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Contents

1 Problem 1

Consider the complex exponential function $f(z) = e^z = e^x(\cos y + i \sin y)$, where $x = \mathrm{Re}(z), y = \mathrm{Im}(z)$. Use the Cauchy-Riemann equations to show that $f(z)$ is analytic in the whole complex plane $\mathbb{C}$, and using the definition of the derivative, show that $f'(z) = f(z)$.

Solution

$$f(z) = e^x \cos y + ie^x \sin y$$

Comparing the above to $f(z) = u + iv$, shows that

$$\begin{aligned} u &= e^x \cos y \\ v &= e^x \sin y \end{aligned}$$

Cauchy-Riemann equations in Cartesian coordinates are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

Since $\frac{\partial u}{\partial x} = e^x \cos y$ and $\frac{\partial v}{\partial y} = e^x \cos y$, then (1) is satisfied. Looking at (2), since $\frac{\partial u}{\partial y} = -e^x \sin y$ and $\frac{\partial v}{\partial x} = e^x \sin y$, then (2) is also satisfied.

In addition, since all these partial derivatives are continuous everywhere because the elementary cos, sin, exp are all continuous everywhere, then $f(z) = e^z$ is entire, or in other words, analytic everywhere.

To show that $f'(z) = f(z)$, by the definition of derivative, which is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

And since $\Delta z = \Delta x + i\Delta y$ and $f(z) = u(x, y) + iv(x, y)$ then the above becomes

$$\begin{aligned} f'(z) &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)) - (u(x, y) + iv(x, y))}{\Delta x + i\Delta y} \\ &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x + i\Delta y} + i\frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i\Delta y} \end{aligned}$$

Since e^z is analytic, then the limit does not depend on the direction, so we can pick any direction to approach z. Let us choose a direction such that the approach is on the x axis only keeping y fixed in order to simplify the above. This implies that now $\Delta y = 0$. The above simplifies to

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i\frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

But $\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} = \frac{\partial u}{\partial x}$ and $\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} = \frac{\partial v}{\partial x}$, then the above reduces to

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

From the first part we obtained that $\frac{\partial u}{\partial x} = e^x \cos y$ and $\frac{\partial v}{\partial x} = e^x \sin y$. Using these in the above gives

$$\begin{aligned} f'(z) &= e^x \cos y + ie^x \sin y \\ &= e^x(\cos y + i \sin y) \\ &= e^x e^{iy} \\ &= e^{x+iy} \\ &= e^z \end{aligned}$$

Therefore $f'(z) = f(z)$. QED.

2 Problem 2

Determine the domain D of the z values on the complex plane where the complex function, given by the following series

$$F(z) = z^{\frac{1}{3}} + z^{\frac{1}{7}} + z^{\frac{1}{11}} + z^{\frac{1}{15}} + \cdots$$

is well defined. What is the set of values $z \in \mathbb{C}$, for which it holds that

$$F'(z) = \frac{1}{3} z^{\frac{-2}{3}} + \frac{1}{7} z^{\frac{-6}{7}} + \frac{1}{11} z^{\frac{-10}{11}} + \frac{1}{15} z^{\frac{-14}{15}} + \cdots$$

Solution

z can be either zero or not zero. When $z = 0$, then clearly $F(z)|_{z=0} = 0$ from the expression given for $F(z)$ above. So $F(z)$ is defined at z.

When $z \neq 0$, then each term in the series will now become multivalued since the terms are of the form $z^{\frac{1}{n}}$ for integer n. So we need to first make $F(z)$ single valued before considering the sum. We need to decide on which branch cut to use. Writing

$$\begin{aligned}
z^{\frac{1}{n}} &= \left(re^{i(\theta+2\pi k)}\right)^{\frac{1}{n}} \qquad k = 0, 1, 2, \cdots, n-1 \\
&= r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)} \\
&= r^{\frac{1}{n}} \left(\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right)
\end{aligned}$$

In order to make the multivalued $z^{\frac{1}{n}}$ function single valued, we select $k = 0$ and limit principal argument θ to

$$-\pi < \theta < \pi$$

with $z \neq 0$ for each term. Hence $z^{\frac{1}{n}}$ simplifies to

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(\cos\left(\frac{\theta}{n}\right) + i \sin\left(\frac{\theta}{n}\right)\right)$$

Where $r = |z|$ is the modulus of z. Now that each term is single valued, we can now look at the sum. Writing $F(z)$ as

$$\begin{aligned}
F(z) &= \sum_{n=0}^{\infty} z^{\frac{1}{4n+3}} \\
&= \sum_{n=0}^{\infty} r^{\frac{1}{4n+3}} e^{i\frac{\theta_0}{4n+3}}
\end{aligned}$$

We start with the preliminarily test to check if the above sum could be converging or not. Since the magnitude of the complex exponential is unity, we only need to check the modulus. Hence let

$$a_n = r^{\frac{1}{4n+3}}$$

Now we check if $\lim_{n\to\infty} a_n = 0$ or not. This is a a necessary condition for convergence but not a sufficient condition.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} r^{\frac{1}{4n+3}} \\
&= 1
\end{aligned}$$

We see that the limit is not zero. Therefore when $z \neq 0$, then $F(z)$ does not converge. Which means

$$\underline{F(z) \text{ is defined only at } z = 0}$$

To answer that last part. Since we showed that $F(z)$ only defined at one point $z = 0$, then its derivative is not defined. Because a derivative requires a small neighborhood region around any point where the derivative to be evaluated due to using the limit as $\Delta z \to 0$ in the definition of derivative. Since there is no such neighborhood around $z = 0$, then it follows immediately that

$$\underline{F'(z) \text{ is not defined anywhere}}$$

3 Problem 3

Consider the real function defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \left(\frac{x}{6}\right)^n$$

Use the results on complex power series to determine the largest open interval on which $f(x)$ is defined. For what values of $a < b$ does $f(x)$ converges uniformly on $[a, b]$?

Solution

Using the ratio test

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{\frac{(4(n+1))!}{((n+1)!)^4} \left(\frac{x}{6}\right)^{n+1}}{\frac{(4n)!}{(n!)^4} \left(\frac{x}{6}\right)^n} \right| \end{aligned}$$

Which simplifies to

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(4(n+1))!\,(n!)^4 \frac{x}{6}}{(4n)!\,((n+1)!)^4} \right| \\ &= \left|\frac{x}{6}\right| \lim_{n\to\infty} \left| \frac{(4(n+1))!\,(n!)^4}{(4n)!\,((n+1)!)^4} \right| \end{aligned}$$

But $((n+1)!)^4 = ((n+1)\,n!)^4 = (n+1)^4 (n!)^4$ and the above simplifies to

$$L = \left|\frac{x}{6}\right| \lim_{n\to\infty} \left| \frac{(4(n+1))!}{(4n)!\,(n+1)^4} \right|$$

But $(4(n+1))! = (4n+4)! = ((4n+4)(4n+3)(4n+2)(4n+1)(4n)!)$ and the above simplifies to

$$\begin{aligned} L &= \left|\frac{x}{6}\right| \lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)(4n)!}{(4n)!\,(n+1)^4} \right| \\ &= \left|\frac{x}{6}\right| \lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)^4} \right| \end{aligned}$$

