= 11c_1 + c_2 e^{2t} + c_3 t e^{2t} \end{aligned} \tag{1}$$

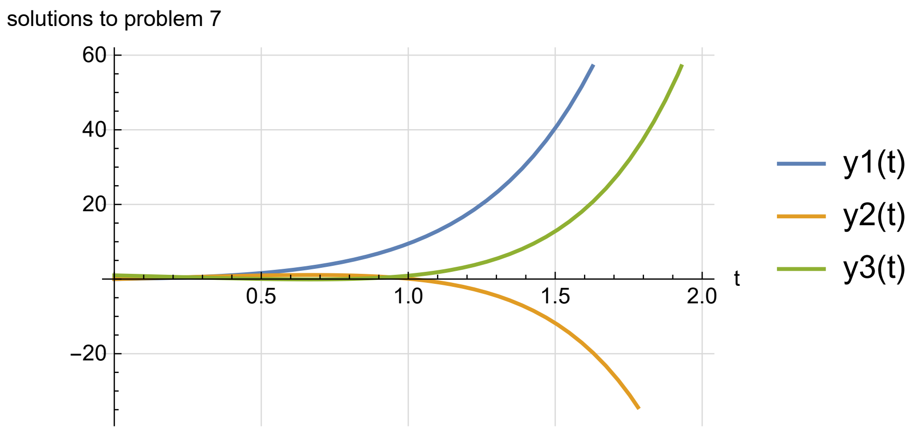
To plot these solutions, the following arbitrary initial conditions $y_1(0) = 0, y_2(0) = 0, y_3(0) = 1$ are used

$$\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + c_3 \\ -7c_1 - c_2 \\ 11c_1 + c_2 \end{pmatrix}$$

Solving, this gives $c_1 = \frac{1}{4}, c_2 = -\frac{7}{4}, c_3 = \frac{3}{2}$. Therefore the above solution (1) becomes

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{4} - \frac{7}{4}e^{2t} + \frac{3}{2}e^{2t}(t+1) \\ -\frac{7}{4} + \frac{7}{4}e^{2t} - \frac{3}{2}te^{2t} \\ \frac{11}{4} - \frac{7}{4}e^{2t} + \frac{3}{2}te^{2t} \end{pmatrix}$$

The following is a plot of the solution for these initial conditions. The solutions are not stable, since they grow in time.




```
ClearAll[t,y1,y2,y3];
myy1=1/4-7/4 Exp[2 t]+3/2 Exp[2 t](t+1);
myy2=-7/4+7/4 Exp[2 t]-3/2 t Exp[2 t];
myy3=11/4-7/4 Exp[2 t]+3/2 t Exp[2 t];
Plot[{myy1,myy2,myy3},{t,0,2},GridLines->Automatic,GridLinesStyle->LightGray,
ImageSize->300,
AxesLabel->{"t","solutions to problem 7"},
PlotLegends->{"y1(t)","y2(t)","y3(t)"}]
```

3.2.8 Problem 8

8. Explain why $y(t) = \cos t + \sin 2t$ cannot be a solution to a constant coefficient ODE of the form $y'' + ay' + by = 0$, with $a, b \in \mathbb{R}$. Find an ODE with real coefficients of order greater than 2 that $y(t)$ does satisfy.

Solution

$y(t) = \cos t + \sin 2t$ can not be a solution to $y'' + ay' + by = 0$, because both basis solutions (these are the linearly independent solutions $\sin$ and $\cos$) must oscillate with the same frequency. The frequency of oscillation of a second order system with no forcing function is called the natural frequency of the system. There is one unique natural frequency for a second order system.

This frequency comes from finding the value of the discriminant of the characteristic equation of the ODE (since it is constant coefficient). To illustrate, the general solution of the second order ODE is found to show that the proposed solution is not possible. The general solution of the above ODE is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Where $\lambda_{1,2}$ are the two roots of the corresponding characteristic equation $\lambda^2 + a\lambda + b = 0$. These roots are

$$\lambda = -\frac{a}{2} \pm \sqrt{a^2 - 4b}$$

Therefore the roots are

$$\begin{aligned}\lambda_1 &= -\frac{a}{2} + \sqrt{a^2 - 4b}\\ \lambda_2 &= -\frac{a}{2} - \sqrt{a^2 - 4b}\end{aligned}$$

The general solution to the given ODE is linear combination of two linearly independent solutions $e^{\lambda_1 t}, e^{\lambda_2 t}$, one for each root, which results in

$$\begin{aligned}y(t) &= c_1 e^{\left(-\frac{a}{2} + \sqrt{a^2-4b}\right)t} + c_2 e^{\left(-\frac{a}{2} - \sqrt{a^2-4b}\right)t}\\ &= e^{-\frac{a}{2}t}\left(c_1 e^{\sqrt{a^2-4b}t} + c_2 e^{-\sqrt{a^2-4b}t}\right)\end{aligned}$$

c_1, c_2 are determined from initial conditions. Since the proposed solution given does not have $e^{-\frac{a}{2}t}$ in it, then this implies that $a = 0$ (this is the damping term), and since $e^{-\frac{a}{2}t} = 1$ then the solution reduces to

$$y(t) = c_1 e^{\sqrt{-4b}t} + c_2 e^{-\sqrt{-4b}t}$$

Since the proposed solution is made up of trigonometric functions, it must be that $b > 0$ in order to make $-4b$ negative and obtain a pair of conjugate complex roots. The solution now becomes

$$y(t) = c_1 e^{2i\sqrt{b}t} + c_2 e^{-2i\sqrt{b}t}$$

Expressing this in terms of trigonometric functions using Euler relation results in

$$y(t) = c_1 \cos\left(\sqrt{b}t\right) + c_2 \sin\left(\sqrt{b}t\right)$$

The above shows that the solution can not be $y(t) = \cos t + \sin 2t$ since $\sqrt{b}$ can not equal 1 and 2 at the same time.

Another way to show that $y(t) = \cos t + \sin 2t$ is not be a solution, is to simply substitute this solution into the ODE and obtain a contradiction as shown below.

Since $y' = -\sin t + 2\cos 2t$ and $y'' = -\cos t - 4\sin 2t$, the ODE now becomes

$$\begin{aligned}
(-\cos t - 4\sin 2t) + a(-\sin t + 2\cos 2t) + b(\cos t + \sin 2t) &= 0 \\
(-1 + b)\cos t - a\sin t + (-4 + b)\sin 2t &= 0
\end{aligned}$$

Because the RHS is zero, this implies that

$$\begin{aligned}
-1 + b &= 0 \\
-4 + b &= 0 \\
-a &= 0
\end{aligned}$$

The first equation gives $b = 1$ and the second equation gives $b = -4$ which is not possible.

To obtain an ODE with such a solution, the ODE has to be of order 4. This is to obtain two different natural frequencies (A 4^{th} order ODE can be written as two separate second order ODE's). Let the ODE be

$$y''''(t) + Ay'''(t) + By''(t) + Cy'(t) + Dy(t) = 0 \tag{1}$$

Given that

$$\begin{aligned}
y &= \cos t + \sin 2t \\
y' &= -\sin t + 2\cos 2t \\
y'' &= -\cos t - 4\sin 2t \\
y''' &= \sin t - 8\cos 2t \\
y'''' &= \cos t + 16\sin 2t
\end{aligned}$$

Substituting the above into (1) gives

$$(\cos t + 16\sin 2t) + A(\sin t - 8\cos 2t) + B(-\cos t - 4\sin 2t) + C(-\sin t + 2\cos 2t) + D(\cos t + \sin 2t) = 0$$

Collecting terms based on the trigonometric function gives

$$(1 - B + D)\cos t + (A - C)\sin t + (16 - 4B + D)\sin 2t + (-8A + 2C)\cos 2t = 0$$

A solution is obtained by setting all the coefficients above to zero which results in the following four equations to solve for A, B, C, D

$$\begin{aligned}
1 - B + D &= 0 \\
A - C &= 0 \\
16 - 4B + D &= 0 \\
-8A + 2C &= 0
\end{aligned}$$

These are solved by elimination. From the second equation $A = C$. The fourth equation gives $-8C + 2C = 0$ or $C = 0$. Hence $A = 0$. From first equation $B = 1 + D$, hence the third equation gives $16 - 4(1 + D) + D = 0$, or $D = 4$ and therefore $B = 5$. The solution is therefore

$$\begin{aligned}
A &= 0 \\
B &= 5 \\
C &= 0 \\
D &= 4
\end{aligned}$$

Using these in (1) gives

$$y''''(t) + 5y''(t) + 4y(t) = 0 \tag{2}$$

The proposed solution $y(t) = \cos t + \sin 2t$ now satisfies the above ODE. There will be four constants of integrations (since this is a 4^{th} order ODE), and therefore two of these constants must be set to zero using the appropriate initial conditions. To find which constants are needed to set to zero, the above ODE is first solved. The characteristic equation of (2) is

$$\lambda^4 + 5\lambda^2 + 4 = 0$$
$$\left(\lambda^2 + 1\right)\left(\lambda^2 + 4\right)$$

The roots are $\lambda_1 = \pm i, \lambda_2 = \pm 2i$. Therefore solution to (2) becomes

$$y(t) = c_1 e^{it} + c_2 e^{-it} + c_3 e^{2it} + c_4 e^{-2it}$$

Using Euler relation the above is written in trigonometric functions as

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 \cos 2t + c_4 \sin 2t \tag{3}$$

To obtain the proposed solution $y(t) = \cos t + \sin 2t$ implies that the constants must have these values

$$c_1 = 1$$
$$c_2 = 0$$
$$c_3 = 0$$
$$c_4 = 1$$

The initial conditions which would lead to these constants having these specific values are now found as follows. From (3)

$$y(0) = c_1 + c_3$$

Since $y'(t) = -c_1 \sin t + c_2 \cos t - 2c_3 \sin 2t + 2c_4 \cos 2t$ then

$$y'(0) = c_2 + 2c_4$$

And since $y''(t) = -c_1 \cos t - c_2 \sin t - 4c_3 \cos 2t - 4c_4 \sin 2t$, then

$$y''(0) = -c_1 - 4c_3$$

and finally since $y'''(t) = c_1 \sin t - c_2 \cos t + 8c_3 \sin 2t - 8c_4 \cos 2t$ then

$$y'''(0) = -c_2 - 8c_4$$

Since $c_1 = 1, c_2 = 0, c_3 = 0, c_4 = 1$, then the above initial conditions become

$$y(0) = 1$$
$$y'(0) = 2$$
$$y''(0) = -1$$
$$y'''(0) = -8$$

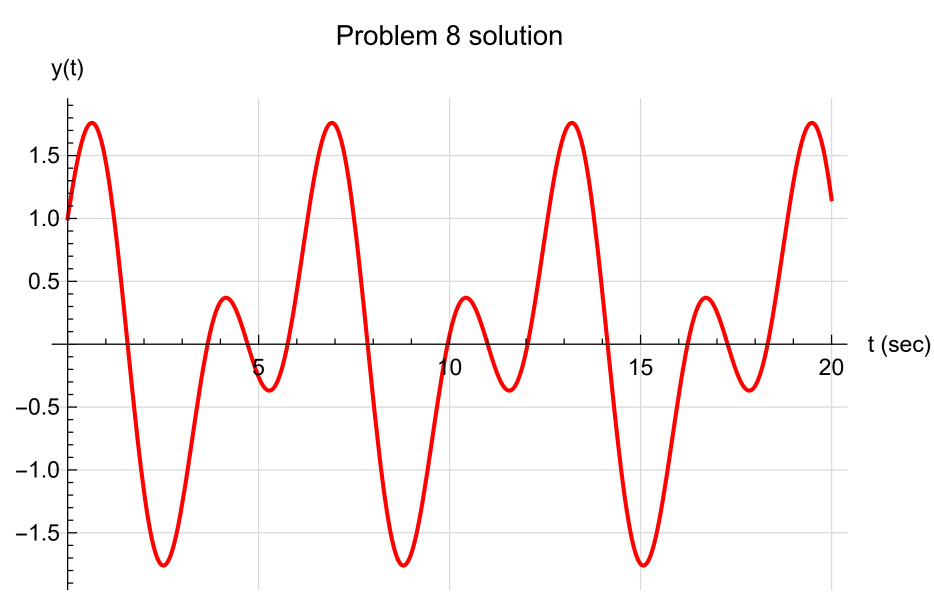
The above initial conditions will now give the solution

$$y(t) = \cos t + \sin 2t$$

For the ODE

$$y''''(t) + 5y''(t) + 4y(t) = 0$$

The following is a plot of the solution



```
Plot[Cos[t] + Sin[2 t], {t, 0, 20}, PlotStyle -> Red,
GridLines -> Automatic, GridLinesStyle -> LightGray,
AxesLabel -> {"t (sec)", "y(t)"},
PlotLabel -> "Problem 8 solution"]
```

3.3 Third exam

3.3.1 Problem 1

1. Let $x(t)$ and $y(t)$ denote the population sizes of two biological species. If the two species are not competing for resources (occupy different biological niches) then a simple logistic model could be feasible to describe the dynamics of their coexistence.

$$
\begin{aligned}
x' &= a_1 x - b_1 x^2 \\
y' &= a_2 y - b_2 y^2.
\end{aligned}
$$

If however the two species are direct competitors, then their access to resources and their population growth rate could be reduced by a quantity that is proportional to the size of the competing species' population, leading to a *competition system* model

$$
\begin{aligned}
x' &= a_1 x - b_1 x^2 - c_1 xy \\
y' &= a_2 y - b_2 y^2 - c_2 xy.
\end{aligned}
$$

Assume that in a competition system (with appropriate units) the coefficients are given as

$$a_1 = 60, \quad a_2 = 42, \quad b_1 = 3, \quad b_2 = 3, \quad c_1 = 4, \quad c_2 = 2,$$

and determine all equilibria of the system as well as their corresponding stability properties. Give a short interpretation of your results in terms of the long term species dynamics (as $t \to \infty$).

Figure 3.1: Problem 1 Statement

<u>Solution</u>

$$
\begin{aligned}
x' &= a_1 x - b_1 x^2 - c_1 xy \\
y' &= a_2 y - b_2 y^2 - c_2 xy
\end{aligned}
$$

Using the values given in the problem, the above equations become

$$
\begin{aligned}
x' &= 60x - 3x^2 - 4xy \qquad (1A) \\
y' &= 42y - 3y^2 - 2xy \qquad (1B)
\end{aligned}
$$

Or

$$
\begin{aligned}
x' &= f(x,y) \\
y' &= g(x,y)
\end{aligned}
$$

Equilibrium points are found by setting $f(x,y) = 0$ and $g(x,y)$. This results in the following two equations to solve for x, y

$$
\begin{aligned}
60x - 3x^2 - 4xy &= 0 \qquad (1) \\
42y - 3y^2 - 2xy &= 0 \qquad (2)
\end{aligned}
$$

The first equation (1A) becomes $x(60 - 3x - 4y) = 0$ which then gives one solution as

$$x = 0 \qquad (3)$$

And $60 - 3x - 4y = 0$ gives another solution as

$$x = \frac{60 - 4y}{3} \qquad (4)$$

The second equation (1B) becomes $y\left(42-3y-2x\right)=0$ which gives one solution as

$$y=0 \tag{5}$$

And $42-3y-2x=0$ gives another solution as

$$y=\frac{42-2x}{3} \tag{6}$$

When $x=0$ then (6) results in $y=\frac{42}{3}=14$. When $x=\frac{60-4y}{3}$ then (6) results in $y=\frac{42-2\left(\frac{60-4y}{3}\right)}{3}=\frac{8}{9}y+\frac{2}{3}$, or $y=6$. Hence in this case $x=\frac{60-4(6)}{3}=12$.