Expanding gives

$$L = \left|\frac{x}{6}\right| \lim_{n\to\infty} \left| \frac{256n^4 + 640n^3 + 560n^2 + 200n + 24}{n^4 + 4n^3 + 6n^2 + 4n + 1} \right|$$

Dividing numerator and denominator by n^4 gives

$$L = \left|\frac{x}{6}\right| \lim_{n\to\infty} \left| \frac{256 + 640\frac{1}{n} + 560\frac{1}{n^2} + 200\frac{1}{n^3} + \frac{24}{n^4}}{1 + 4\frac{1}{n} + 6\frac{1}{n^2} + 4\frac{1}{n^3} + \frac{1}{n^4}} \right|$$

Now we can take the limit which gives 256. Hence

$$L = \frac{256}{6} |x|$$

For convergence, we want $|L| < 1$, which implies

$$\begin{aligned} \frac{256}{6} |x| &< 1 \\ |x| &< \frac{6}{256} \\ |x| &< \frac{3}{128} \end{aligned}$$

Therefore $f(x)$ is defined and absolutely converges for $\frac{-3}{128} < x < \frac{3}{128}$. Therefore by using theorem 1, page 699 in the textbook, we conclude that for uniform convergence we need

$$|x| \leq |r| < \frac{3}{128}$$

Or

$$\underline{x \geq a > -\tfrac{3}{128} \text{ and } x \leq b < \tfrac{3}{128}}$$

Hence the series converges uniformly on $[a, b]$ where

$$\begin{aligned} a &> -\frac{3}{128} \\ b &< \frac{3}{128} \end{aligned}$$

4 Problem 5

Let $f(z)$ be given as

$$f(z)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\frac{z}{4}\right)^{n+1}$$

(a) Find the domain D on which $f\left(\frac{1}{z}\right)$ is analytic. (b) For what z values does $g(z)$ defined by the Laurent series

$$g(z)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\frac{1}{4z}\right)^{n+1}+\sum_{n=1}^{\infty}\frac{n!}{n^n}z^n$$

Converge?

Solution

4.1 part (a)

First we find where $f(z)$ converges.

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\frac{z}{4}\right)^{n+1}\\
&= \sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}\left(\frac{z}{4}\right)^{n+2}\\
&= \left(\frac{z}{4}\right)^{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}\left(\frac{z}{4}\right)^{n}\\
&= \left(\frac{z}{4}\right)^{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)4^n}(z-z_0)^{n}
\end{aligned}$$

$f(z)$ converges in a disk centered at $z_0=0$ if the series $\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)6^n}z^n$ converges there. Using the ratio test to find L gives

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\\
&= \lim_{n\to\infty}\left|\frac{\frac{1}{(n+2)(n+3)4^{n+1}}}{\frac{1}{(n+1)(n+2)4^n}}\right|\\
&= \lim_{n\to\infty}\left|\frac{(n+1)(n+2)4^n}{(n+2)(n+3)4^{n+1}}\right|\\
&= \frac{1}{4}\lim_{n\to\infty}\left|\frac{n^2+3n+2}{n^2+5n+6}\right|\\
&= \frac{1}{4}\lim_{n\to\infty}\left|\frac{1+3\frac{1}{n}+2\frac{1}{n^2}}{1+5\frac{1}{n}+\frac{6}{n^2}}\right|\\
&= \frac{1}{4}
\end{aligned}$$

Since $L=\frac{1}{4}$ then the radius of convergence $R=\frac{1}{L}$ or $R=4$. This means $f(z)$ converges inside disk centered at zero of radius $R=4$. Therefore $f\left(\frac{1}{z}\right)$ converges everywhere outside this disk. Since there are no other singularities in the function given by $f\left(\frac{1}{z}\right)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\frac{1}{4z}\right)^{n+1}$ outside disk of radius 4 then it is analytic there everywhere (it is differentiable everywhere outside this disk). Therefore, we conclude this part by saying that

$$f\left(\frac{1}{z}\right)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\frac{1}{4z}\right)^{n+1}$$

is analytic outside disk of radius 4.

4.2 Part (b)

$$g(z)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\frac{1}{4z}\right)^{n+1}+\sum_{n=1}^{\infty}\frac{n!}{n^n}z^n$$

The first series in the right side above, we found from part (a) where it converges, which is for $|z| > 4$. Now we need to find where the second series converges.

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)!n^n}{n!\,(n+1)^{n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)\,n!n^n}{n!\,(n+1)^{n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)\,n^n}{(n+1)^{n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)\,n^n}{(n+1)\,(n+1)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n}\right| \\
&= \frac{1}{e}
\end{aligned}$$

Hence the radius of convergence is $R = \frac{1}{L} = e \approx 2.718$. This means the second series $\sum_{n=1}^{\infty}\frac{n!}{n^n}z^n$ convergence for $|z| < e$, or <u>inside</u> disk of radius $R = e$. But the first series converges <u>outside</u> disk of radius 4. Therefore, there is <u>no common annulus</u> where <u>both</u> series converge. Therefore

<u>There are no z values where $g(z)$ converges</u>

5 Problem 6

Determine the MacLaurin series for the following special functions for $z \in \mathbb{R}$. The resulting series defines the functions for complex numbers as well. Give the radius of convergence of the resulting series. Determine whether any of them is even $f(-z) = f(z)$ or odd $f(-z) = f(z)$.

(a) $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ (b) $\operatorname{Si}(z) = \int_0^z \frac{\sin t}{t} dt$

solution

5.1 Part (a)

Starting with MacLaurin series for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, hence e^{-t^2} series expansion around zero becomes

$$
\begin{aligned}
e^{-t^2} &= 1 + \left(-t^2\right) + \frac{\left(-t^2\right)^2}{2!} + \frac{\left(-t^2\right)^3}{3!} + \frac{\left(-t^2\right)^4}{43!} + \cdots \\
&= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \cdots
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\operatorname{erf}(z) &= \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^z \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \cdots\right) dt
\end{aligned}
$$

Since $\exp(x)$ is analytic everywhere, we can integrate the above term by term, which gives

$$
\begin{aligned}
\operatorname{erf}(z) &= \frac{2}{\sqrt{\pi}} \left(t - \frac{1}{3}t^3 + \frac{1}{5}\frac{t^5}{2!} - \frac{1}{7}\frac{t^7}{3!} + \frac{1}{9}\frac{t^9}{4!} - \cdots\right)_0^z \\
&= \frac{2}{\sqrt{\pi}} \left(z - \frac{1}{3}z^3 + \frac{1}{5}\frac{z^5}{2!} - \frac{1}{7}\frac{z^7}{3!} + \frac{1}{9}\frac{z^9}{4!} - \cdots\right)
\end{aligned}
$$

To find its radius of convergence, we need to first find closed form for the above. The general term is seen to be

$$
\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}
$$

Hence

$$
\begin{aligned}
\operatorname{erf}(z) &= \frac{2}{\sqrt{\pi}} z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!(2n+1)} \\
&= \frac{2}{\sqrt{\pi}} z \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \left(z^2\right)^n
\end{aligned}
$$

Now let $z^2 = s$. We now find the radius of convergence R for $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} s^n$ and then find $\sqrt{R}$ to find radius of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n}$.