Similarly, when $y=0$ then from (4) $x=\frac{60-4(0)}{3}=20$. The above shows that there are 4 equilibrium points. These are

$$\begin{aligned} x&=0,y=0\\ x&=0,y=14\\ x&=12,y=6\\ x&=20,y=0 \end{aligned}$$

To determine the type of stability of each equilibrium point, and since this is a nonlinear system, we must first linearize the system around each equilibrium point in order to determine the Jacobian matrix.

Once the system is linearized, then the eigenvalues of the Jacobian matrix are found in each case. From the values of eigenvalues we can then determine if the system is stable or not at each one of the above four equilibrium points.

The first step is then to linearize $f\left(x,y\right)$ and $g\left(x,y\right)$ around each of the equilibrium points. If we assume the equilibrium point is given by x_0,y_0 then expanding $f\left(x,y\right)$ in Taylor series around this point gives

$$f\left(x_0+\Delta x,y_0+\Delta y\right)=f\left(x_0,y_0\right)+\left.\frac{\partial f\left(x,y\right)}{\partial x}\right|_{x_0,y_0}\left(\Delta x+\Delta y\right)+\left.\frac{\partial f\left(x,y\right)}{\partial y}\right|_{x_0,y_0}\left(\Delta x+\Delta y\right)+\cdots$$

But $f\left(x_0,y_0\right)=0$ since it is what defines an equilibrium point, the above becomes, after ignoring higher order terms since we are assuming small $\Delta x,\Delta y$

$$f\left(x_0+\Delta x,y_0+\Delta y\right)=\left.\frac{\partial f\left(x,y\right)}{\partial x}\right|_{x_0,y_0}\left(\Delta x+\Delta y\right)+\left.\frac{\partial f\left(x,y\right)}{\partial y}\right|_{x_0,y_0}\left(\Delta x+\Delta y\right)$$

Similarly for $g\left(x,y\right)$ we obtain the following

$$g\left(x_0+\Delta x,y_0+\Delta y\right)=\left.\frac{\partial g\left(x,y\right)}{\partial x}\right|_{x_0,y_0}\left(\Delta x+\Delta y\right)+\left.\frac{\partial g\left(x,y\right)}{\partial y}\right|_{x_0,y_0}\left(\Delta x+\Delta y\right)$$

Therefore a linearized f,g functions at the equilibrium point become

$$\begin{pmatrix} f\left(x_0+\Delta x,y_0+\Delta y\right)\\ g\left(x_0+\Delta x,y_0+\Delta y\right)\end{pmatrix}=\begin{pmatrix} \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} & \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0}\\ \left.\frac{\partial g}{\partial x}\right|_{x_0,y_0} & \left.\frac{\partial g}{\partial y}\right|_{x_0,y_0}\end{pmatrix}\begin{pmatrix}\Delta x+\Delta y\\ \Delta x+\Delta y\end{pmatrix}$$

Replacing the original nonlinear $f\left(x,y\right),g\left(x,y\right)$ by the above linearized (approximation), the system can now be written as

$$\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial x}\end{pmatrix}_{\substack{x=x_0\\ y=y_0}}\begin{pmatrix}x\\ y\end{pmatrix}$$

Where $\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial x}\end{pmatrix}$ is called the the Jacobian J matrix. Hence the system now can be written as

$$\vec{x}'=[J]\vec{x}$$

Now J is determined. From

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(60x - 3x^2 - 4xy\right) = 60 - 6x - 4y \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(60x - 3x^2 - 4xy\right) = -4x \\
\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left(42y - 3y^2 - 2xy\right) = -2y \\
\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left(42y - 3y^2 - 2xy\right) = 42 - 6y - 2x
\end{aligned}$$

The Jacobian matrix becomes

$$J = \begin{pmatrix} 60 - 6x - 4y & -4x \\ -2y & 42 - 6y - 2x \end{pmatrix}_{x=x_0, y=y_0}$$

And the linearized system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 60 - 6x - 4y & -4x \\ -2y & 42 - 6y - 2x \end{pmatrix}_{x=x_0, y=y_0} \begin{pmatrix} x \\ y \end{pmatrix}$$

Now each equilibrium point is examined using the above linearized system to determine the type of stability a that point.

<u>case $x_0 = 0, y_0 = 0$</u>

$$J = \begin{pmatrix} 60 - 6(0) - 4(0) & -4(0) \\ -2(0) & 42 - 6(0) - 2(0) \end{pmatrix} = \begin{pmatrix} 60 & 0 \\ 0 & 42 \end{pmatrix}$$

Hence the linearized system at this specific equilibrium point is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 60 & 0 \\ 0 & 42 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Since J is a diagonal matrix, its eigenvalues are the values on the diagonal. Therefore $\lambda_1 = 60, \lambda_2 = 42$. Since the eigenvalues are positive, then this equilibrium point is <u>not stable.</u>

<u>case $x_0 = 0, y_0 = 14$</u>

$$J = \begin{pmatrix} 60 - 6(0) - 4(14) & -4(0) \\ -2(14) & 42 - 6(14) - 2(0) \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -28 & -42 \end{pmatrix}$$

Therefore linearized system at this specific equilibrium point is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -28 & -42 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues can be found by solving $\begin{vmatrix} 4 - \lambda & 0 \\ -28 & -42 - \lambda \end{vmatrix} = 0$ to be $\lambda_1 = 4, \lambda_2 = -42$. Because one of the eigenvalues is positive, then this equilibrium point is <u>not stable.</u>

<u>case $x_0 = 12, y_0 = 6$</u>

$$J = \begin{pmatrix} 60 - 6(12) - 4(6) & -4(12) \\ -2(6) & 42 - 6(6) - 2(12) \end{pmatrix} = \begin{pmatrix} -36 & -48 \\ -12 & -18 \end{pmatrix}$$

The linearized system at this specific equilibrium point is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -36 & -48 \\ -12 & -18 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues can be found to be $\lambda_1 = -52.632, \lambda_2 = -1.368$. Since both eigenvalues are now negative, then this equilibrium point is <u>stable.</u>

<u>case $x_0 = 20, y_0 = 0$</u>

$$J = \begin{pmatrix} 60 - 6(20) - 4(0) & -4(20) \\ -2(0) & 42 - 6(0) - 2(20) \end{pmatrix} = \begin{pmatrix} -60 & -80 \\ 0 & 2 \end{pmatrix}$$

The linearized system at this specific equilibrium point is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -60 & -80 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 2, \lambda_2 = -60$. Since one of the eigenvalues is positive, then this equilibrium point is not stable.

Summary of results obtained so far

equilibrium point	eigenvalues	type of stability
$x = 0, y = 0$	$\lambda_1 = 60, \lambda_2 = 42$	not stable (nodal source)
$x = 0, y = 14$	$\lambda_1 = 4, \lambda_2 = -42$	not stable (Saddle point)
$x = 12, y = 6$	$\lambda_1 = -52.632, \lambda_2 = -1.368$	stable (Nodal sink)
$x = 20, y = 0$	$\lambda_1 = 2, \lambda_2 = -60$	not stable (Saddle point)

To verify the above result, the phase plot for the original nonlinear system was plotted on the computer and the equilibrium points locations highlighted. The plot below agrees with the above result when looking at direction of arrows around each point. We see that the direction field arrows are all moving toward the stable point from any location near it. The stable equilibrium point was colored as green while the unstable ones colored in red.

```
In[•]:= ClearAll[x, y];
f1 = 60 x - 3 x^2 - 4 x y;
f2 = 42 y - 3 y^2 - 2 x y;
StreamPlot[{f1, f2}, {x, -5, 25}, {y, -5, 20},
 Epilog → {{Red, PointSize[0.03], Point[{{0, 0}, {0, 14}, {20, 0}}]},
   {Green, PointSize[.035], Point[{{12, 6}}]}},
 PlotLabel → Column[{"Phase plot for problem 1", "Showing location of all equilibrium points"},
   Alignment → Center],
 BaseStyle → 12,
 ImageSize → 400]
```

Out[•]=

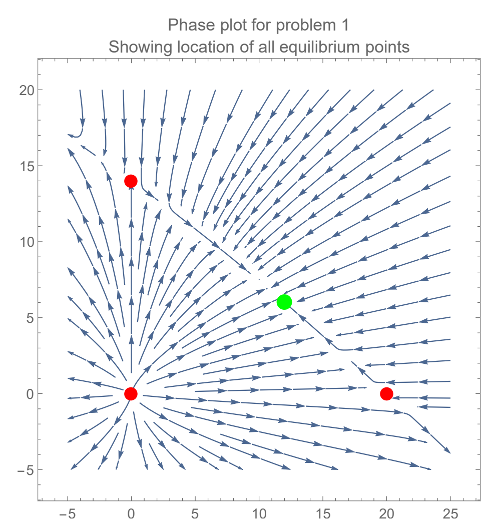


Figure 3.2: Phase plot for problem 1

Interpretation of results Since the solution of the linearized system can be written as linear combination of solutions made up of terms that look like $c_i e^{\lambda_i t}$ where c_i are constants of integration and λ_i are the eigenvalues found above, then this implies when the real part of the eigenvalue is positive the solution will increase with time, moving away from the equilibrium point. Similarly, if the eigenvalue has a negative real part, it means it is a stable solution because solution will decay with time when perturbed slightly from the equilibrium.

Since this is second order system, there are two eigenvalues. Even if one eigenvalue is stable (i.e. negative), if the other eigenvalue is positive, then the system is unstable since one part of the solution will keep growing with time.

In terms of the dynamics of species, it means if the populations $x = 12$ and population $y = 6$, (this is the stable equilibrium) then these population will remain the same in long term even when one population becomes a little more or less than the other population. But for all other equilibrium populations sizes, such as $x = 20, y = 0$, then if the population y were to change slightly to become say $y = 1$ (may be by external influence) then this will cause both population to start changing, moving it away from $x = 20, y = 0$ as time increases, hence $x = 20, y = 0$ is not stable population size.

This seems to be sensitive to the parameters a_i, b_i, c_i given in the problem. It is not easy to give a more physical reasoning as why some population values is stable while other are not, other than to also note that all the unstable ones had at least one population at zero.

3.3.2 Problem 2

2. Use Laplace transform to solve the following initial value problem for $y(t)$:

$$y'' - 5y' + 6y = \begin{cases} 4e^t & if\ 0 < t < 2 \\ 0 & if \quad t \geq 2 \end{cases}$$

$$y(0) = 1, \qquad y'(0) = -2.$$

Figure 3.3: Problem 2 Statement

Solution

The ODE can be written as

$$\begin{aligned} y'' - 5y' + 6y &= 4e^t \left(U(t) - U(t-2)\right) \\ &= 4\left(e^t U(t) - e^t U(t-2)\right) \end{aligned}$$

Where $U(t)$ is the unit step function. In the following solutions, these Laplace transform relations obtained from table are used

$$\begin{aligned} U(t) &\Leftrightarrow \frac{1}{s} \\ U(t-\tau) &\Leftrightarrow \frac{1}{s} e^{-\tau s} \\ e^{-\alpha t} U(t) &\Leftrightarrow \frac{1}{s+\alpha} \\ \sin(\omega t) &\Leftrightarrow \frac{\omega}{s^2+\omega^2} \\ \cos(\omega t) &\Leftrightarrow \frac{s}{s^2+\omega^2} \end{aligned}$$

Assuming $\mathscr{L}\left[y(t)\right] = Y(s)$, and using the above relations of Laplace transform we find

$$\begin{aligned} \mathscr{L}\left[e^t U(t)\right] &= \frac{1}{s-1} \\ \mathscr{L}\left[e^t U(t-2)\right] &= \frac{e^{-2(s-1)}}{s-1} \end{aligned}$$

Now, taking the Laplace transform of the ODE results in

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) - 5\left(sY(s) - y(0)\right) + 6Y(s) = 4\left(\frac{1}{s-1} - \frac{e^{-2(s-1)}}{s-1}\right)$$

Using $y(0) = 1, y'(0) = -2$ the above simplifies to

$$\begin{aligned} \left(s^2 Y(s) - s + 2\right) - 5\left(sY(s) - 1\right) + 6Y(s) &= \frac{4}{s-1} - \frac{4e^{-2(s-1)}}{s-1} \\ s^2 Y(s) - s + 2 - 5sY(s) + 5 + 6Y(s) &= \frac{4}{s-1} - \frac{4e^{-2(s-1)}}{s-1} \\ Y(s)\left(s^2 - 5s + 6\right) - s + 7 &= \frac{4}{s-1} - \frac{4e^{-2(s-1)}}{s-1} \\ Y(s)\left(s^2 - 5s + 6\right) &= \frac{4}{s-1} - \frac{4e^{-2(s-1)}}{s-1} + (s-7) \end{aligned}$$

But $\left(s^2 - 5s + 6\right) = (s-3)(s-2)$ and the above becomes

$$Y(s) = \frac{4}{(s-1)(s-3)(s-2)} - \frac{4e^2 e^{-2s}}{(s-1)(s-3)(s-2)} + \frac{(s-7)}{(s-3)(s-2)} \tag{1}$$

These are now simplified by partial fractions. The final result is only shown for brevity, since the process of performing partial fraction is a standard one.

$$\frac{1}{(s-1)(s-3)(s-2)} = \frac{1}{2(s-3)} - \frac{1}{s-2} + \frac{1}{2(s-1)}$$

And

$$\frac{s-7}{(s-3)(s-2)} = \frac{-4}{s-3} + \frac{5}{s-2}$$

Using the above result back in (1) results in

$$\begin{aligned} Y(s) &= \frac{2}{s-3} - \frac{4}{s-2} + \frac{2}{s-1} - e^2 e^{-2s}\left(\frac{2}{(s-3)} - \frac{4}{s-2} + \frac{2}{(s-1)}\right) - \frac{4}{s-3} + \frac{5}{s-2} \\ &= \frac{-2}{s-3} + \frac{1}{s-2} + \frac{2}{s-1} - e^2\left(\frac{2e^{-2s}}{(s-3)} - \frac{4e^{-2s}}{s-2} + \frac{2e^{-2s}}{(s-1)}\right) \end{aligned} \tag{2}$$

Now we apply the inverse Laplace transform. lookup table is also used for this purpose to obtain

$$\begin{aligned} -2\mathscr{L}^{-1}\left(\frac{1}{s-3}\right) &= -2e^{3t} \\ \mathscr{L}^{-1}\left(\frac{1}{s-2}\right) &= e^{2t} \\ 2\mathscr{L}^{-1}\left(\frac{1}{s-1}\right) &= 2e^{t} \end{aligned}$$

And

$$\begin{aligned} 2\mathscr{L}^{-1}\left(\frac{e^{-2s}}{s-3}\right) &= 2e^{3(t-2)}U(t-2) \\ 4\mathscr{L}^{-1}\left(\frac{e^{-2s}}{s-2}\right) &= 4e^{2(t-2)}U(t-2) \\ 2\mathscr{L}^{-1}\left(\frac{e^{-2s}}{s-1}\right) &= 2e^{(t-2)}U(t-2) \end{aligned}$$

Putting all these results back into (2) gives the response in time domain as

$$y(t) = -2e^{3t} + e^{2t} + 2e^{t} - e^2\left(2e^{3(t-2)} - 4e^{2(t-2)} + 2e^{(t-2)}\right)U(t-2)$$

The above can also be written as

$$y(t) = \begin{cases} -2e^{3t} + e^{2t} + 2e^{t} & 0 < t < 2 \\ -2e^{3t} + e^{2t} + 2e^{t} - e^2\left(2e^{3t} - 4e^{2t} + 2e^{t}\right) & t \geq 2 \end{cases}$$

Since the original ODE is not stable (due to damping term -5 negative in the given ODE, the solution will blow up with time). This is seen by the solution above, where the exponential are all positive, hence growing with time. The following is a plot of the above solution for up to $t = 2.2$

```
ClearAll[t];
mySol = -2 e^(3 t) + e^(2 t) + 2 e^t - e^2 (2 e^(3 (t-2)) - 4 e^(2 (t-2)) + 2 e^(t-2)) UnitStep[t - 2];
Plot[mySol, {t, 0, 2.2},
 Frame -> True,
 FrameLabel → {{"y(t)", None}, {"t (sec)", "Solution to problem 2"}},
 GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red]
```

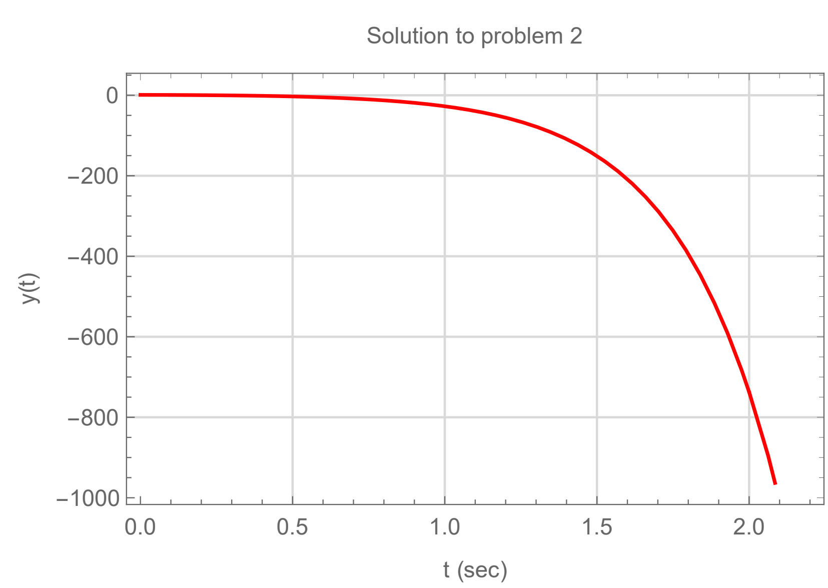


Figure 3.4: Plot of solution for problem 2

3.3.3 Problem 3

3. Solve the following linear system using Laplace transform

$$
\begin{aligned}
y_1'' + 5y_1 + y_2 &= 0\\
y_2'' - 2y_1 + 2y_2 &= 0\\
y_1(0) &= 3\\
y_1'(0) &= 0\\
y_2(0) &= 1\\
y_2'(0) &= 0
\end{aligned}
$$

Figure 3.5: Problem 3 Statement

Solution

Let $Y_1(s) = \mathscr{L}\left[y_1(t)\right]$ and let $Y_2(s) = \mathscr{L}\left[y_2(t)\right]$. Taking Laplace transform of the two ODE's gives

$$
\begin{aligned}
s^2Y_1(s) - sy_1(0) - y_1'(0) + 5Y_1(s) + Y_2(s) &= 0\\
s^2Y_2(s) - sy_2(0) - y_2'(0) - 2Y_1(s) + 2Y_2(s) &= 0
\end{aligned}
$$

Substituting the given initial conditions results in

$$
\begin{aligned}
s^2Y_1 - 3s + 5Y_1 + Y_2 &= 0 \qquad (1)\\
s^2Y_2 - s - 2Y_1 + 2Y_2 &= 0 \qquad (2)
\end{aligned}
$$

The above two ODE's are now solved for $Y_1(s), Y_2(s)$

$$
\begin{aligned}
Y_1\left(s^2+5\right) + Y_2 &= 3s\\
Y_2\left(s^2+2\right) - 2Y_1 &= s
\end{aligned}
$$

or

$$
\begin{pmatrix} s^2+5 & 1\\ -2 & s^2+2 \end{pmatrix}\begin{pmatrix} Y_1\\ Y_2\end{pmatrix} = \begin{pmatrix} 3s\\ s\end{pmatrix}
$$

Using Gaussian elimination: Adding $\left(\frac{2}{s^2+5}\right)$ times first row to second row gives

$$
\begin{pmatrix} s^2+5 & 1\\ 0 & s^2+2+\frac{2}{s^2+5} \end{pmatrix}\begin{pmatrix} Y_1\\ Y_2\end{pmatrix} = \begin{pmatrix} 3s\\ s+\frac{6s}{s^2+5}\end{pmatrix}
$$

$$
\begin{pmatrix} s^2+5 & 1\\ 0 & \frac{1}{s^2+5}\left(s^4+7s^2+12\right) \end{pmatrix}\begin{pmatrix} Y_1\\ Y_2\end{pmatrix} = \begin{pmatrix} 3s\\ \frac{s}{s^2+5}\left(s^2+11\right)\end{pmatrix}
$$

Back substitution: From last row

$$
\begin{aligned}
Y_2(s) &= \frac{\frac{s}{s^2+5}\left(s^2+11\right)}{\frac{1}{s^2+5}\left(s^4+7s^2+12\right)}\\
&= \frac{s\left(s^2+11\right)}{s^4+7s^2+12}\\
&= \frac{s\left(s^2+11\right)}{\left(s^2+4\right)\left(s^2+3\right)} \qquad (3)
\end{aligned}
$$

First row gives

$$
\left(s^2+5\right)Y_1 + Y_2 = 3s
$$

Using $Y_2(s)$ found from (3), the above becomes

$$
\begin{aligned}
\left(s^2+5\right)Y_1 &= 3s - \frac{s\left(s^2+11\right)}{\left(s^2+4\right)\left(s^2+3\right)} \\
Y_1(s) &= \frac{3s}{\left(s^2+5\right)} - \frac{s\left(s^2+11\right)}{\left(s^2+5\right)\left(s^2+4\right)\left(s^2+3\right)} \qquad (4)
\end{aligned}
$$

To obtain the time domain solution we need to inverse Laplace transform (3,4). Starting with (3), and applying partial fractions gives

$$
Y_2(s) = \frac{s\left(s^2+11\right)}{\left(s^2+4\right)\left(s^2+3\right)} = \frac{8s}{3+s^2} - \frac{7s}{4+s^2} \qquad (5)
$$

From tables we see that

$$
\begin{aligned}
8\mathscr{L}^{-1}\left[\frac{s}{3+s^2}\right] &= 8\cos\left(\sqrt{3}t\right) \\
7\mathscr{L}^{-1}\left[\frac{s}{4+s^2}\right] &= 7\cos(2t)
\end{aligned}
$$

Hence (5) becomes in time domain as

$$
y_2(t) = 8\cos\left(\sqrt{3}t\right) - 7\cos(2t)
$$

Similarly for $Y_1(s)$, from (4) and applying partial fractions

$$
\begin{aligned}
Y_1(s) &= \frac{3s}{\left(s^2+5\right)} - \frac{s\left(s^2+11\right)}{\left(s^2+5\right)\left(s^2+4\right)\left(s^2+3\right)} \\
&= \frac{3s}{\left(s^2+5\right)} - \left(\frac{4s}{s^2+3} - \frac{7s}{s^2+4} + \frac{3s}{5+s^2}\right) \qquad (6)
\end{aligned}
$$

From inverse Laplace transform table

$$
\begin{aligned}
3\mathscr{L}^{-1}\left[\frac{s}{\left(s^2+5\right)}\right] &= 3\cos\left(\sqrt{5}t\right) \\
4\mathscr{L}^{-1}\left[\frac{s}{s^2+3}\right] &= 4\cos\left(\sqrt{3}t\right) \\
7\mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right] &= 7\cos(2t) \\
3\mathscr{L}^{-1}\left[\frac{s}{5+s^2}\right] &= 3\cos\left(\sqrt{5}t\right)
\end{aligned}
$$

Using these in (6), the solution $y_1(t)$ becomes

$$
\begin{aligned}
y_1(t) &= 3\cos\left(\sqrt{5}t\right) - \left(4\cos\left(\sqrt{3}t\right) - 7\cos(2t) + 3\cos\left(\sqrt{5}t\right)\right) \\
&= -4\cos\left(\sqrt{3}t\right) + 7\cos(2t)
\end{aligned}
$$

In summary

$$
\begin{aligned}
y_1(t) &= -4\cos\left(\sqrt{3}t\right) + 7\cos(2t) \\
y_2(t) &= 8\cos\left(\sqrt{3}t\right) - 7\cos(2t)
\end{aligned}
$$

The following is a plot of the solutions for 10 seconds.

```
ClearAll[y1, y2, t]
y1 = -4 Cos[√3 t] + 7 Cos[2 t];
y2 = 8 Cos[√3 t] - 7 Cos[2 t];
Plot[{y1, y2}, {t, 0, 10}, Frame → True,
 FrameLabel → {{"y1(t),y2(t)", None}, {"t (sec)", "Problem 3 solution"}},
 PlotLegends → {"y1(t)", "y2(t)"},
 BaseStyle → 14,
 GridLines → Automatic,
 GridLinesStyle → LightGray]
```

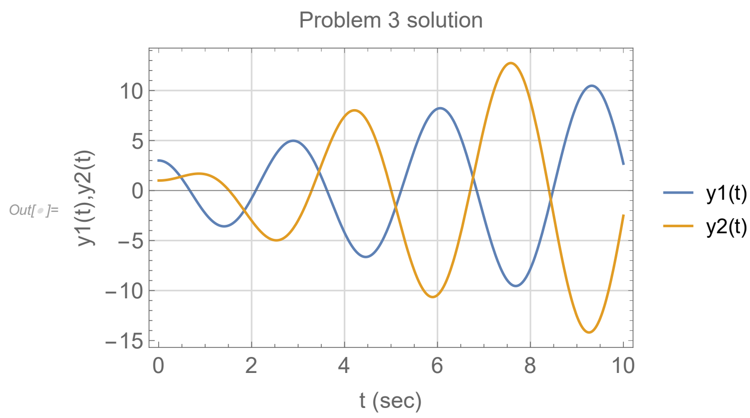


Figure 3.6: Plot of solution for problem 3

3.3.4 Problem 4

4. Consider an $n \times n$ real matrix A. Show that A can be uniquely written as a sum of a symmetric and a skew symmetric matrix

$$A = A_{sy} + A_{sk}.$$

We say that A is positive definite if $q(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ where the quadratic form $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined as $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

Show that A is positive definite if and only if A_{sy} is positive definite.

Figure 3.7: Problem 4 Statement

Solution

Part a

Note: In all the following it is assumed that x is a vector and that $x \neq 0$

Let

$$A = A_{sy} + A_{sk} \tag{1}$$

Where A_{sy} is a symmetric matrix, which means $A_{sy}^T = A_{sy}$ and A_{sk} is skew symmetric matrix which means $A_{sk}^T = -A_{sk}$. Taking the transpose of (1) gives

$$\begin{aligned} A^T &= \left(A_{sy} + A_{sk}\right)^T \\ &= A_{sy}^T + A_{sk}^T \\ &= A_{sy} - A_{sk} \end{aligned} \tag{2}$$

Adding (1)+(2) gives

$$\begin{aligned} A + A^T &= 2A_{sy} \\ A_{sy} &= \frac{A + A^T}{2} \end{aligned} \tag{3}$$

Subtracting (2)-(1) gives

$$A^T - A = -2A_{sk}$$
$$A_{sk} = \frac{A - A^T}{2} \tag{4}$$

Therefore for any A,

$$A_{sy} = \frac{1}{2}\left(A + A^T\right) \tag{4A}$$
$$A_{sk} = \frac{1}{2}\left(A - A^T\right) \tag{4B}$$

To show that A_{sy} is indeed symmetric, this is done by construction :

$$\begin{aligned} A_{sy}^T &= \frac{1}{2}\left(A + A^T\right)^T \\ &= \frac{1}{2}\left(A^T + \left(A^T\right)^T\right) \end{aligned}$$

But $\left(A^T\right)^T = A$, and the above becomes

$$\begin{aligned} A_{sy}^T &= \frac{1}{2}\left(A^T + A\right) \\ &= A_{sy} \end{aligned}$$

Therefore A_{sy} is indeed symmetric.

To show that A_{sk} is skew symmetric matrix :

$$\begin{aligned} A_{sk}^T &= \frac{1}{2}\left(A - A^T\right)^T \\ &= \frac{1}{2}\left(A^T - \left(A^T\right)^T\right) \\ &= \frac{1}{2}\left(A^T - A\right) \\ &= -\frac{1}{2}\left(A - A^T\right) \\ &= -A_{sk} \end{aligned}$$

Hence A_{sk} is indeed skew symmetric.

Therefore any A matrix can be written as $A = A_{sy} + A_{sk}$ where A_{sy}, A_{sk} are given by (4A,4B).

Now we need to show that this is a unique was to write A. Proof is by contradictions. Let there be $\tilde{A}_{sy}$ matrix which is symmetric and $\tilde{A}_{sy} \neq A_{sy}$ and let there be $\tilde{A}_{sy}$ matrix which is skew matrix and $\tilde{A}_{sk} \neq A_{sk}$. And also let $A = \tilde{A}_{sy} + \tilde{A}_{sy}$ in addition to $A = A_{sy} + A_{sk}$. Then

$$\begin{aligned} A^T &= \left(\tilde{A}_{sy} + \tilde{A}_{sk}\right)^T \\ &= \tilde{A}_{sy}^T + \tilde{A}_{sk}^T \end{aligned}$$

Since $\tilde{A}_{sy}$ is assumed to be symmetric, then $\tilde{A}_{sy}^T = \tilde{A}_{sy}$ and since $\tilde{A}_{sk}$ is assumed to be skew symmetric, then $\tilde{A}_{sk}^T = -\tilde{A}_{sk}$ and the above becomes

$$A^T = \tilde{A}_{sy} - \tilde{A}_{sk}$$

Therefore

$$\begin{aligned} \frac{1}{2}\left(A + A^T\right) &= \frac{1}{2}\left(\tilde{A}_{sy} + \tilde{A}_{sk} + \tilde{A}_{sy} - \tilde{A}_{sk}\right) \\ &= \tilde{A}_{sy} \end{aligned}$$

But from (4A) above, we showed that $\frac{1}{2}\left(A + A^T\right) = A_{sy}$. Hence

$$A_{sy} = \tilde{A}_{sy}$$

Which is a contradiction to our assumption that $\tilde{A}_{sy} \neq A_{sy}$. Therefore A_{sy} is unique. The same is done for $\tilde{A}_{sk}$. From

$$\begin{aligned} \frac{1}{2}\left(A - A^T\right) &= \frac{1}{2}\left(\tilde{A}_{sy} + \tilde{A}_{sk} - \left(\tilde{A}_{sy} - \tilde{A}_{sk}\right)\right) \\ &= \tilde{A}_{sk} \end{aligned}$$

But from (4) above, we showed that $\frac{1}{2}\left(A - A^T\right) = A_{sk}$. Hence $A_{sk} = \tilde{A}_{sk}$ which is a contradiction. Therefore there is only way to write A as sum of symmetric and skew symmetric way, which is

$$A = \overbrace{\frac{A + A^T}{2}}^{A_{sy}} + \overbrace{\frac{1}{2}\left(A - A^T\right)}^{A_{sk}}$$

QED.