Applying the ratio test to $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} s^n$ to find its L. Since we are using absolute values, the $(-1)^n$

does not affect the result, hence

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!(2n+2)}}{\frac{1}{n!(2n+1)}}\right| \\
&= \lim_{n\to\infty}\left|\frac{n!\,(2n+1)}{(n+1)!\,(2n+2)}\right| \\
&= \lim_{n\to\infty}\left|\frac{n!\,(2n+1)}{(n+1)\,n!\,(2n+2)}\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n+1)}{(n+1)\,(2n+2)}\right| \\
&= \lim_{n\to\infty}\left|\frac{2n+1}{2n^2+4n+2}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{2}{n}+\frac{1}{n^2}}{2+\frac{4}{n}+\frac{2}{n^2}}\right| \\
&= \frac{0}{2} \\
&= 0
\end{aligned}$$

Therefore $R = \frac{1}{L} = \infty$. But $\sqrt{\infty} = \infty$. Hence

$$\underline{\operatorname{erf}(z) \text{ is analytic on the whole complex plane}}$$

Now, to find if it is even or odd. Using the above series definition

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!\,(2n+1)}$$

Lets check if it odd. i.e. if $f(-z) = -f(z)$. From above

$$\operatorname{erf}(-z) = \frac{2}{\sqrt{\pi}}(-z) \sum_{n=0}^{\infty} \frac{(-1)^n (-z)^{2n}}{n!\,(2n+1)}$$

But $(-z)^{2n} = z^{2n}$ since the exponent is even, and the above simplifies to

$$\operatorname{erf}(-z) = \frac{-2}{\sqrt{\pi}} z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!\,(2n+1)} \tag{1}$$

Now lets find $-f(z)$. From the definition

$$-\operatorname{erf}(z) = \frac{-2}{\sqrt{\pi}} z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!\,(2n+1)} \tag{2}$$

Since (1) and (2) are the same, then

$$\underline{\operatorname{erf}(z) \text{ is odd}}$$

5.2 Part (b)

Starting with MacLaurin series for

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Hence $\frac{\sin(x)}{x}$ becomes

$$\begin{aligned}
\frac{\sin(x)}{x} &= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) \\
&= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots
\end{aligned}$$

Hence

$$\operatorname{Si}(x)=\int_0^z\left(1-\frac{t^2}{3!}+\frac{t^4}{5!}-\frac{t^6}{7!}+\frac{t^8}{9!}-\cdots\right)dt$$

Since $1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\cdots$ is analytic everywhere, we can integrate the above term by term, which gives

$$\begin{aligned}\operatorname{Si}(z)&=\left(t-\frac{1}{3}\frac{t^3}{3!}+\frac{1}{5}\frac{t^5}{5!}-\frac{1}{7}\frac{t^7}{7!}+\frac{1}{9}\frac{t^9}{9!}-\cdots\right)_0^z\\&=z-\frac{1}{3}\frac{z^3}{3!}+\frac{1}{5}\frac{z^5}{5!}-\frac{1}{7}\frac{z^7}{7!}+\frac{1}{9}\frac{z^9}{9!}-\cdots\end{aligned}$$

In closed form, this can be written as

$$\begin{aligned}\operatorname{Si}(z)&=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)(2n+1)!}z^{2n+1}\\&=z\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)(2n+1)!}z^{2n}\\&=z\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)(2n+1)!}\left(z^2\right)^n\end{aligned}$$

So we need to find radius of convergence R for $\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)(2n+1)!}s^n$ and then find $\sqrt{R}$ as we did in part (a).

$$\begin{aligned}L&=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\\&=\lim_{n\to\infty}\left|\frac{\frac{1}{(2n+2)(2n+2)!}}{\frac{1}{(2n+1)(2n+1)!}}\right|\\&=\lim_{n\to\infty}\left|\frac{(2n+1)(2n+1)!}{(2n+2)(2n+2)!}\right|\\&=\lim_{n\to\infty}\left|\frac{(2n+1)(2n+1)!}{(2n+2)(2n+2)(2n+1)!}\right|\\&=\lim_{n\to\infty}\left|\frac{(2n+1)}{(2n+2)(2n+2)}\right|\\&=\lim_{n\to\infty}\left|\frac{2n+1}{4n^2+8n+4}\right|\\&=\lim_{n\to\infty}\left|\frac{\frac{2}{n}+\frac{1}{n^2}}{4+\frac{8}{n}+\frac{4}{n^2}}\right|\\&=\frac{0}{4}\\&=0\end{aligned}$$

Hence $R=\frac{1}{L}=\infty$. But $\sqrt{\infty}=\infty$. Hence

$$\underline{\operatorname{Si}(z)\text{ is analytic on the whole complex plane}}$$

Now, to find if it is even or odd. Using the above series definition

$$\operatorname{Si}(z)=z\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)(2n+1)!}z^{2n}$$

Lets check if it odd. i.e. $f(-z)=-f(z)$. From above

$$\operatorname{Si}(-z)=-z\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n+1)(2n+1)!}(-z)^{2n}$$

But $(-z)^{2n}=z^{2n}$ since the exponent is even, so the above becomes

$$\operatorname{Si}(-z)=-z\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n+1)(2n+1)!}z^{2n}\tag{1}$$

Now lets find what $-f(z)$ gives

$$-\operatorname{Si}(z)=-z\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)(2n+1)!}z^{2n}\tag{2}$$

Comparing (1,2) we see they are the same. Hence

$$\underline{\operatorname{Si}(z)\text{ is odd}}$$

6 Problem 7

(a) Determine the Laurent series of the function

$$f(z)=\frac{2z+6}{z^2-6z+5}$$

In the annulus $1<z<5$ and in $|z|>5$.

(b) Determine the Taylor series representation of the function

$$g(z)=e^{-\frac{z^2}{2}}$$

with center $z_0=0$. What is the radius of convergence?

Solution

6.1 Part (a)

$$\begin{aligned} f(z) &= \frac{2z+6}{z^2-6z+5} \\ &= \frac{2z+6}{(z-5)(z-1)} \\ &= \frac{A}{(z-5)}+\frac{B}{(z-1)} \end{aligned}$$