Part b

Starting with the forward direction. We need to show that given A is positive definite (p.d.) then this implies A_{sy} is also p.d.

From part (a) we found that A can be written as $A = A_{sy} + A_{sk}$. Since A is now assumed to be p.d. then this implies

$$\begin{aligned} x^T A x &> 0 \\ x^T \left(A_{sy} + A_{sk}\right) x &> 0 \\ x^T A_{sy} x + x^T A_{sk} x &> 0 \end{aligned} \tag{1}$$

Now we will show that $x^T A_{sk} x = 0$ to finish the above proof. First we observe that

$$\begin{aligned} \left(x^T A_{sk} x\right)^T &= \left(A_{sk} x\right)^T x \\ &= x^T A_{sk}^T x \end{aligned}$$

But $A_{sk} = -A_{sk}^T$ by definition of skew symmetric matrix. Therefore the above becomes

$$\left(x^T A_{sk} x\right)^T = -\left(x^T A_{sk} x\right)$$

But $x^T A_{sk} x$ is a single number, say q. (To be precise, q is 1×1 matrix. but since it is 1×1 we can treat it as a number, since it is one element). But the transpose of a number (or 1×1 matrix) is itself. Hence the above relation says that

$$q^T = -q$$

For a number, this is the same as saying $q = -q$ and this only possible if $q = 0$ or in other words

$$x^T A_{sk} x = 0 \tag{2}$$

Using (2) in (1) shows immediately that

$$x^T A_{sy} x > 0$$

Therefore A_{sy} is positive definite.

Now we need to show the reverse direction. That is, we need to show that if A_{sy} is p.d. then this implies A is also p.d.

Since A_{sy} is now assumed to be p.d. then we can write

$$x^T A_{sy} x > 0$$

But $A = A_{sy} + A_{sk}$ therefore $A_{sy} = A - A_{sk}$ and the above becomes

$$\begin{aligned} x^T \left(A - A_{sk}\right) x &> 0 \\ x^T A x - x^T A_{sk} x &> 0 \end{aligned}$$

But we showed in (2) that $x^T A_{sk} x = 0$. Therefore the above becomes

$$x^T A x > 0$$

Which implies that A is positive definite, which is what we are asked to show. QED

3.3.5 Problem 5

5. Consider

$$\mathbf{u_1} = [2 \quad 3 \quad 0 \quad 1], \qquad \mathbf{u_2} = [-1 \quad 0 \quad 3 \quad 2]$$
$$\mathbf{u_3} = [2 \quad 2 \quad 1 \quad 4], \qquad \mathbf{u_4} = [-6 \quad 4 \quad -2 \quad 0].$$

Show that three vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ out of the above four form an orthogonal set in $\mathbb{R}^4$. What are these vectors?
Find the best approximation $\mathbf{w} \in Span(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$ to the fourth vector using Gramm's theorem.
Find a basis $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ of orthogonal vectors for $\mathbb{R}^4$.

Figure 3.8: Problem 5 Statement

Solution

Two vectors $\vec{v},\vec{u}$ are orthogonal if their dot product is zero. This is because $\vec{v}\cdot\vec{u}=\left\|\vec{v}\right\|\left\|\vec{u}\right\|\cos\theta$ where θ is the inner angle between the two vectors. Since the vectors are orthogonal, then $\cos 90^0=0$ and therefore $\vec{v}\cdot\vec{u}=0$. To find which pairs are orthogonal to each others, we compute the inner product between all possible pairs :

$$
\begin{aligned}
\vec{u}_1\cdot\vec{u}_2 &= -2+0+0+2=0\\
\vec{u}_1\cdot\vec{u}_3 &= 4+6+0+4=14\\
\vec{u}_1\cdot\vec{u}_4 &= -12+12+0+0=0\\
\vec{u}_2\cdot\vec{u}_3 &= -2+0+3+8=9\\
\vec{u}_2\cdot\vec{u}_4 &= 6+0-6+0=0\\
\vec{u}_3\cdot\vec{u}_4 &= -12+8-2+0=-2
\end{aligned}
$$

We see from the above that $\vec{u}_1\cdot\vec{u}_2=0,\vec{u}_1\cdot\vec{u}_4=0,\vec{u}_2\cdot\vec{u}_4=0$. Therefore

$$
\begin{aligned}
S &= \left\{\vec{u}_1,\vec{u}_2,\vec{u}_4\right\}\\
&\equiv \left\{\vec{v}_1,\vec{v}_2,\vec{v}_3\right\}
\end{aligned}
$$

Or

$$
\begin{aligned}
\vec{v}_1 &= (2,3,0,1) \qquad (1B)\\
\vec{v}_2 &= (-1,0,3,2)\\
\vec{v}_3 &= (-6,4,-2,0)
\end{aligned}
$$

Form an orthogonal set in $\mathbb{R}^4$.

Now we need to find the best approximation of $\vec{w}=\vec{u}_3=(2,2,1,4)$ using the above orthogonal vectors $\vec{v}_1,\vec{v}_2,\vec{v}_3$. Using Gram's theorem, this approximation is

$$
\vec{w}=c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3 \tag{1A}
$$

Where the constants c_i are found from solving the system

$$
\begin{pmatrix}\vec{v}_1\cdot\vec{v}_1 & \vec{v}_1\cdot\vec{v}_2 & \vec{v}_1\cdot\vec{v}_3\\ \vec{v}_2\cdot\vec{v}_1 & \vec{v}_2\cdot\vec{v}_2 & \vec{v}_2\cdot\vec{v}_3\\ \vec{v}_3\cdot\vec{v}_1 & \vec{v}_3\cdot\vec{v}_2 & \vec{v}_3\cdot\vec{v}_3\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}\vec{v}_1\cdot\vec{w}\\ \vec{v}_2\cdot\vec{w}\\ \vec{v}_3\cdot\vec{w}\end{pmatrix}
$$

But since $\vec{v}_i$ are all orthogonal to each others then $\vec{v}_i\cdot\vec{v}_j=0$ for $i\neq j$, and $\vec{v}_i\cdot\vec{v}_i=\|v_i\|^2$ and the above becomes

$$
\begin{pmatrix}\left\|\vec{v}_1\right\|^2 & 0 & 0\\ 0 & \left\|\vec{v}_2\right\|^2 & 0\\ 0 & 0 & \left\|\vec{v}_3\right\|^2\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}\vec{v}_1\cdot\vec{w}\\ \vec{v}_2\cdot\vec{w}\\ \vec{v}_3\cdot\vec{w}\end{pmatrix} \tag{1}
$$

But

$$
\begin{aligned}
\vec{v}_1\cdot\vec{w} &= (2,3,0,1)\cdot(2,2,1,4)=4+6+4=14\\
\vec{v}_2\cdot\vec{w} &= (-1,0,3,2)\cdot(2,2,1,4)=-2+3+8=9\\
\vec{v}_3\cdot\vec{w} &= (-6,4,-2,0)\cdot(2,2,1,4)=-12+8-2=-6
\end{aligned}
$$

Hence (1) becomes

$$
\begin{pmatrix}\left\|\vec{v}_1\right\|^2 & 0 & 0\\ 0 & \left\|\vec{v}_2\right\|^2 & 0\\ 0 & 0 & \left\|\vec{v}_3\right\|^2\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}14\\9\\-6\end{pmatrix} \tag{2}
$$

Since

$$
\begin{aligned}
\left\|\vec{v}_1\right\|^2 &= \|(2,3,0,1)\|^2=4+9+1=14\\
\left\|\vec{v}_2\right\|^2 &= \|(-1,0,3,2)\|^2=1+9+4=14\\
\left\|\vec{v}_3\right\|^2 &= \|(-6,4,-2,0)\|^2=36+16+4=56
\end{aligned}
$$

Then (2) becomes

$$
\begin{pmatrix}14 & 0 & 0\\ 0 & 14 & 0\\ 0 & 0 & 56\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}14\\9\\-6\end{pmatrix} \tag{3}
$$

From the above we see that

$$
\begin{aligned}
c_1 &= 1\\
c_2 &= \frac{9}{14}\\
c_3 &= \frac{-3}{28}
\end{aligned}
$$

Hence the best approximation using (1A) becomes

$$
\begin{aligned}
\vec{w} &= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3\\
&= (2,3,0,1) + \frac{9}{14}(-1,0,3,2) - \frac{3}{28}(-6,4,-2,0)\\
&= \left(2,\frac{18}{7},\frac{15}{7},\frac{16}{7}\right)
\end{aligned}
$$

Therefore

$$
\vec{w} = \frac{1}{7}(14,18,15,16)
$$

Now we need to find basis $\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4$ of orthogonal vectors in $\mathbb{R}^4$. We already found that from (1B) that $\vec{v}_1,\vec{v}_2,\vec{v}_3$ are three such vectors. So we just need to find another $\vec{v}_4 = [a_1,a_2,a_3,a_4]$ such that it is orthogonal to the other three, in other words we need to solve

$$
\begin{aligned}
\vec{v}_1\cdot\vec{v}_4 &= 0\\
\vec{v}_2\cdot\vec{v}_4 &= 0\\
\vec{v}_3\cdot\vec{v}_4 &= 0
\end{aligned}
$$

This implies

$$
\begin{aligned}
[2,3,0,1]\cdot[a_1,a_2,a_3,a_4] &= 0\\
[-1,0,3,2]\cdot[a_1,a_2,a_3,a_4] &= 0\\
[-6,4,-2,0]\cdot[a_1,a_2,a_3,a_4] &= 0
\end{aligned}
$$

Or

$$
\begin{aligned}
2a_1 + 3a_2 + a_4 &= 0\\
-a_1 + 3a_3 + 2a_4 &= 0\\
-6a_1 + 4a_2 - 2a_3 &= 0
\end{aligned}
$$

Or

$$
\begin{pmatrix} 2 & 3 & 0 & 1\\ -1 & 0 & 3 & 2\\ -6 & 4 & -2 & 0 \end{pmatrix}
\begin{pmatrix} a_1\\ a_2\\ a_3\\ a_4 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0 \end{pmatrix}
$$

This system has three equations and 4 unknowns. Therefore it will have one free parameter giving an infinite number of solutions. Using Gaussian elimination:

$$
\begin{pmatrix} 2 & 3 & 0 & 1\\ -1 & 0 & 3 & 2\\ -6 & 4 & -2 & 0 \end{pmatrix}
\xrightarrow{R_2=R_2+\frac{1}{2}R_1}
\begin{pmatrix} 2 & 3 & 0 & 1\\ 0 & \frac{3}{2} & 3 & \frac{5}{2}\\ -6 & 4 & -2 & 0 \end{pmatrix}
\xrightarrow{R_3=R_3+3R_1}
$$

$$
\begin{pmatrix} 2 & 3 & 0 & 1\\ 0 & \frac{3}{2} & 3 & \frac{5}{2}\\ 0 & 13 & -2 & 3 \end{pmatrix}
\xrightarrow{R_3=R_3-\frac{26}{3}R_2}
\begin{pmatrix} 2 & 3 & 0 & 1\\ 0 & \frac{3}{2} & 3 & \frac{5}{2}\\ 0 & 0 & -28 & -\frac{56}{3} \end{pmatrix}
$$

We stop the elimination here since no more elimination is possible. We have now this system

$$
\begin{pmatrix} 2 & 3 & 0 & 1\\ 0 & \frac{3}{2} & 3 & \frac{5}{2}\\ 0 & 0 & -28 & -\frac{56}{3} \end{pmatrix}
\begin{pmatrix} a_1\\ a_2\\ a_3\\ a_4 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0 \end{pmatrix}
$$

Back substitution: From last row we obtain the equation

$$
\begin{aligned}
-28a_3 - \frac{56}{3}a_4 &= 0\\
a_4 &= -\frac{3}{2}a_3
\end{aligned}
$$

The second row gives

$$
\begin{aligned}
\frac{3}{2}a_2 + 3a_3 + \frac{5}{2}a_4 &= 0 \\
a_2 &= \frac{2}{3}\left(-3a_3 - \frac{5}{2}a_4\right) \\
&= -2a_3 - \frac{5}{3}a_4
\end{aligned}
$$

Since $a_4 = -\frac{3}{2}a_3$ the above becomes

$$
\begin{aligned}
a_2 &= -2a_3 - \frac{5}{3}\left(-\frac{3}{2}a_3\right) \\
&= \frac{1}{2}a_3
\end{aligned}
$$

First row gives

$$
\begin{aligned}
2a_1 + 3a_2 + a_4 &= 0 \\
a_1 &= \frac{1}{2}\left(-3a_2 - a_4\right)
\end{aligned}
$$

Since $a_2 = \frac{1}{2}a_3$ and $a_4 = -\frac{3}{2}a_3$ the above becomes

$$
\begin{aligned}
a_1 &= \frac{1}{2}\left(-3\left(\frac{1}{2}a_3\right) - \left(-\frac{3}{2}a_3\right)\right) \\
&= 0
\end{aligned}
$$

Therefore the solution is

$$
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \\ -\frac{3}{2} \end{pmatrix} a_3
$$

The above means that for any arbitrary a_3 value there is a solution. a_3 is just a scalar which only stretches or shrinks the vector but does not change its direction (orientation). Therefore the vector remains orthogonal to all others for any a_3. Let us pick $a_3 = 1$. Using this $\vec{v}_4$ becomes

$$
\vec{v}_4 = \left[0, \frac{1}{2}, 1, -\frac{3}{2}\right]
$$

To verify the result found, we will check that $\vec{v}_4$ is indeed orthogonal with the other three vectors :

$$
\begin{aligned}
\vec{v}_1 \cdot \vec{v}_4 &= [2,3,0,1] \cdot \left[0, \frac{1}{2}, 1, -\frac{3}{2}\right] = 0 \\
\vec{v}_2 \cdot \vec{v}_4 &= [-1,0,3,2] \cdot \left[0, \frac{1}{2}, 1, -\frac{3}{2}\right] = 0 \\
\vec{v}_3 \cdot \vec{v}_4 &= [-6,4,-2,0] \cdot \left[0, \frac{1}{2}, 1, -\frac{3}{2}\right] = 0
\end{aligned}
$$

QED.

3.3.6 Problem 6

6. For a general $n \times n$ nonsingular matrix A compute the total number of multiplications/divisions necessary to solve the linear system

$$
A\mathbf{x} = \mathbf{b},
$$

a) using Gauss elimination with back-substitution,
b) using Cramer's rule.