Hence

$$\begin{aligned} 2z+6 &= A(z-1)+B(z-5) \\ &= z(A+B)-A-5B \end{aligned}$$

Solving the above two equations for A,B gives

$$\begin{aligned} 2 &= A+B \\ 6 &= -A-5B \end{aligned}$$

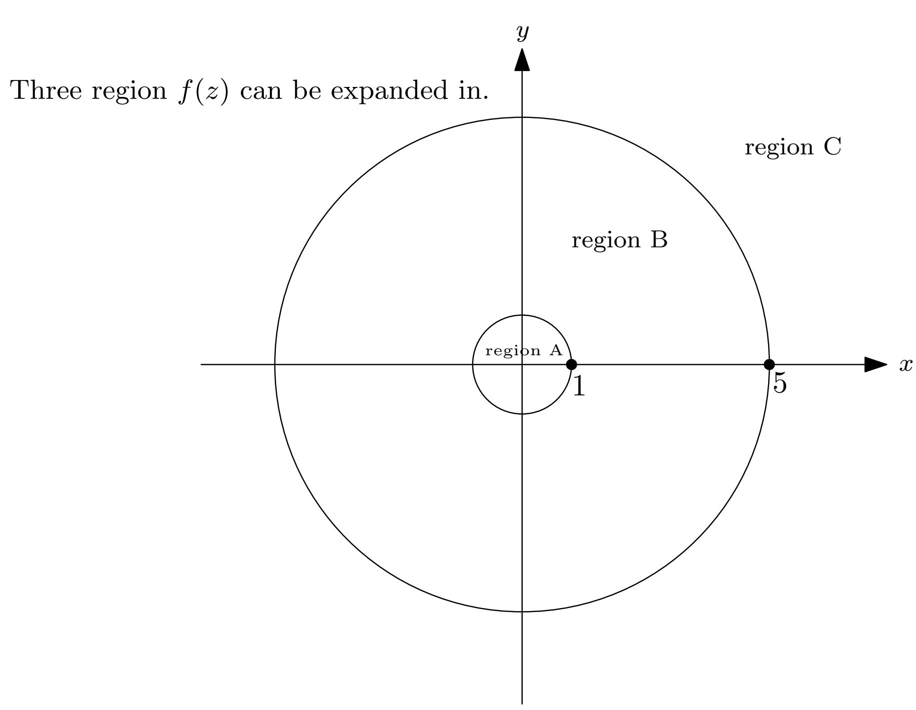
First equations gives $A=2-B$. Second equation becomes

$$\begin{aligned} 6 &= -(2-B)-5B \\ 6 &= -2-4B \\ B &= -\frac{8}{4} \\ &= -2 \end{aligned}$$

Hence $A=4$. Therefore (1) becomes

$$f(z)=\frac{4}{(z-5)}-\frac{2}{(z-1)} \tag{2}$$

We now see there is a pole at $z=5$ and at $z=1$. So there are three regions. The following diagram shows these three different regions



For region B, which is annulus $1 < z < 5$, we need to expand $f(z) = \frac{4}{(z-5)} - \frac{2}{(z-1)}$. Looking at first part

$$\begin{aligned}\frac{4}{(z-5)} &= \frac{-4}{5-z} \\ &= \frac{-4}{5\left(1-\frac{z}{5}\right)}\end{aligned}$$

This can be expanded for $\left|\frac{z}{5}\right| < 1$ or $|z| < 5$. Using Binomial series it gives

$$\begin{aligned}\frac{-4}{5\left(1-\frac{z}{5}\right)} &= \frac{-4}{5}\left(1+\left(\frac{z}{5}\right)+\left(\frac{z}{5}\right)^2+\left(\frac{z}{5}\right)^3+\cdots\right) \\ &= \frac{-4}{5}\sum_{n=0}^{\infty}\left(\frac{z}{5}\right)^n \qquad (3)\end{aligned}$$

We now consider the second term in (2), which is $\frac{2}{(z-1)}$

$$\frac{2}{(z-1)} = -\frac{2}{1-z}$$

This can be expanded only when $|z| < 1$. But we want $|z| > 1$, therefore we need to convert it to negative power. We write

$$\begin{aligned}\frac{2}{(z-1)} &= \frac{-2}{z\left(\frac{1}{z}-1\right)} \\ &= \frac{2}{z}\frac{1}{\left(1-\frac{1}{z}\right)}\end{aligned}$$

Now $\frac{1}{\left(1-\frac{1}{z}\right)}$ can be expanded for $\left|\frac{1}{z}\right| < 1$ or $z > 1$, which puts in region B. Hence the second term can now be expanded as

$$\begin{aligned}\frac{2}{(z-1)} &= \frac{2}{z}\frac{1}{\left(1-\frac{1}{z}\right)} \\ &= \frac{2}{z}\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right) \\ &= \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{2}{z^{n+1}} \qquad (4)\end{aligned}$$

Therefore (3,4) gives us the expansion of $f(z)$ valid in region B. Substituting results from (3,4) into (2) gives

$$\begin{aligned}f(z) &= \frac{4}{(z-5)} - \frac{2}{(z-1)} \\ &= \frac{-4}{5}\sum_{n=0}^{\infty}\left(\frac{z}{5}\right)^n - \sum_{n=0}^{\infty}\frac{2}{z^{n+1}} \\ &= \frac{-4}{5}\left(1+\frac{z}{5}+\frac{z^2}{5^2}+\cdots\right) - 2\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) \\ &= \left(-\frac{4}{5}-\frac{4z}{5^2}-\frac{4z^2}{5^3}+\cdots\right)+\left(-\frac{2}{z}-\frac{2}{z^2}-\frac{2}{z^3}+\cdots\right)\end{aligned}$$

The above shows that residue is -2, which is the coefficient for the $\frac{1}{z}$ term.

For region C

This is for $|z| > 5$ .For the first term in (2), which is $\frac{4}{(z-5)}$ we write it as

$$\frac{4}{(z-5)} = \frac{4}{z}\frac{1}{\left(1-\frac{5}{z}\right)}$$

We can expand this for $\left|\frac{5}{z}\right| < 1$ or $|z| > 5$ which is what we want. Hence it becomes

$$\begin{aligned}\frac{4}{z}\frac{1}{\left(1-\frac{5}{z}\right)} &= \frac{4}{z}\left(1+\left(\frac{5}{z}\right)+\left(\frac{5}{z}\right)^2+\left(\frac{5}{z}\right)^3+\cdots\right) \\ &= \frac{4}{z}\sum_{n=0}^{\infty}\left(\frac{5}{z}\right)^n \\ &= 4\sum_{n=0}^{\infty}\frac{5^n}{z^{n+1}}\end{aligned}$$

For the second in (2), which is $\frac{2}{(z-1)}$, we can use the expansion found earlier since it is valid for $|z| > 1$, hence also valid for $|z| > 5$ as well. which is $\frac{2}{(z-1)} = \sum_{n=0}^{\infty} \frac{2}{z^{n+1}}$.

Therefore, in region C, the expansion is

$$\begin{aligned}
f(z) &= \frac{4}{(z-5)} - \frac{2}{(z-1)} \\
&= 4\sum_{n=0}^{\infty} \frac{5^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2}{z^{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(4)(5^n) - 2}{z^{n+1}} \\
&= \frac{2}{z} + \frac{18}{z^2} + \frac{98}{z^3} + \frac{498}{z^4} + \cdots
\end{aligned}$$

The residue is 2.