Figure 3.9: Problem 6 Statement

Solution

Let

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

Part (a)

The first step in Gaussian elimination is to reduce the above matrix to row echelon form :

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \vdots \\ \tilde{b}_n \end{pmatrix}$$

Row echelon form has zeros in its the lower left triangle. After this, back substitution starts by solving for x_n from the last row, then solving for x_{n-1} from the row above the last row and so on until we reach the first row.

Counting operations for forward pass

The first step is to zero out all entries in first column below a_{11} using a_{11} as pivot. Next is to zero out all entries in second column below the (updated) a_{22} value and so on.

To zero out an entry, for example a_{21}, we first need to do one division $\frac{a_{21}}{a_{11}} = \Delta$ and store this in memory, then do $a_{2i} = a_{2i} - \Delta a_{1i}$ for all entries in that row, which means for $i = 1 \cdots n$. (no need to count a_{21} since we know it will be zero). We have to remember that this is being applied to the b vector as well and not just for A matrix rows.

Hence we need one division to find Δ, and then $2n$ multiplication and addition/subtraction operations per row. The division is only needed once per row to find the pivot scaling Δ.

Since there are $n-1$ rows then there are $(n-1)$ divisions and $(2n)(n-1)$ multiplications/addition to zero out the first column. After this we have the following system reached

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \vdots \\ \tilde{b}_n \end{pmatrix}$$

The total cost now is therefore $(n-1) + (2n)(n-1)$.

We now switch to the second row and use the new value of a_{22} as pivot and repeat the same as above. The only difference now is that there are $n-2$ rows to process and $(n-2)$ divisions and therefore $2(n-1)(n-2)$ multiplications/addition to zero out the second column entries below the second row. After this we reach the following system

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \vdots \\ \tilde{b}_n \end{pmatrix}$$

The total cost of the above is therefore $(n-2) + 2(n-1)(n-2)$.

We now switch to the third row and use the new value of a_{33} as pivot. Now there are now $(n-3)$ divisions and $2(n-2)(n-3)$ multiplications/additions to obtain the following system

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \vdots \\ \tilde{b}_n \end{pmatrix}$$

The total cost of the above is therefore $(n-3)+2(n-2)(n-3)$.

And so on until we reach the row before the last row, where there is only one row below it to process. The cost then is just one division and 2 additions and 2 multiplications. Therefore the number of total number of multiplication and additions operations for the forward pass is the sum of all the above operations, which can be written as

$$\overbrace{[(n-1)+(2n)(n-1)]}^{\text{row 2}}+\overbrace{[(n-2)+2(n-1)(n-2)]}^{\text{row 3}}+\cdots+\overbrace{[1+4]}^{\text{last row}}$$

Writing the above as $\sum_{k=1}^{n-1}(n-k)+2(n-k+1)(n-k)$ then we need to calculate this sum using known formulas for summations. Let this sum be Δ, hence

$$\begin{aligned}
\Delta &= \sum_{k=1}^{n-1} -3k+2k^2+3n-4kn+2n^2\\
&= -3\sum_{k=1}^{n-1}k+2\sum_{k=1}^{n-1}k^2+3\sum_{k=1}^{n-1}n-4\sum_{k=1}^{n-1}kn+2\sum_{k=1}^{n-1}n^2\\
&= -3\left(\frac{n(n-1)}{2}\right)+2\left(\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}\right)+3\left(n^2-n\right)-4\left(\frac{n^2(n-1)}{2}\right)+2\left(n^2(n-1)\right)\\
&= \frac{2}{3}n^3+\frac{1}{2}n^2-\frac{7}{6}n
\end{aligned} \tag{1}$$

The above is the number of operations just for for the forward pass (elimination phase).

For example for matrix of size 3×3 the above gives 19 operations, and for matrix of size 4×4, it gives 46 operations and for 5×5 it gives 90 operations and so on.

Counting operations for backward pass In back substitution, we start from the end of the elimination phase above, which will be

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ 0 & a_{22} & a_{23} & \cdots & a_{2n}\\ 0 & 0 & a_{33} & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & a_{nn}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\\\vdots\\b_n\end{pmatrix}$$

First step is to solve for x_n by finding $x_n=\frac{b_n}{a_{nn}}$. This requires only one division. Next is to solve for x_{n-1} by finding $a_{n-1,n-1}x_{n-1}+a_{n-1,n}x_n=b_{n-1}$, or

$$x_{n-1}=\frac{b_{n-1}-\left(a_{n-1,n}\right)x_n}{\left(a_{n-1,n-1}\right)}$$

We see that this needs one subtraction, one multiplication and one division, or 3 operations. The next step is to solve for x_{n-2} from

$$\left(a_{n-2,n-2}\right)x_{n-2}+\left(a_{n-2,n-1}\right)x_{n-1}+\left(a_{n-2,n}\right)x_n=b_2$$

Hence

$$x_{n-2}=\frac{b_2-\left(a_{n-2,n-1}\right)x_{n-1}-\left(a_{n-2,n}\right)x_n}{\left(a_{n-2,n-2}\right)}$$

Therefore we need 2 subtractions, 2 multiplication and one division, or 5 operations. And so on until we reach the first row to solve for x_1. Therefore the total number of operations can be seen as

$$1+3+5+7+\cdots$$

The above can be written as the sum

$$\begin{aligned}
\sum_{k=0}^{n-1}(2k+1) &= 2\sum_{k=0}^{n-1}k+\sum_{k=0}^{n-1}1\\
&= 2\left(\frac{n(n-1)}{2}\right)+n\\
&= n(n-1)+n\\
&= n^2-n+n\\
&= n^2
\end{aligned} \tag{2}$$

We see that the cost of the elimination is much greater than the cost of back substitution. One is $O\left(n^3\right)$ while the other is $O\left(n^2\right)$.

From (1,2), the total number of operations for the complete Gaussian elimination process is

$$\begin{aligned}\Delta &= \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n + n^2\\ &= \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n\end{aligned}$$

For large n the above is $O\left(n^3\right)$.

Part (b)

Given a system of equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\ \vdots\\x_n\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\\ \vdots\\b_n\end{pmatrix}$$

Cramer method works as follows :

$$\begin{aligned}x_1 &= \frac{|A|}{|A_1|}\\ x_2 &= \frac{|A|}{|A_2|}\\ &\vdots\\ x_n &= \frac{|A|}{|A_n|}\end{aligned}$$

Where $|A|$ is the determinant of coefficient matrix $A_{n\times n}$ and $|A_i|$ is determinant of coefficient matrix but with the i^{th} column replaced by the column vector b.

An efficient way to find the determinant is to convert the matrix to row echelon form. In this form, the matrix is upper triangle. Hence the determinant is the product of all elements along the diagonal. This is more efficient than using the matrix cofactor expansion method.

In doing these row operations on the matrix to find $|A|$ the only difference from the elimination steps we did for part(a), is that we have to remember the following rules now during the elimination process

1. When adding multiple of one row to another row, the determinant is not affected.
2. When switching two rows, the determinant is multiplied by -1
3. When multiplying one row by some scalar, the determinant is also multiplied by the same scalar.

Given the above, let us assume that for each elimination step of a row, we do one multiplication to account for a possible multiplication by -1 or possible multiplication by a scalar. Since we do not know if this will happen every time as this clearly depends on the data in the matrix, then this will be the worst case counting.

This means there is an additional $(n-1)$ multiplications to add to the cost of doing the elimination step to reach row echelon form at the end.

Another small difference from part(a), is that now we do not have the b vector added during the forward step.

Therefore, as we did in part(a), the cost to reach this form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ 0 & a_{22} & a_{23} & \cdots & a_{2n}\\ 0 & a_{32} & a_{33} & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn}\end{pmatrix}$$

Is now $(n-1)+(2(n-1))(n-1)$. Recalling from part(a) the cost at this stage was $(n-1)+(2n)(n-1)$ here. So we changed $2n$ to $2(n-1)$, since there is no b vector, hence one less element. And as was done in part (a), the cost to reach

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Now becomes $(n-2) + 2(n-2)(n-2)$. Recalling from part(a) the cost at this stage was $(n-1) + 2(n-1)(n-1)$. So we changed $2(n-1)$ to $2(n-2)$, since there is no b vector. This continues to the row before the last as in part (a). Therefore the number of total multiplication and additions operations for just the forward pass is

$$\overbrace{[(n-1) + (2(n-1))(n-1)]}^{\text{row 2}} + \overbrace{[(n-2) + 2(n-2)(n-2)]}^{\text{row 3}} + \cdots + \overbrace{[1+2]}^{\text{last row}}$$

Hence the cost to put the matrix in row echelon form is

$$\begin{aligned} \sum_{k=1}^{n-1}(n-k) + 2(n-k)^2 &= \sum_{k=1}^{n-1} n - \sum_{k=1}^{n-1} k + 2\sum_{k=1}^{n-1}(n-k)^2 \\ &= n(n-1) - \frac{n(n-1)}{2} + 2\frac{n-3n^2+2n^3}{6} \\ &= \frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n \end{aligned} \tag{1}$$

The above cost is very close to part(a) elimination phase as expected which was $\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n$. Current cost is smaller because in part(a) we had the b vector there which added more operations, while here we just operated on A itself.

Let us now add the $(n-1)$ multiplication we mentioned earlier to the result above. The cost now becomes

$$\begin{aligned} \Delta &= \frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n + (n-1) \\ &= \frac{2}{3}n^3 - \frac{1}{2}n^2 + \frac{5}{6}n - 1 \end{aligned}$$

We still need to calculate the product of the diagonal elements to find the determinant. For $n \times n$ matrix, this takes $n-1$ multiplications. Adding these to the above gives

$$\begin{aligned} \Delta &= \frac{2}{3}n^3 - \frac{1}{2}n^2 + \frac{5}{6}n - 1 + (n-1) \\ &= \frac{2}{3}n^3 - \frac{1}{2}n^2 + \frac{11}{6}n - 2 \\ &\approx \frac{2}{3}n^3 - \frac{1}{2}n^2 + \frac{11}{6}n \end{aligned}$$

We will use the above as <u>the cost of finding the determinant.</u>

How many times do we need to find determinants? We need to do it one time to find $|A|$ and then n more time for each $|A_i|$. Hence $(n+1)$ times. This is the main reason why Cramer method becomes much more costly compared to Gaussian elimination.

The number of operations now becomes

$$\begin{aligned} \Delta &= \left(\frac{2}{3}n^3 - \frac{1}{2}n^2 + \frac{11}{6}n\right)(n+1) \\ &= \frac{2}{3}n^4 + \frac{1}{6}n^3 + \frac{4}{3}n^2 + \frac{11}{6}n \end{aligned}$$

We also need to add the cost of the final divisions $\frac{|A|}{|A_i|}$ to find each x_i. So we add n divisions to the above, giving the final cost as

$$\begin{aligned} \Delta &= \frac{2}{3}n^4 + \frac{1}{6}n^3 + \frac{4}{3}n^2 + \frac{11}{6}n + n \\ &= \frac{2}{3}n^4 + \frac{1}{6}n^3 + \frac{4}{3}n^2 + \frac{17}{6}n \end{aligned}$$

We see from above, that Cramer rule for large n is $O\left(n^4\right)$ while Gaussian elimination was $O\left(n^3\right)$. Hence Gaussian elimination is <u>much more efficient</u> for large n.

In summary

n	cost of Gaussian elimination $\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$	cost of Cramer $\frac{2}{3}n^4 + \frac{1}{6}n^3 + \frac{4}{3}n^2 + \frac{17}{6}n$
2	5	23
3	19	79
4	46	214
5	90	485
6	155	965
7	245	1743
8	364	2924
9	516	4629
10	705	6995

The following is a graphical illustration of the above

```
In[◦]:= ge = 2/3 n^3 + 3/2 n^2 - 7/6 n;
cramer = 2/3 n^4 + 1/6 n^3 + 4/3 n^2 + 17/6 n;
dataGE = Table[{n, ge}, {n, 2, 10}];
dataCramer = Table[{n, cramer}, {n, 2, 10}];

ListLinePlot[{dataCramer, dataGE}, Mesh → Full, PlotRange → All,
 MeshStyle → {Red, PointSize@Large}, GridLines → Automatic, GridLinesStyle → LightGray,
 PlotLegends → {"Cramer", "G.E."}, AxesLabel → {"n", "cost"}, BaseStyle → 14,
 AxesOrigin → {1, 0}, Ticks → {Range[2, 10], Automatic}]
```

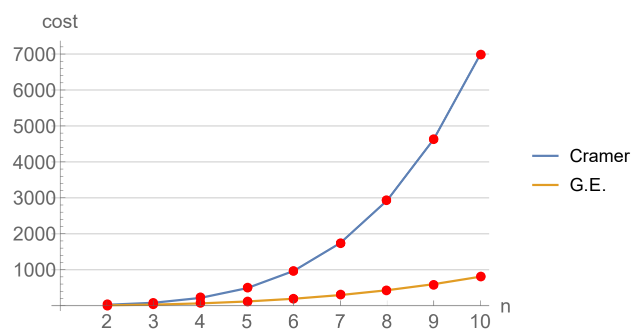


Figure 3.10: Cost of Gaussian elimination vs. Cramer method. Problem 6

3.3.7 Problem 7

7. Consider the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & -1 & 0 \end{bmatrix}.$$

a) Show that its eigenvalues lie on the unit circle in $\mathbb{C}$.

b) Show that the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ preserves the angles in $\mathbb{R}^3$, i.e. for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$,

$$\angle(\mathbf{x}, \mathbf{y}) = \angle(A\mathbf{x}, A\mathbf{y}).$$

(Here $\angle(\mathbf{u}, \mathbf{v})$ denotes the angle between $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$. Note that $\angle(\mathbf{u}, \mathbf{v})$ can be expressed in terms of the norms and dot product of the vectors $\mathbf{u}$ and $\mathbf{v}$.)