6.2 Part (b)

$$g(z) = e^{-\frac{z^2}{2}}$$

Taylor series for $g(z)$ expanded around z_0 is given by

$$g(z) = g(z_0) + g'(z_0)\,z + g''(z_0)\frac{z^2}{2!} + g'''(z_0)\frac{z^3}{3!} + g^{(4)}(z_0)\frac{z^4}{4!} + \cdots$$

But

$$\begin{aligned}
g(z_0) &= g(0) = 1 \\
g'(z_0) &= -\frac{2z}{2}e^{-\frac{z^2}{2}}\Big|_{z=z_0=0} \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
g''(z_0) &= \frac{d}{dz}\left(-ze^{-\frac{z^2}{2}}\right)_{z=z_0} \\
&= \left(-e^{-\frac{z^2}{2}} - z^2e^{-\frac{z^2}{2}}\right)_{z=z_0=0} \\
&= -1
\end{aligned}$$

And

$$\begin{aligned}
g'''(z_0) &= \frac{d}{dz}\left(-e^{-\frac{z^2}{2}} - z^2e^{-\frac{z^2}{2}}\right)_{z=z_0} \\
&= \left(ze^{-\frac{z^2}{2}} - 2ze^{-\frac{z^2}{2}} - z^3e^{-\frac{z^2}{2}}\right)_{z=z_0=0} \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
g^{(4)}(z_0) &= \frac{d}{dz}\left(ze^{-\frac{z^2}{2}} - 2ze^{-\frac{z^2}{2}} - z^3e^{-\frac{z^2}{2}}\right)_{z=z_0} \\
&= \left(e^{-\frac{z^2}{2}} - z^3e^{-\frac{z^2}{2}} - 2e^{-\frac{z^2}{2}} + 2z^3e^{-\frac{z^2}{2}} - 3z^2e^{-\frac{z^2}{2}} + z^4e^{-\frac{z^2}{2}}\right)_{z=z_0=0} \\
&= 1
\end{aligned}$$

And so on. We can see the sequence pattern as

$$\begin{aligned}
g(z) &= g(z_0) + g'(z_0)\,z + g''(z_0)\frac{z^2}{2!} + g'''(z_0)\frac{z^3}{3!} + g^{(4)}(z_0)\frac{z^4}{4!} + \cdots \\
&= 1 + 0 - \frac{z^2}{2} + 0 + \frac{z^4}{4!} + 0 - \frac{z^6}{6!} + \cdots \\
&= 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}
\end{aligned}$$

To find radius of convergence, we write the above as

$$
\begin{aligned}
g(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \left(z^2\right)^n \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} s^n \qquad (1)
\end{aligned}
$$

And find R for s then take $\sqrt{R}$. Hence for (1)

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{1}{(2(n+1))!}}{\frac{1}{(2n)!}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2n)!}{(2(n+1))!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2n)!}{(2n+2)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+1)} \right| \\
&= \lim_{n\to\infty} \left| \frac{1}{4n^2+6n+2} \right| \\
&= 0
\end{aligned}
$$

Hence $R = \frac{1}{L} = \infty$. Therefore $\sqrt{R} = \infty$. The expansion is valid in the whole complex plane.

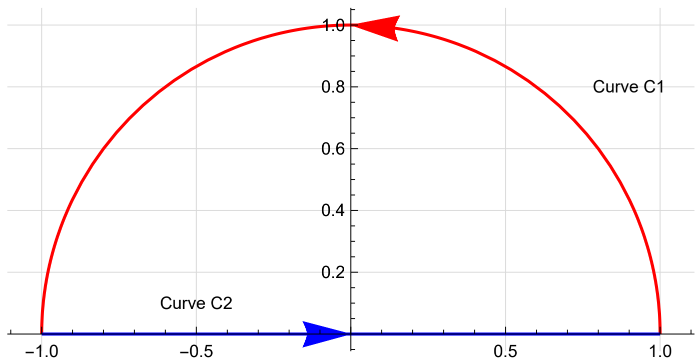
7 Problem 8

Evaluate the integral below on the curve $C = C_1 \cup C_2$ where $C_1 : z(t) = e^{i\pi t}, 0 \leq t \leq 1$ and $C_2 : z(t) = 2t - 1, 0 \leq t \leq 1$.

$$\oint_C \operatorname{Re}(z)\, dz$$

Solution

The diagram below shows the curves



$$\oint_C \operatorname{Re}(z)\, dz = \int_{C_1} \operatorname{Re}(z(t))\, z'(t)\, dt + \int_{C_2} \operatorname{Re}(z(t))\, z'(t)\, dt \tag{1}$$

But on C_1, $z(t) = e^{i\pi t} = \cos(\pi t) + i \sin(\pi t)$, then $\operatorname{Re}(z(t)) = \cos(\pi t)$ and $z'(t) = i\pi e^{i\pi t}$, therefore the integral on C_1 becomes

$$\begin{aligned}\int_0^1 \cos(\pi t)\, i\pi e^{i\pi t} dt &= i\pi \int_0^1 \cos(\pi t)\, e^{i\pi t} dt \\ &= i\pi I \end{aligned} \tag{2}$$

Where $I = \int_0^1 \cos(\pi t)\, e^{i\pi t} dt$. We now evaluate I. Since $e^{i\pi t} = \cos(\pi t) + i \sin(\pi t)$, then

$$I = \int_0^1 \cos(\pi t) \cos(\pi t)\, dt + i \int_0^1 \cos(\pi t) \sin(\pi t)\, dt \tag{3}$$

But first integral in above is

$$\begin{aligned}\int_0^1 \cos(\pi t) \cos(\pi t)\, dt &= \int_0^1 \cos^2(\pi t)\, dt \\ &= \int_0^1 \frac{1}{2} + \frac{1}{2} \cos(2\pi t)\, dt \\ &= \frac{1}{2} + \frac{1}{2} (\sin(2\pi t))_0^1 \\ &= \frac{1}{2}\end{aligned}$$

And for second integral in (3), and using $\sin A \cos A = \frac{1}{2} \sin(2A)$, it becomes

$$\begin{aligned}\int_0^1 \cos(\pi t) \sin(\pi t)\, dt &= \int_0^1 \frac{1}{2} \sin(2\pi t)\, dt \\ &= \frac{1}{2} \left[\frac{\cos(2\pi t)}{2\pi} \right]_0^1 \\ &= \frac{1}{4\pi} [\cos(2\pi t)]_0^1 \\ &= \frac{1}{4\pi} [\cos(2\pi) - 1] \\ &= 0\end{aligned}$$

Therefore integral on C_1 from (2) becomes

$$\begin{aligned}\int_0^1 \cos(\pi t)\, i\pi e^{i\pi t} dt &= i\pi I \\ &= \frac{i\pi}{2}\end{aligned}$$

Now the second integral on C_2 is found, where $z(t) = 2t - 1$, and $z'(t) = 2$. Hence

$$\begin{aligned}\int_0^1 \operatorname{Re}(z(t))\, z'(t)\, dt &= \int_0^1 (2t-1)\, 2\, dt \\ &= 2\left(t^2 - t\right)_0^1 \\ &= 0\end{aligned}$$

Therefore contribution comes only from the integration over C_1 which is

$$\oint_C \operatorname{Re}(z)\, dz = \frac{i\pi}{2}$$

8 Optional choice Problem 4

4. We refer to an open connected set $D \subset \mathbb{C}$ as a domain of the complex plane, and if $F(z)$ is an analytic function on D we call the set

$$F(D) = \{F(z) : z \in D\}$$

the analytic transformation of D by F. (E.g.: if the open unit disk centered at 1 is given as

$$U^1 = \{z \in \mathbb{C} : |z - 1| < 1\},$$

and $F(z) = z + i$, $H(z) = iz$, then the analytic transformations $F(U^1)$ and $H(U^1)$ are a shift by i of U^1, and a counterclockwise rotation by 90 degrees – or $\pi/2$ radian – around the origin of U^1, respectively.)

a) Consider

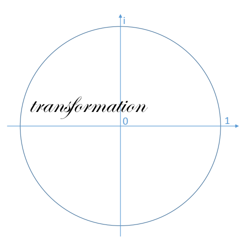
$$U^0 = \{z \in \mathbb{C} : |z| < 1\},$$

and $f_1(z) = \frac{z+1}{1-z}$. Show that $f_1(z)$ is a 1-1 (and analytic) function on U^0, and argue that f_1 , also referred to as a Möbius transformation, transforms U^0 to the right half plane, i.e. $f_1(U^0) = \{z \in \mathbb{C} : Re\, z > 0\}$.

b) Write the function

$$f(z) = e^{-i \ln\left\{\left[i\left(\frac{z+1}{1-z}\right)\right]^{\frac{1}{2}}\right\}}$$

as a composite of 6 analytic functions $f(z) = f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1(z)$ and show that f transforms U^0 into an annulus $f(U^0)$. [Hint: Determine in order $f_1(U^0)$, $f_2 \circ f_1(U^0)$,..., $f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1(U^0)$.]



c) *(Extra credit/fun:)* If you have access to a software performing complex arithmetic (e.g. Matlab), compute the transformation of the *"transformation"* text shaped domain T given inside the unit disk (as black text) on the adjacent image.

Solution

8.1 Part (a)

$$f_1(z) = \frac{z+1}{1-z} \tag{1}$$

The domain U^0 is the unit disk centered at origin. To show $f_1(z)$ is 1-1 on U^0 means to show that if

$$f_1(z_1) = f_1(z_2) \tag{2}$$

The this implies that

$$z_1 = z_2$$

Applying (1,2) to both points gives

$$
\begin{aligned}
f_1(z_1) &= f_1(z_2) \\
\frac{z_1+1}{1-z_1} &= \frac{z_2+1}{1-z_2} \\
\frac{(x_1+iy_1)+1}{1-(x_1+iy_1)} &= \frac{(x_2+iy_2)+1}{1-(x_2+iy_2)} \\
((x_1+iy_1)+1)(1-(x_2+iy_2)) &= (1-(x_1+iy_1))((x_2+iy_2)+1) \\
(x_1+iy_1+1)(1-x_2-iy_2) &= (1-x_1-iy_1)(x_2+iy_2+1) \\
x_1-x_1x_2-ix_1y_2+iy_1-iy_1x_2+y_1y_2+1-x_2-iy_2 &= x_2+iy_2+1-x_1x_2-iy_2x_1-x_1-iy_1x_2+y_1y_2-iy_1
\end{aligned}
$$

Collecting real and imaginary parts

$$
(x_1-x_1x_2+y_1y_2+1-x_2)+i(-x_1y_2+y_1-y_1x_2-y_2) = (x_2+1-x_1x_2-x_1+y_1y_2)+i(y_2-y_2x_1-y_1x_2-y_1) \tag{3}
$$

If two complex numbers are equal, then the real part and the imaginary part must be equal. Hence in equation (3), equating real parts gives

$$
\begin{aligned}
x_1-x_1x_2+y_1y_2+1-x_2 &= x_2+1-x_1x_2-x_1+y_1y_2 \\
x_1-x_2 &= x_2-x_1 \\
2x_1 &= 2x_2 \\
x_1 &= x_2 \qquad (4)
\end{aligned}
$$

And equating imaginary parts in (3) gives

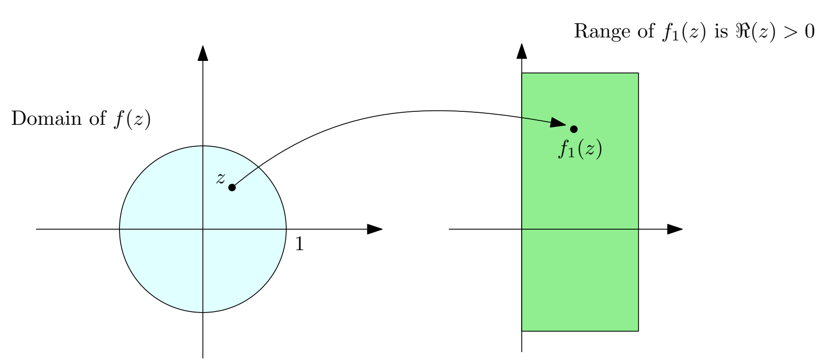
$$
\begin{aligned}
-x_1y_2+y_1-y_1x_2-y_2 &= y_2-y_2x_1-y_1x_2-y_1 \\
y_1-y_2 &= y_2-y_1 \\
2y_1 &= 2y_2 \\
y_1 &= y_2 \qquad (5)
\end{aligned}
$$

From (4,5) we see that $z_1 = x_1 + iy_1$ is the same point as $z_2 = x_2 + iy_2$. This shows that

$$\underline{f_1(z) \text{ is 1-1 on } U^0}$$

To show that $f_1(z)$ is analytic, we see that there is a pole at $z = 1$. But this is <u>outside</u> the disk $|z| < 1$. So there is no singularity inside the disk. And since $\frac{z+1}{z-1}$ is differentiable as many times as we wish, then it is analytic. We can also apply Cauchy Riemann equations also to verify this, but it is not needed for this simple function.

The last part is to show that f_1 is a Mobius transformation.



To show this, we apply $f_1(z)$ to an arbitrary point in the domain $|z| < 1$ and see if the real part of $f_1(z)$ comes out to be always positive or not. Let z_0 be any point inside the disk $|z| < 1$ where $z = x + iy$. Hence

$$
\begin{aligned}
f_1(z) &= \frac{z+1}{1-z} \\
&= \frac{(x+iy)+1}{1-(x+iy)} \\
&= \frac{(x+1)+iy}{(1-x)-iy}
\end{aligned}
$$

Multiplying the numerator and denominator by the complex conjugate of the denominator gives

$$
\begin{aligned}
f_1(z_0) &= \frac{((x+1)+iy)((1-x)+iy)}{((1-x)-iy)((1-x)+iy)} \\
&= \frac{(x+1)(1-x)+iy(x+1)+iy(1-x)-y^2}{(1-x)^2+y^2} \\
&= \frac{(x+1)(1-x)+iyx+iy+iy-iyx-y^2}{(1-x)^2+y^2} \\
&= \frac{\left(1-x^2\right)+2iy-y^2}{(1-x)^2+y^2} \\
&= \frac{\left(1-x^2\right)-y^2}{(1-x)^2+y^2} + i\frac{-2yx}{(1-x)^2+y^2} \\
&= u+iv
\end{aligned}
$$

Hence

$$
\begin{aligned}
u(x,y) &= \frac{\left(1-x^2\right)-y^2}{(1-x)^2+y^2} \\
&= \frac{1-\left(x^2+y^2\right)}{(1-x)^2+y^2} \\
&= \frac{1-|z|^2}{(1-x)^2+y^2}
\end{aligned}
$$

We now need to show that $u(x,y)$ is always positive. Now, Since $|z|<1$, then $|x|<1$ and also $|y|<1$. This shows that the denominator is always positive and can not be zero even when $x=0, y=0$ in which case the denominator is 1. The only problem comes when $x=1$ and $y=0$ in which case the mapping goes to infinity. More on this below. But this point is on the boundary itself, and not inside the disk.