Figure 3.11: Problem 7 Statement

Solution

Part a

The eigenvalues of A are found by solving $|A - \lambda I| = 0$ or

$$\begin{vmatrix} \frac{1}{\sqrt{5}} - \lambda & 0 & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & -\lambda & \frac{1}{\sqrt{5}} \\ 0 & -1 & -\lambda \end{vmatrix} = 0$$

$$\left(\frac{1}{\sqrt{5}} - \lambda\right)\begin{vmatrix} -\lambda & \frac{1}{\sqrt{5}} \\ -1 & -\lambda \end{vmatrix} - 0 + \frac{2}{\sqrt{5}}\begin{vmatrix} \frac{-2}{\sqrt{5}} & -\lambda \\ 0 & -1 \end{vmatrix} = 0$$

$$\left(\frac{1}{\sqrt{5}} - \lambda\right)\left(\lambda^2 + \frac{1}{\sqrt{5}}\right) + \frac{2}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}\right) = 0$$

$$\frac{1}{5}(\lambda - 1)\left(-5\lambda - 5\lambda^2 + \sqrt{5}\lambda - 5\right) = 0$$

$$\frac{1}{5}(\lambda - 1)\left(-5\lambda^2 + \lambda\left(\sqrt{5} - 5\right) - 5\right) = 0$$

$$(\lambda - 1)\left(-5\lambda^2 + \lambda\left(\sqrt{5} - 5\right) - 5\right) = 0$$

Hence $\lambda = 1$. The quadratic formula is used to solve $-5\lambda^2 + \lambda\left(\sqrt{5} - 5\right) - 5 = 0$. First it is normalized

$$5\lambda^2 - \lambda\left(\sqrt{5} - 5\right) + 5 = 0$$

$$\lambda^2 - \lambda\left(\frac{1}{5}\sqrt{5} - 1\right) + 1 = 0$$

Then $\lambda = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$ where $b = -\left(\frac{1}{5}\sqrt{5} - 1\right), c = 1, a = 1$ and the roots are

$$\begin{aligned}
\lambda &= \frac{\left(\frac{1}{5}\sqrt{5} - 1\right)}{2} \pm \frac{1}{2}\sqrt{\left(-\left(\frac{1}{5}\sqrt{5} - 1\right)\right)^2 - 4} \\
&= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{5}\sqrt{5} - 1\right)^2 - 4} \\
&= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{1 + \frac{5}{25} - \frac{2}{5}\sqrt{5} - 4} \\
&= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{1 + \frac{1}{5} - \frac{2}{5}\sqrt{5} - 4} \\
&= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{-\frac{14}{5} - \frac{2\sqrt{5}}{5}} \\
&= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{\frac{-14 - 2\sqrt{5}}{5}} \\
&= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) \pm \frac{1}{2}i\sqrt{\frac{14 + 2\sqrt{5}}{5}} \\
&= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) \pm \frac{1}{2}i\sqrt{\frac{5\left(14 + 2\sqrt{5}\right)}{(5)\,5}} \\
&= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) \pm \frac{1}{(2)\,(5)}i\sqrt{5\left(14 + 2\sqrt{5}\right)} \\
&= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) \pm \frac{1}{10}i\sqrt{10\sqrt{5} + 70}
\end{aligned}$$

Therefore the roots are

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) + \frac{1}{10}i\sqrt{10\sqrt{5} + 70} \\
\lambda_3 &= \left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right) - \frac{1}{10}i\sqrt{10\sqrt{5} + 70}
\end{aligned}$$

Numerically the above becomes

$$
\begin{aligned}
\lambda_1 &= 1\\
\lambda_2 &= -0.276 + 0.961i\\
\lambda_3 &= -0.276\ - 0.961i
\end{aligned}
$$

The following plot shows the locations on the complex plane

```
eigs = {1, -0.276 + 0.961 I, -0.276 - 0.961 I};
eigs = ReIm@eigs;
img = Graphics[{
    {FaceForm[White], EdgeForm[Black], Disk[]},
    {Red, PointSize[.05], Point[eigs]}
   }, ImageSize → 300, Axes → True];
grid = Graphics[{}, GridLines → Automatic, PlotRangePadding → None,
   GridLinesStyle → LightGray, ImageSize → ImageDimensions@img];
Overlay[{img, grid}]
```

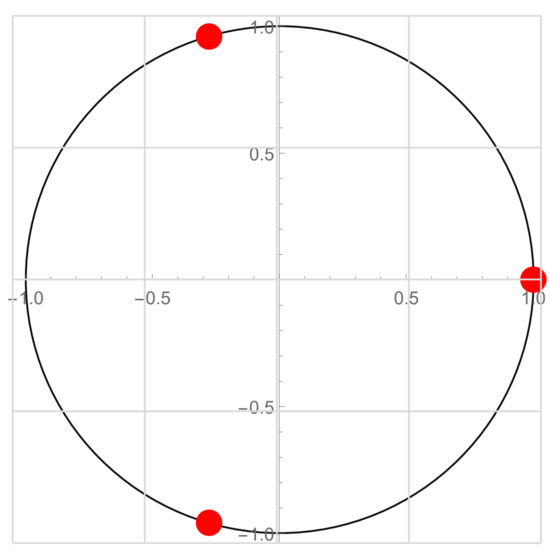


Figure 3.12: Graphical location of eigenvalues for problem 7

To show analytically that the eigenvalues lie on the unit circle means to show that the magnitude of each complex number is 1. Clearly λ_1 already satisfy this condition. We need to check now that $\|\lambda_2\| = 1$ and that $\left\|\lambda_3\right\| = 1$

$$
\begin{aligned}
\|\lambda_2\| &= \sqrt{\operatorname{Re}(\lambda_2)^2 + \operatorname{Im}(\lambda_2)^2}\\
&= \sqrt{\left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right)^2 + \left(\frac{1}{10}\sqrt{10\sqrt{5}+70}\right)^2}\\
&= \sqrt{\left(\frac{3}{10} - \frac{1}{10}\sqrt{5}\right) + \left(\frac{1}{10}\sqrt{5} + \frac{7}{10}\right)}\\
&= \sqrt{\frac{10}{10}}\\
&= 1
\end{aligned}
$$

Similarly for λ_3 since it is the same except for the sign on the complex part (complex conjugate) which does not affect the norm. Therefore all the eigenvalues lie on unit circle in $\mathbb{C}$. QED.

Part b

Let two vectors in the domain of A be $\mathrm{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathrm{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. And let the two vector after the mapping, which now lie in the range of A be $\tilde{\mathrm{x}} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}$ and $\tilde{\mathrm{y}} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{pmatrix}$. Since $\mathrm{x} \cdot \mathrm{y} = \|\mathrm{x}\| \|\mathrm{y}\| \cos\theta$ where

θ is the inner angle between the vectors, and since $\tilde{x} \cdot \tilde{y} = \|\tilde{x}\| \|\tilde{y}\| \cos \tilde{\theta}$, then we need to show that

$$\theta = \tilde{\theta}$$

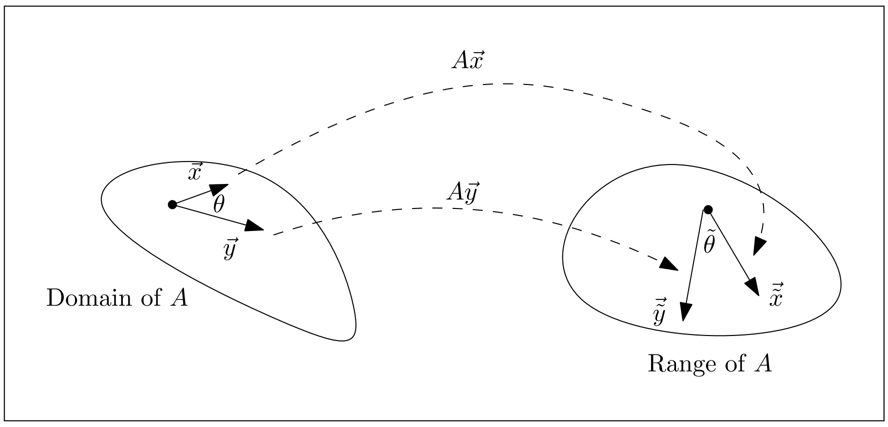


Figure 3.13: Linear transformation Ax preserves angles. Problem 7

Let

$$A\mathrm{x} = \tilde{\mathrm{x}}$$
$$A\mathrm{y} = \tilde{\mathrm{y}}$$

Using the A given, then

$$\begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_3 \\ \frac{-2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_3 \\ -x_2 \end{pmatrix}$$

Hence

$$\tilde{\mathrm{x}} = \begin{pmatrix} \frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_3 \\ \frac{-2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_3 \\ -x_2 \end{pmatrix}$$

We see from the above that

$$\tilde{x}_1 = \frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_3$$
$$\tilde{x}_2 = \frac{-2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_3$$
$$\tilde{x}_3 = -x_2$$

Similarly

$$\tilde{\mathrm{y}} = \begin{pmatrix} \frac{1}{\sqrt{5}}y_1 + \frac{2}{\sqrt{5}}y_3 \\ \frac{1}{\sqrt{5}}y_1 - \frac{2}{\sqrt{5}}y_3 \\ -y_2 \end{pmatrix}$$

We see from the above that

$$\tilde{y}_1 = \frac{1}{\sqrt{5}}y_1 + \frac{2}{\sqrt{5}}y_3$$
$$\tilde{y}_2 = \frac{-2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_3$$
$$\tilde{y}_3 = -y_2$$

We now need to determine θ and $\tilde{\theta}$ and show they are the same. From the definition above

$$\theta = \arccos\left(\frac{\mathrm{x} \cdot \mathrm{y}}{\|\mathrm{x}\| \|\mathrm{y}\|}\right)$$

But $\mathrm{x} \cdot \mathrm{y} = x_1y_1 + x_2y_2 + x_3y_3$ and $\|\mathrm{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $\|\mathrm{y}\| = \sqrt{y_1^2 + y_2^2 + y_3^2}$, therefore the above becomes

$$\theta = \arccos\left(\frac{x_1y_1 + x_2y_2 + x_3y_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}\sqrt{y_1^2 + y_2^2 + y_3^2}}\right) \tag{1}$$

Similarly, $\tilde{x} \cdot \tilde{y} = \tilde{x}_1\tilde{y}_1 + \tilde{x}_2\tilde{y}_2 + \tilde{x}_3\tilde{y}_3$. Using the values of $\tilde{x}_i, \tilde{y}_i$ found above the dot product becomes

$$\begin{aligned}
\tilde{x} \cdot \tilde{y} &= \left(\frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_3\right)\left(\frac{1}{\sqrt{5}}y_1 + \frac{2}{\sqrt{5}}y_3\right) + \left(\frac{-2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_3\right)\left(\frac{-2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_3\right) + (-x_2)\left(-y_2\right) \\
&= \frac{1}{5}\left(x_1 + 2x_3\right)\left(y_1 + 2y_3\right) + \frac{1}{5}\left(x_3 - 2x_1\right)\left(y_3 - 2y_1\right) + x_2y_2 \\
&= \frac{1}{5}x_1y_1 + \frac{2}{5}x_1y_3 + \frac{2}{5}x_3y_1 + \frac{4}{5}x_3y_3 + \frac{4}{5}x_1y_1 - \frac{2}{5}x_1y_3 - \frac{2}{5}x_3y_1 + \frac{1}{5}x_3y_3 + x_2y_2
\end{aligned}$$

Which simplifies to

$$\tilde{x} \cdot \tilde{y} = x_1y_1 + x_2y_2 + x_3y_3$$

And $\|\tilde{x}\| = \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2}$. Using the values of $\tilde{x}_i$ found above, this becomes

$$\begin{aligned}
\|\tilde{x}\| &= \sqrt{\left(\frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_3\right)^2 + \left(\frac{-2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_3\right)^2 + (-x_2)^2} \\
&= \sqrt{\frac{1}{5}x_1^2 + \frac{4}{5}x_1x_3 + \frac{4}{5}x_3^2 + \frac{4}{5}x_1^2 - \frac{4}{5}x_1x_3 + \frac{1}{5}x_3^2 + x_2^2}
\end{aligned}$$

Which simplifies to

$$\|\tilde{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Similarly, $\|\tilde{y}\| = \sqrt{\tilde{y}_1^2 + \tilde{y}_2^2 + \tilde{y}_3^2}$ and using the values of $\tilde{y}_i$ found above, then this becomes

$$\begin{aligned}
\|\tilde{y}\| &= \sqrt{\left(\frac{1}{\sqrt{5}}y_1 + \frac{2}{\sqrt{5}}y_3\right)^2 + \left(\frac{-2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_3\right)^2 + \left(-y_2\right)^2} \\
&= \sqrt{\frac{1}{5}y_1^2 + \frac{4}{5}y_1y_3 + \frac{4}{5}y_3^2 + \frac{4}{5}y_1^2 - \frac{4}{5}y_1y_3 + \frac{1}{5}y_3^2 + y_2^2}
\end{aligned}$$

Which simplifies to

$$\|\tilde{y}\| = \sqrt{y_1^2 + y_2^2 + y_3^2}$$

Therefore

$$\begin{aligned}
\cos\tilde{\theta} &= \frac{\tilde{x} \cdot \tilde{y}}{\|\tilde{x}\|\,\|\tilde{y}\|} \\
\tilde{\theta} &= \arccos\left(\frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}\sqrt{y_1^2 + y_2^2 + y_3^2}}\right) \qquad (2)
\end{aligned}$$

Comparing (1) and (2) shows they are the same. Therefore $\theta = \tilde{\theta}$. QED.

3.3.8 Problem 8

8. Write the expressions

$$\cos(nx - mx) - \cos(nx + mx), \cos(nx - mx) - \cos(nx + mx),$$

$$\sin(nx - mx) - \sin(nx + mx), \sin(nx - mx) + \sin(nx + mx)$$

with integers n and m in terms of

$$\cos nx, \sin nx, \cos mx, \sin mx.$$

Consider the Hilbert space of real square integrable functions $L^2[-\pi\ \pi]$ on the $[-\pi, \pi]$ interval, equipped with the inner product $< f, g >= \int_{-\pi}^{\pi} fg\, dx$. Show that the set of functions

$$S = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots, \frac{\cos 5x}{\sqrt{\pi}}, \frac{\sin 5x}{\sqrt{\pi}} \right\} \subset L^2[-\pi, \pi]$$

is an orthogonal but not an orthonormal set in $L^2[-\pi, \pi]$. How would you change S to make it orthonormal? Use Gram's theorem to compute the best approximating function $H_5(x)$ from the subspace of S-linear combinations $span(S)$ of $L^2[-\pi, \pi]$ to the function

$$H(x) = \pi - \mid x \mid \quad \in L^2[-\pi, \pi].$$

Figure 3.14: Problem 8 Statement

Correction: The set S shown above should be

$$S = \left\{ \frac{1}{2\pi}, \frac{\cos x}{\pi}, \frac{\sin x}{\pi}, \frac{\cos 2x}{\pi}, \frac{\sin 2x}{\pi}, \cdots, \frac{\cos 5x}{\pi}, \frac{\sin x}{\pi} \right\}$$

<u>Solution</u>

Two functions f, g are orthogonal on $[-\pi, \pi]$ if $\int_{-\pi}^{\pi} fg dx = 0$. To show this for the set of functions given, we pick $f = \frac{1}{2\pi}$ and then for g we pick $\frac{\cos mx}{\pi}$ and then $\frac{\sin mx}{\pi}$. i.e.

$$I_1 = \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{\cos mx}{\pi} dx$$

$$I_2 = \int_{-\pi}^{\pi} \frac{1}{2\pi} \frac{\sin mx}{\pi} dx$$

For the rest, we have to determine the following 3 cases

$$I_3 = \int_{-\pi}^{\pi} \frac{\cos mx}{\pi} \frac{\cos nx}{\pi} dx$$

$$I_4 = \int_{-\pi}^{\pi} \frac{\cos mx}{\pi} \frac{\sin nx}{\pi} dx$$

$$I_5 = \int_{-\pi}^{\pi} \frac{\sin mx}{\pi} \frac{\sin nx}{\pi} dx$$

These will take care of all possible combination of any two function in the set S. We could always replace m, n by a number from $1 \cdots 5$ after evaluating the integrals in order to obtain a specific case. Starting with I_1

$$I_1 = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \cos mx dx$$

But cos function has period 2π and therefore the integral above is zero $\underline{I_1 = 0}$. This shows that $f = \frac{1}{2\pi}$ is orthogonal with all $\frac{\cos mx}{\pi}$ functions in the set.

$$I_2 = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \sin mx dx$$

As above, sin function has period of 2π and therefore the integral above is zero $\underline{I_2 = 0}$. This shows that $f = \frac{1}{2\pi}$ is orthogonal with all $\frac{\sin mx}{\pi}$ functions in the set.

$$I_3 = \frac{1}{\pi^2}\int_{-\pi}^{\pi} \cos mx \cos nx dx$$

From tables, using $\underline{\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))}$, then

$$\cos mx \cos nx = \frac{1}{2}(\cos((m-n)x) + \cos((m+n)x))$$

And I_3 now becomes

$$\begin{aligned} I_3 &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi} \cos((m-n)x) + \cos((m+n)x)\,dx \\ &= \frac{1}{2\pi^2}\left(\int_{-\pi}^{\pi} \cos(m-n)x dx + \int_{-\pi}^{\pi} \cos((m+n)x)\,dx\right) \end{aligned}$$

Since the problem is asking us to show orthogonality of different functions in the set, then we assume $m \neq n$, otherwise the integral will have to be handled as special case when $m = n$ due to the division.

$$I_3 = \frac{1}{2\pi^2}\left(\frac{1}{m-n}[\sin(m-n)x]_{-\pi}^{\pi} + \frac{1}{m+n}[\sin(m+n)x]_{-\pi}^{\pi}\right)$$

But since n, m are integers, then both terms above are zero since $\sin(N\pi) = 0$ for integer N. Hence $\underline{I_3 = 0}$. This shows that $\frac{\cos mx}{\pi}$ is orthogonal with $\frac{\cos nx}{\pi}$ when $m \neq n$.

$$I_4 = \frac{1}{\pi^2}\int_{-\pi}^{\pi} \sin nx \cos mx dx$$

Using $\underline{\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))}$, the above becomes

$$\begin{aligned} I_4 &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi} (\sin(n-m)x + \sin(n+m)x)\,dx \\ &= \frac{1}{2\pi^2}\left(\int_{-\pi}^{\pi} \sin(n-m)x dx + \int_{-\pi}^{\pi} \sin(n+m)x dx\right) \end{aligned}$$

Again, since $n \neq m$, then the above becomes

$$\begin{aligned} I_4 &= \frac{1}{2\pi^2}\left(\frac{-1}{n-m}[\cos(n-m)x]_{-\pi}^{\pi} + \frac{-1}{n+m}[\cos(m+n)x]_{-\pi}^{\pi}\right) \\ &= \frac{1}{2\pi^2}\left(\frac{-1}{n-m}[\cos((n-m)\pi) - \cos((n-m)(-\pi))] + \frac{-1}{n+m}[\cos((n+m)\pi) - \cos((n+m)(-\pi))]\right) \end{aligned}$$

But $\cos(-x) = \cos(x)$ and the above becomes

$$\begin{aligned} I_4 &= \frac{1}{2\pi^2}\left(\frac{-1}{n-m}[\cos((n-m)\pi) - \cos((n-m)\pi)] + \frac{-1}{n+m}[\cos((n+m)\pi) - \cos((n+m)\pi)]\right) \\ &= \frac{1}{2\pi^2}\left(\frac{-1}{n-m}[0] + \frac{-1}{n+m}[0]\right) \\ &= 0 \end{aligned}$$

Hence $\underline{I_4 = 0}$. This shows that $\frac{\sin mx}{\pi}$ is orthogonal with $\frac{\sin nx}{\pi}$ when $m \neq n$.

The final integral is

$$I_5 = \frac{1}{\pi^2}\int_{-\pi}^{\pi} \sin mx \sin nx dx$$

Using $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ the above becomes

$$\begin{aligned} I_5 &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi} \cos((m-n)x) - \cos((m+n)x)\,dx \\ &= \frac{1}{2\pi^2}\left(\int_{-\pi}^{\pi} \cos((m-n)x)\,dx - \int_{-\pi}^{\pi} \cos((m+n)x)\,dx\right) \end{aligned}$$

<u>Case $n = m$</u>

$$\begin{aligned} I_5 &= \frac{1}{2\pi^2}\left(\int_{-\pi}^{\pi} dx - \int_{-\pi}^{\pi} dx\right) \\ &= 0 \end{aligned}$$

This shows that $\frac{\sin mx}{\pi}$ is orthogonal with all $\frac{\cos nx}{\pi}$ when $m = n$.

Case $n \neq m$

$$I_5 = \frac{1}{2\pi^2}\left(\frac{1}{m-n}\left[\sin\left(\left(m-n\right)x\right)\right]_{-\pi}^{\pi} - \frac{1}{m+n}\left[\sin\left(\left(m+n\right)x\right)\right]_{-\pi}^{\pi}\right)$$

But since n, m are integers, then both terms above are zero since $\sin\left(N\pi\right) = 0$ for integer N. Hence $I_5 = 0$. This shows that $\frac{\sin mx}{\pi}$is orthogonal $\frac{\cos nx}{\pi}$.

The above shows that all the functions in S are pairwise orthogonal.

To make the set S orthonormal, we need to find weight k such that $\left\|kf\left(x\right)\right\| = 1$ or for functions, this is the same as

$$\sqrt{\int_{-\pi}^{\pi}\left(kf\left(x\right)\right)^2 dx} = 1$$

For $f = \frac{1}{2\pi}$, this becomes

$$\begin{aligned}
\sqrt{\int_{-\pi}^{\pi}\left(k\frac{1}{2\pi}\right)^2 dx} &= 1\\
\frac{k}{2\pi}\sqrt{\int_{-\pi}^{\pi} dx} &= 1\\
\frac{k}{2\pi}\sqrt{2\pi} &= 1\\
k &= \sqrt{2\pi}
\end{aligned}$$

For $f = \frac{\cos mx}{\pi}$

$$\begin{aligned}
\sqrt{\int_{-\pi}^{\pi}\left(k\frac{\cos mx}{\pi}\right)^2 dx} &= 1\\
\frac{k}{\pi}\sqrt{\int_{-\pi}^{\pi}\cos^2 mx dx} &= 1\\
\frac{k}{\pi}\sqrt{\int_{-\pi}^{\pi}\frac{1}{2} + \frac{1}{2}\cos 2mx dx} &= 1\\
\frac{k}{\pi}\sqrt{\left(\int_{-\pi}^{\pi}\frac{1}{2}dx + \frac{1}{2}\int_{-\pi}^{\pi}\cos 2mx dx\right)} &= 1\\
\frac{k}{\pi}\sqrt{\left(\pi + \frac{1}{2}\overbrace{\left[\frac{\sin\left(2mx\right)}{2m}\right]_{-\pi}^{\pi}}^{0}\right)} &= 1\\
\frac{k}{\pi}\sqrt{\pi} &= 1\\
k &= \sqrt{\pi}
\end{aligned}$$

For $f = \frac{\sin mx}{\pi}$

$$\begin{aligned}
\sqrt{\int_{-\pi}^{\pi}\left(k\frac{\sin mx}{\pi}\right)^2 dx} &= 1\\
\frac{k}{\pi}\sqrt{\int_{-\pi}^{\pi}\sin^2 mx dx} &= 1\\
\frac{k}{\pi}\sqrt{\int_{-\pi}^{\pi}\frac{1}{2} - \frac{1}{2}\cos 2mx dx} &= 1\\
\frac{k}{\pi}\sqrt{\left(\int_{-\pi}^{\pi}\frac{1}{2}dx - \frac{1}{2}\int_{-\pi}^{\pi}\cos 2mx dx\right)} &= 1\\
\frac{k}{\pi}\sqrt{\left(\pi - \frac{1}{2}\overbrace{\left[\frac{\sin\left(2mx\right)}{2m}\right]_{-\pi}^{\pi}}^{0}\right)} &= 1\\
\frac{k}{\pi}\sqrt{\pi} &= 1\\
k &= \sqrt{\pi}
\end{aligned}$$

Therefore the orthonormal set now becomes, after using the weights found above as

$$
\begin{aligned}
\tilde{S} &= \left\{\sqrt{2\pi}\frac{1}{2\pi}, \sqrt{\pi}\frac{\cos x}{\pi}, \sqrt{\pi}\frac{\sin x}{\pi}, \sqrt{\pi}\frac{\cos 2x}{\pi}, \sqrt{\pi}\frac{\sin 2x}{\pi}, \cdots, \sqrt{\pi}\frac{\cos 5x}{\pi}, \sqrt{\pi}\frac{\sin x}{\pi}\right\} \\
&= \left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \cdots, \frac{\cos 5x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}\right\}
\end{aligned}
$$

We now need to approximate $H(x) = \pi - |x|$ using $\tilde{S}$. The following is a plot of $H(x)$ over $[-\pi, \pi]$

```
f = Pi - Abs[x];
Plot[f, {x, -Pi, Pi}, Frame -> True, GridLines → Automatic,
 GridLinesStyle → LightGray, PlotStyle → Red,
 FrameLabel → {{"H(x)", None}, {"x", "Function to approximate"}},
 BaseStyle → 12,
 FrameTicks → {{Automatic, None}, {{-Pi, -Pi / 2, 0, Pi / 2, Pi}, None}}]
```

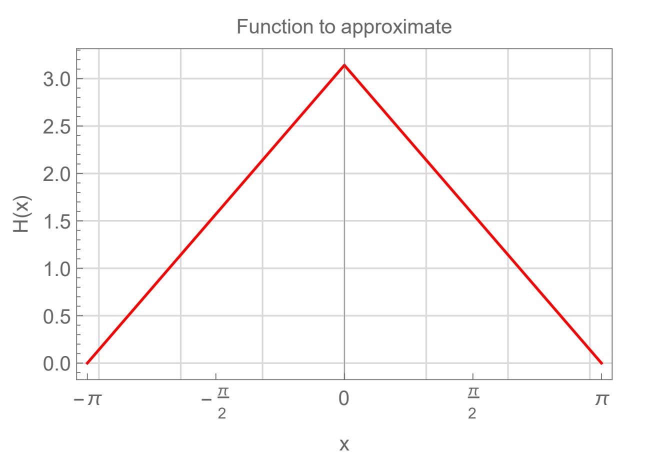


Figure 3.15: Function $H(x)$ to approximate. Problem 8

Counting the number of functions in $\tilde{S}$, there are 11 of them. Using Gram's theorem, this approximation is

$$
H(x) \approx c_1 S_1 + c_2 S_2 + c_3 S_3 + \cdots \tag{1}
$$

Where $S_1 = \frac{1}{\sqrt{2\pi}}, S_2 = \frac{\cos x}{\sqrt{\pi}}, \cdots, S_{10} = \frac{\cos 5x}{\sqrt{\pi}}, S_{11} = \frac{\sin x}{\sqrt{\pi}}$. Hence

$$
H(x) \approx c_1 \frac{1}{\sqrt{2\pi}} + c_2 \frac{\cos x}{\sqrt{\pi}} + c_3 \frac{\sin x}{\sqrt{\pi}} + c_4 \frac{\cos 2x}{\sqrt{\pi}} + c_5 \frac{\sin 2x}{\sqrt{\pi}} + \cdots + c_{10} \frac{\cos 5x}{\sqrt{\pi}} + c_{11} \frac{\sin 5x}{\sqrt{\pi}}
$$

where the constants c_i are found from solving

$$
\begin{pmatrix}
\langle S_1, S_1 \rangle & \langle S_1, S_2 \rangle & \langle S_1, S_3 \rangle & \cdots & \langle S_1, S_{11} \rangle \\
\langle S_2, S_1 \rangle & \langle S_2, S_2 \rangle & \langle S_2, S_3 \rangle & \cdots & \langle S_2, S_{11} \rangle \\
\langle S_3, S_1 \rangle & \langle S_3, S_2 \rangle & \langle S_3, S_3 \rangle & \cdots & \langle S_3, S_{11} \rangle \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\langle S_{11}, S_1 \rangle & \langle S_{11}, S_2 \rangle & \langle S_{11}, S_3 \rangle & \cdots & \langle S_{11}, S_{11} \rangle
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{11} \end{pmatrix}
=
\begin{pmatrix} \langle S_1, H(x) \rangle \\ \langle S_2, H(x) \rangle \\ \langle S_3, H(x) \rangle \\ \vdots \\ \langle S_{11}, H(x) \rangle \end{pmatrix}
$$

But since $\left\langle S_i, S_j \right\rangle = 0$ for $i \neq j$, because we showed above they are orthogonal to each others, and since S_i are all normalized now, then $\langle S_i, S_i \rangle = \|S_i\|^2 = 1$. Hence the above reduces to

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{11} \end{pmatrix}
=
\begin{pmatrix} \langle S_1, H(x) \rangle \\ \langle S_2, H(x) \rangle \\ \langle S_3, H(x) \rangle \\ \vdots \\ \langle S_{11}, H(x) \rangle \end{pmatrix} \tag{2}
$$

So we just need to evaluate $\langle S_i, H(x) \rangle$. But we need to do this only for three cases. These are

$\left\langle \frac{1}{\sqrt{2\pi}}, H(x) \right\rangle, \left\langle \frac{\cos mx}{\sqrt{\pi}}, H(x) \right\rangle, \left\langle \frac{\sin mx}{\sqrt{\pi}}, H(x) \right\rangle$ and then set $m = 1 \cdots 5$.

$$
\begin{aligned}
\left\langle \frac{1}{\sqrt{2\pi}}, H(x) \right\rangle &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} H(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} H(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\pi}^{0} (\pi + x)\, dx + \int_{0}^{\pi} (\pi - x)\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\left[\pi x + \frac{x^2}{2} \right]_{-\pi}^{0} + \left[\pi x - \frac{x^2}{2} \right]_{0}^{\pi} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\left[0 - \left(-\pi^2 + \frac{\pi^2}{2} \right) \right] + \left[\pi^2 - \frac{\pi^2}{2} \right] \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\left[\frac{\pi^2}{2} \right] + \left[\pi^2 - \frac{\pi^2}{2} \right] \right) \\
&= \frac{1}{\sqrt{2\pi}} \pi^2 \\
&= \frac{\pi^{\frac{3}{2}}}{\sqrt{2}}
\end{aligned}
$$

And

$$
\begin{aligned}
\left\langle \frac{\cos mx}{\sqrt{\pi}}, H(x) \right\rangle &= \int_{-\pi}^{\pi} \frac{\cos mx}{\sqrt{\pi}} H(x)\, dx \\
&= \frac{1}{\sqrt{\pi}} \left(\int_{-\pi}^{0} (\pi + x) \cos mx dx + \int_{0}^{\pi} (\pi - x) \cos mx dx \right) \\
&= \frac{1}{\sqrt{\pi}} \left(\int_{-\pi}^{0} \pi \cos mx + \int_{-\pi}^{0} x \cos mx dx + \int_{0}^{\pi} \pi \cos mx dx - \int_{-\pi}^{0} x \cos mx dx \right) \\
&= \frac{1}{\sqrt{\pi}} \left(\int_{-\pi}^{\pi} \pi \cos mx + \int_{-\pi}^{0} x \cos mx dx - \int_{-\pi}^{0} x \cos mx dx \right) \qquad (3)
\end{aligned}
$$

$\int x \cos mx dx$ can be evaluated by integration by parts. Let $u = x, dv = \cos mx \rightarrow du = 1, v = \frac{\sin mx}{m}$ hence

$$
\begin{aligned}
\int_{-\pi}^{0} x \cos mx dx &= \left[x \frac{\sin mx}{m} \right]_{-\pi}^{0} - \int_{-\pi}^{0} \frac{\sin mx}{m} dx \\
&= 0 - \frac{1}{m} \int_{-\pi}^{0} \sin mx dx \\
&= -\frac{1}{m} \left(-\frac{\cos mx}{m} \right)_{-\pi}^{0} \\
&= \frac{1}{m^2} (1 - \cos m\pi) \qquad (4)
\end{aligned}
$$