Now, for the numerator, since $|z|<1$ then $1-|z|^2$ is always positive. Only when $|z|=1$ (boundary points), then will $u(x,y)=0$. Therefore we conclude that

$$\underline{\text{Each point inside the disk maps to right side of the complex plane}}$$

For example, the center of the disk, $x=0, y=0$, maps to the right side complex plane, since

$$
\begin{aligned}
f_1(x,y) &= \frac{\left(1-x^2\right)-y^2}{(1-x)^2+y^2} + i\frac{-2yx}{(1-x)^2+y^2} \\
f_1(0,0) &= 1+0i
\end{aligned}
$$

What about points on the boundary of the disk where $|z|=1$? Lets pick the point $x=1, y=0$, then we see that

$$\lim_{y\to 0}\frac{(x+iy)+1}{1-(x+iy)} = \frac{1+x}{1-x}$$

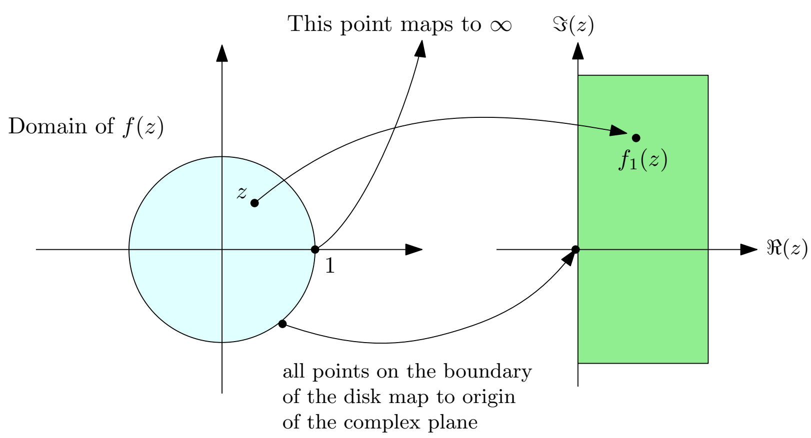
Hence as $x\to 1$ it will blow up and it goes to infinity. How about the point $x=0, y=1$, then this point maps to

$$
\begin{aligned}
f_1(0,1) &= \frac{1-1}{1+1} + i\frac{0}{1+1} \\
&= 0+i0
\end{aligned}
$$

So it maps to origin in the complex plane. The point $x=-1, y=0$ maps to

$$f_1(-1,0) = 0+i0$$

All other points on the boundary of disk $|z|=1$ map to the origin of the complex plane, <u>except for the point</u> $x=1, y=0$ which maps to infinity.



Transformation given by $\frac{z+1}{1-z}$

8.2 Part (b)

$$f(z) = e^{-i \ln \sqrt{i \frac{z+1}{1-z}}}$$

Since $f_1(z) = \frac{z+1}{1-z}$, where from part(a), we know it maps all points inside disk $|z| < 1$ to the right side of the complex plane. Then the above can be written as

$$f(z) = e^{-i \ln \sqrt{i f_1(z)}}$$

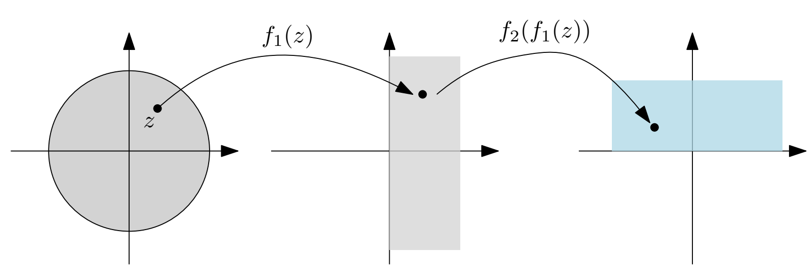
Now let

$$f_2(z) = i f_1(z)$$

The effect of this is to rotate each point in the right half plane clockwise by 90^0. This can be seen by considering an arbitrary point $z_0 = re^{i\theta}$, then

$$\begin{aligned} iz_0 &= e^{i\frac{\pi}{2}} \left(re^{i\theta} \right) \\ &= re^{i\left(\theta + \frac{\pi}{2}\right)} \end{aligned}$$

Hence the result of applying $f_2(z) = i f_1(z)$ is to rotate the right side plane to the upper half plane as shown below



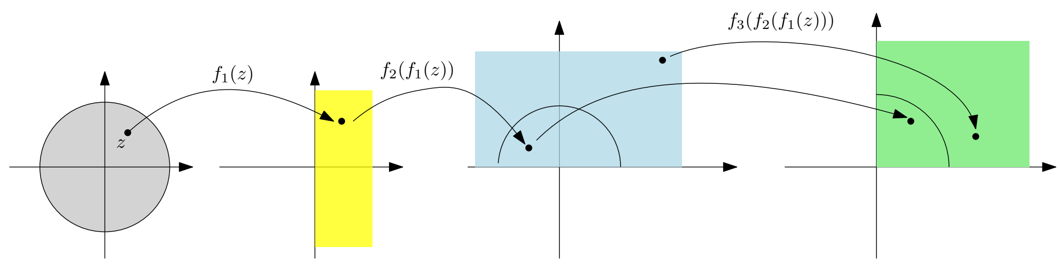
The next step is to apply the square root function. This means

$$f_3(z) = \sqrt{f_2(f_1(z))}$$

What does applying a square root to a point z_0 in the complex plane do? Since $z^{\frac{1}{2}} = e^{\frac{1}{2}(\ln|z| + i \operatorname{Arg}(z))}$ where here we used the principal argument of z, therefore

$$\begin{aligned} z^{\frac{1}{2}} &= e^{\frac{1}{2}(\ln r + i\theta)} \\ &= r^{\frac{1}{2}} e^{i\frac{\theta}{2}} \end{aligned}$$

Hence the effect is to take the square root of the module and to reduce the argument by half. Points inside a unit circle will increase their module and move closer to the inner edge of the unit circle, and points outside the unit circle will decrease their modulus and move closer to the outside edge of the unit circle. Points on the unit circle will not change their modulus. But all points will have their argument halved. The result of this is all points will move and end up in the first quadrant of the complex plane



The next step is to apply the ln function on the resulting points. Hence

$$f_4(z) = \ln(f_3(z))$$

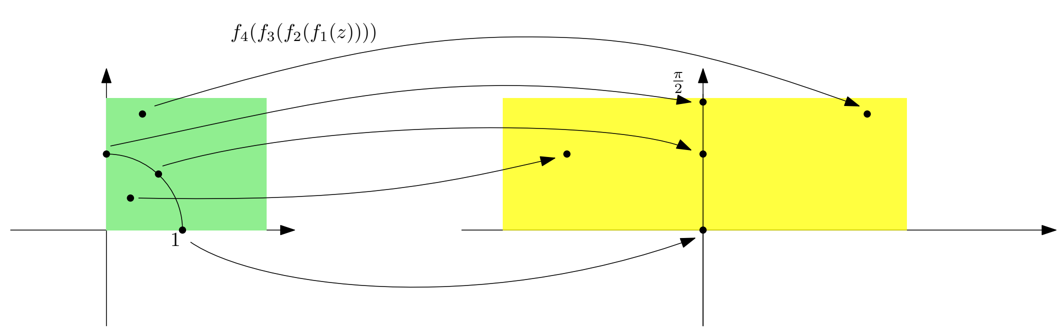
Now we ask, what does $\ln(z)$ do to a point z? Let $z = re^{i\theta}$ then

$$\begin{aligned}\ln(z) &= \ln\left(re^{i\theta}\right)\\ &= \ln r + \ln e^{i\theta}\\ &= \ln r + i\theta\end{aligned}$$

This gives a complex variable whose real part is $\ln|z|$ and whose imaginary part is the argument of z. Since $\ln|z|$ is negative for $|z| < 1$, then all points inside the unit circle will have their real part move to the negative half plane, and all points outside the unit circle will have their real part in the right half plane. And all points on the unit circle will have their real part be zero. So all point on the unit circle will move to the imaginary axis. For example, the point $(1, 0)$ will move to $(0, 0)$ and the point $(0, 1)$ will move to $\left(0, \frac{\pi}{2}\right)$.

As a point is closer to the origin, it will map closer to $-\infty$ in negative half plane, since $\lim_{r\to 0}\ln(r)$ is $-\infty$.

The imaginary part of each point be the argument of the point z. Since all points now reside in the first quadrant as seen in the above diagram, then the imaginary part will extend from $0\cdots\frac{\pi}{2}$. The following diagram just shows the transformation by $f_4(z)$ for selected points



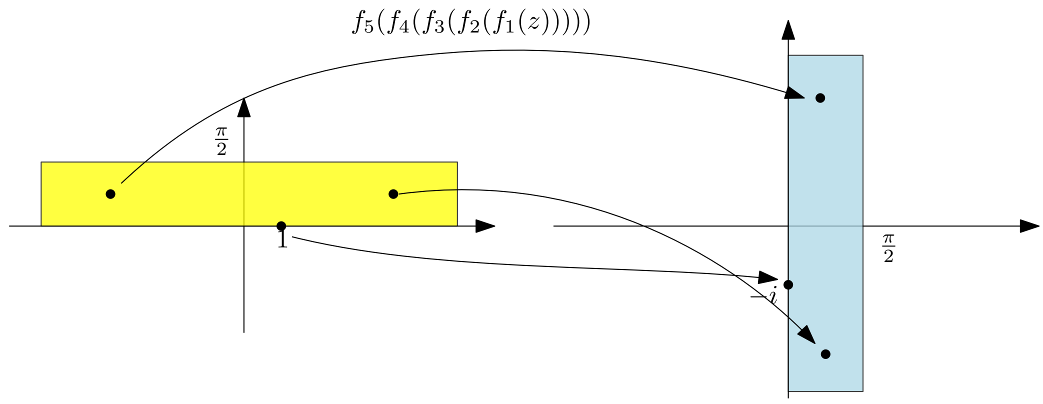
The next step is to apply $-i$ to each point. Hence

$$\begin{aligned}f_5(z) &= -if_4(z)\\ &= e^{-i\frac{\pi}{2}}f_4(z)\end{aligned}$$

Let $z = re^{i\theta} = f_4(z)$ and the above becomes

$$\begin{aligned}f_5(z) &= e^{-i\frac{\pi}{2}}re^{i\theta}\\ &= re^{i\left(\theta-\frac{\pi}{2}\right)}\end{aligned}$$

So the effect of multiplying by $-i$ is rotate each point clockwise by 90^0. Hence the whole strip shown above $(f_4(z))$, will now rotate by 90^0 clockwise. The arguments of each new point location will now be in the range $\frac{\pi}{2}\cdots-\frac{\pi}{2}$ as shown below



The final step is to apply $\exp(z)$ to each point in generated by applying $f_5(z)$. Let a point be $z = x + iy$, then

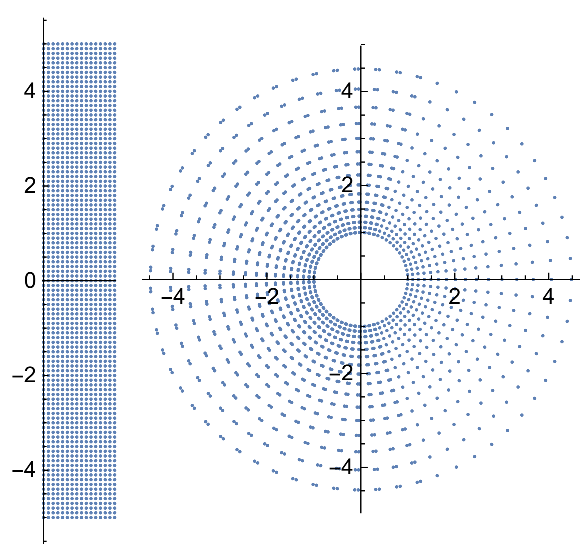
$$
\begin{aligned}
f_6(z) &= e^{x+iy} \\
&= e^x e^{iy} \\
&= e^x (\cos y + i \sin y) \\
&= e^x \cos y + i e^x \sin y
\end{aligned}
$$

Hence the real part of each new point become $e^x \cos y$ and imaginary part become $e^x \sin y$.

All points on imaginary line, with $x = 0$ will map to $\cos y + i \sin y$. All point on the x axis, where $y = 0$ will map to $e^x + 0i$.

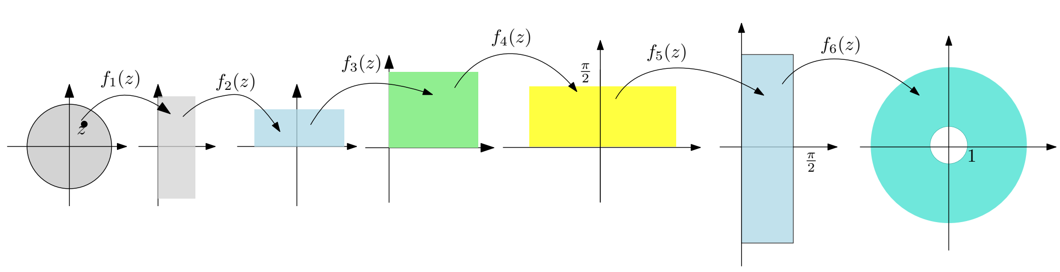
All points on the vertical line $\left(\frac{\pi}{2}, y\right)$ will map to $e^{\frac{\pi}{2}} \cos y + i e^{\frac{\pi}{2}} \sin y$. To better see the mapping, I wrote a small program to plot the above transformation. The function samples points from $x = 0$ to $x = \frac{\pi}{2}$ and samples points from $y = -5$ to $y = 5$. For each such point (x, y) it transforms it to $(e^x \cos y, e^x \sin y)$.

The result shows that all points map to concentric rings outside the unit circle as shown in the plot below



Result of applying $f_6(z)$. annulus outside unit circle.

Hence the final mapping is to an annulus outside disk on radius 1. The following shows all the transformation applied on the same diagram.



8.3 Part (c)

Using Matlab, the code provided was run after applying the function $f_6(f_5(f_4(f_3(f_2(f_1(z))))))$ on it.

```
close all
load 'TransPoints.mat';
TR=exp(-1i.*log( sqrt( 1i*((COMPLD+1)./(1-COMPLD) ))));
IMtr=imag(TR);
REtr=real(TR);
plot(REtr,IMtr,'k.')
axis equal
title('Math 601, problem 4 result. Nasser M. Abbasi')
grid
```

The following shows the original image, and the transformed image below it.