And

$$
\begin{aligned}
\int_{0}^{\pi} x \cos mx dx &= \left[x \frac{\sin mx}{m} \right]_{0}^{\pi} - \int_{0}^{\pi} \frac{\sin mx}{m} dx \\
&= 0 - \frac{1}{m} \int_{0}^{\pi} \sin mx dx \\
&= -\frac{1}{m} \left(-\frac{\cos mx}{m} \right)_{0}^{\pi} \\
&= \frac{1}{m^2} (\cos m\pi - 1) \qquad (5)
\end{aligned}
$$

And $\int_{-\pi}^{\pi} \pi \cos mx = \pi \int_{-\pi}^{\pi} \pi \cos mx = 0$. Using (4,5) in (3), then

$$
\begin{aligned}
\left\langle \frac{\cos mx}{\sqrt{\pi}}, H(x) \right\rangle &= \frac{1}{\sqrt{\pi}} \left(\frac{1}{m^2} (1 - \cos m\pi) - \frac{1}{m^2} (\cos m\pi - 1) \right) \\
&= \frac{1}{m^2 \sqrt{\pi}} (1 - \cos m\pi - \cos m\pi + 1) \\
&= \frac{2 (1 - \cos m\pi)}{m^2 \sqrt{\pi}}
\end{aligned}
$$

Hence

$$\begin{aligned}
\left\langle \frac{\cos mx}{\sqrt{\pi}}, H(x) \right\rangle_{m=1} &= \left\langle \frac{\cos x}{\sqrt{\pi}}, H(x) \right\rangle = \frac{2(1-\cos\pi)}{\sqrt{\pi}} = \frac{2(1+1)}{\sqrt{\pi}} = \frac{4}{\sqrt{\pi}} \\
\left\langle \frac{\cos mx}{\sqrt{\pi}}, H(x) \right\rangle_{m=2} &= \left\langle \frac{\cos 2x}{\sqrt{\pi}}, H(x) \right\rangle = \frac{2(1-\cos 2\pi)}{4\sqrt{\pi}} = 0 \\
\left\langle \frac{\cos mx}{\sqrt{\pi}}, H(x) \right\rangle_{m=3} &= \left\langle \frac{\cos 3x}{\sqrt{\pi}}, H(x) \right\rangle = \frac{2(1-\cos 3\pi)}{9\sqrt{\pi}} = \frac{2(1+1)}{9\sqrt{\pi}} = \frac{4}{9\sqrt{\pi}} \\
\left\langle \frac{\cos mx}{\sqrt{\pi}}, H(x) \right\rangle_{m=4} &= \left\langle \frac{\cos 4x}{\sqrt{\pi}}, H(x) \right\rangle = \frac{2(1-\cos 4\pi)}{16\sqrt{\pi}} = 0 \\
\left\langle \frac{\cos mx}{\sqrt{\pi}}, H(x) \right\rangle_{m=5} &= \left\langle \frac{\cos 5x}{\sqrt{\pi}}, H(x) \right\rangle = \frac{2(1-\cos 5\pi)}{25\sqrt{\pi}} = \frac{4}{25\sqrt{\pi}}
\end{aligned}$$

Similarly (we expect all the following integrals to be zero, this is because we see from above that $H(x)$ is an even function and $\sin$ is odd, hence the product is an odd function and the integral is over the period). This is the same as when in doing Fourier series expansion (which is what we are doing here essentially but using Gram's theorem instead), all the b_n terms will be zero when the function being approximated is even and all the a_n terms will be zero when the function being approximation is odd.

But we will go ahead and do the integrals to show that this is indeed the case.

$$\begin{aligned}
\left\langle \frac{\sin mx}{\sqrt{\pi}}, H(x) \right\rangle &= \int_{-\pi}^{\pi} \frac{\sin mx}{\sqrt{\pi}} H(x)\,dx \\
&= \frac{1}{\sqrt{\pi}} \left(\int_{-\pi}^{0} (\pi + x)\sin mx dx + \int_{0}^{\pi} (\pi - x)\sin mx dx \right) \\
&= \frac{1}{\sqrt{\pi}} \left(\int_{-\pi}^{0} \pi \sin mx + \int_{-\pi}^{0} x \sin mx dx + \int_{0}^{\pi} \pi \sin mx dx - \int_{-\pi}^{0} x \sin mx dx \right) \\
&= \frac{1}{\sqrt{\pi}} \left(\int_{-\pi}^{\pi} \pi \sin mx + \int_{-\pi}^{0} x \sin mx dx - \int_{-\pi}^{0} x \sin mx dx \right) \qquad (6)
\end{aligned}$$

$\int x \sin mx dx$ is evaluated by integration by parts. Let $u = x, dv = \sin mx \rightarrow du = 1, v = \frac{-\cos mx}{m}$ hence

$$\begin{aligned}
\int_{-\pi}^{0} x \sin mx dx &= -\frac{1}{m} [x \cos mx]_{-\pi}^{0} - \int_{-\pi}^{0} \frac{-\cos mx}{m} dx \\
&= -\frac{1}{m} [0 - (-\pi \cos m\pi)] + \frac{1}{m} \int_{-\pi}^{0} \cos mx dx \\
&= -\frac{\pi}{m} [\cos m\pi] + \frac{1}{m} \left(\frac{\sin mx}{m} \right)_{-\pi}^{0} \\
&= -\frac{\pi}{m} [\cos m\pi] \qquad (7)
\end{aligned}$$

And

$$\begin{aligned}
\int_{0}^{\pi} x \sin mx dx &= -\frac{1}{m} [x \cos mx]_{0}^{\pi} - \int_{0}^{\pi} \frac{-\cos mx}{m} dx \\
&= -\frac{1}{m} [\pi \cos m\pi] + \frac{1}{m} \int_{0}^{\pi} \cos mx dx \\
&= -\frac{\pi}{m} [\cos m\pi] \qquad (8)
\end{aligned}$$

And $\int_{-\pi}^{\pi} \pi \sin mx = 0$. Using (7,8) in (6), then

$$\begin{aligned}
\left\langle \frac{\sin mx}{\sqrt{\pi}}, H(x) \right\rangle &= \frac{1}{\sqrt{\pi}} \left(-\frac{\pi}{m} [\cos m\pi] + \frac{\pi}{m} [\cos m\pi] \right) \\
&= 0
\end{aligned}$$

Hence as expected all the inner products now are zero

$$\left\langle \frac{\sin mx}{\sqrt{\pi}}, H(x) \right\rangle_{m} = 0 \qquad m = 1,2,3,4,5$$

Using all the above results in (2) gives

$$\begin{pmatrix}1&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\\c_5\\c_6\\c_7\\c_8\\c_9\\c_{10}\\c_{11}\end{pmatrix}=\begin{pmatrix}\left\langle\frac{1}{\sqrt{2\pi}},H(x)\right\rangle\\\left\langle\frac{\cos x}{\sqrt{\pi}},H(x)\right\rangle\\\left\langle\frac{\sin x}{\sqrt{\pi}},H(x)\right\rangle\\\left\langle\frac{\cos 2x}{\sqrt{\pi}},H(x)\right\rangle\\\left\langle\frac{\sin 2x}{\sqrt{\pi}},H(x)\right\rangle\\\left\langle\frac{\cos 3x}{\sqrt{\pi}},H(x)\right\rangle\\\left\langle\frac{\sin 3x}{\sqrt{\pi}},H(x)\right\rangle\\\left\langle\frac{\cos 4x}{\sqrt{\pi}},H(x)\right\rangle\\\left\langle\frac{\sin 4x}{\sqrt{\pi}},H(x)\right\rangle\\\left\langle\frac{\cos 5x}{\sqrt{\pi}},H(x)\right\rangle\\\left\langle\frac{\sin 5x}{\sqrt{\pi}},H(x)\right\rangle\end{pmatrix}$$

Using the results found above, the above becomes

$$\begin{pmatrix}1&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\\c_5\\c_6\\c_7\\c_8\\c_9\\c_{10}\\c_{11}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\pi^{\frac{3}{2}}\\\frac{4}{\sqrt{\pi}}\\0\\0\\0\\\frac{4}{9\sqrt{\pi}}\\0\\0\\0\\\frac{4}{25\sqrt{\pi}}\\0\end{pmatrix}$$

Therefore we see that

$$\begin{pmatrix}c_1\\c_2\\c_3\\c_4\\c_5\\c_6\\c_7\\c_8\\c_9\\c_{10}\\c_{11}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\pi^{\frac{3}{2}}\\\frac{4}{\sqrt{\pi}}\\0\\0\\0\\\frac{4}{9\sqrt{\pi}}\\0\\0\\0\\\frac{4}{25\sqrt{\pi}}\\0\end{pmatrix}$$

The above shows that $c_1=\frac{1}{\sqrt{2}}\pi^{\frac{3}{2}}, c_2=\frac{4}{\sqrt{\pi}}, c_6=\frac{4}{9\sqrt{\pi}}, c_{10}=\frac{4}{25\sqrt{\pi}}$ and all other $c's$ are zero. Therefore the best approximation is

$$\begin{aligned}H(x)&\approx c_1\frac{1}{\sqrt{2\pi}}+c_2\frac{\cos x}{\sqrt{\pi}}+c_3\frac{\sin x}{\sqrt{\pi}}+c_4\frac{\cos 2x}{\sqrt{\pi}}+c_5\frac{\sin 2x}{\sqrt{\pi}}+\cdots+c_{10}\frac{\cos 5x}{\sqrt{\pi}}+c_{11}\frac{\sin 5x}{\sqrt{\pi}}\\&=\frac{1}{\sqrt{2}}\pi^{\frac{3}{2}}\frac{1}{\sqrt{2\pi}}+\frac{4}{\sqrt{\pi}}\frac{\cos x}{\sqrt{\pi}}+\frac{4}{9\sqrt{\pi}}\frac{\cos 3x}{\sqrt{\pi}}+\frac{4}{25\sqrt{\pi}}\frac{\cos 5x}{\sqrt{\pi}}\\&=\frac{1}{2}\pi+\frac{4}{\pi}\cos x+\frac{4}{9\pi}\cos 3x+\frac{4}{25\pi}\cos 5x\end{aligned}$$

Or

$$\boxed{H(x)\approx\frac{1}{2}\pi+\frac{4}{\pi}\cos x+\frac{4}{9\pi}\cos 3x+\frac{4}{25\pi}\cos 5x}$$

To verify the approximation, the above was plotted against the original $H(x)$, first using one term

$H_1(x) \approx \frac{1}{2}\pi$ then using 2 terms $H_2(x) \approx \frac{1}{2}\pi + \frac{4}{\pi}\cos x$ then using 3 terms $H_3(x) \approx \frac{1}{2}\pi + \frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x$ and then using all terms $H_4(x) \approx \frac{1}{2}\pi + \frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x + \frac{4}{25\pi}\cos 5x$. The plot below shows that the approximation improved as more terms added giving the best approximation when all terms are added as expected.

```
ClearAll[x, n];
f = Pi - Abs[x];
approx = {1/2 π, 4/π Cos[x], 4/(9 π) Cos[3 x], 4/(25 π) Cos[5 x]};
data = Table[
   Plot[{f, Total[approx[[1 ;; n]]]}, {x, -Pi, Pi}, Frame -> True,
    GridLines → Automatic, GridLinesStyle → LightGray,
    PlotStyle → {Red, Blue}, FrameLabel → {{"H(x)", None}, {"x", Total[approx[[1 ;; n]]]}},
    BaseStyle → 12,
    FrameTicks → {{Automatic, None}, {{-Pi, -Pi / 2, 0, Pi / 2, Pi}, None}},
    ImageSize → 300],
   {n, 1, 4}
  ];
Grid[Partition[data, 2]]
```

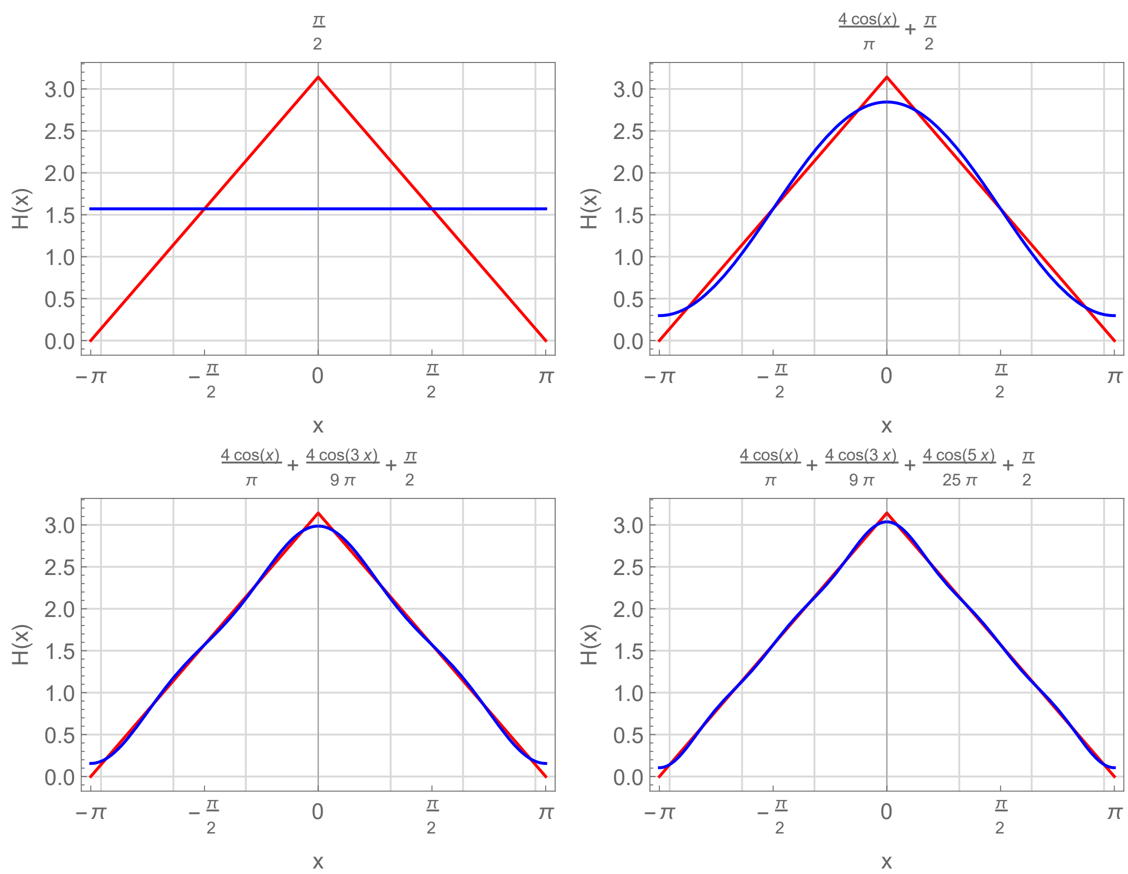


Figure 3.16: $H(x)$ approximation final resul. Problem 8t