HW 6 (3rd HW on ODE's), Math 601
Fall 2018
University Of Wisconsin, Milwaukee

Nasser M. Abbasi

December 30, 2018 Compiled on December 30, 2018 at 3:33pm

Contents

1 problems description

Homework #3 | ODEs MATH 601 oct 23, 2018.

[1] Show that the initial value problem $\begin{cases} xy' = 4y \\ y(0) = 1 \end{cases}$ has no solution. Does this contradict the existence theorem we considered?

[2] State the definition of a Lipschitz condition. Show that $\begin{cases} y' = |\sin y| + x \\ y(x_0) = y_0 \end{cases}$ satisfies the Lipschitz condition with Lipschitz constant 1 on the whole x-y plane.

[3] Apply the Picard iteration to $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$ and show that the aproximation approaches the exact solution $y = e^x$.

[4] Solve the following initial value problems

a) $\begin{cases} y'' + 2y' + y = 0 \\ y(0) = 1, \ y'(0) = 0 \end{cases}$ b) $\begin{cases} y'' + 9y = 0 \\ y(0) = 4, \ y'(0) = -6 \end{cases}$

[5] Solve the following problem (catenary)

$\begin{cases} y'' = \sqrt{1 + (y')^2} \\ y(-1) = y(1) = 0 \end{cases}$ and plot the solution.

2 Key solution

Homework #3 solutions Math 601 (1)

$\boxed{1}$ $\begin{cases} xy' = 4y \\ y(0)=1 \end{cases}$ $\quad xy' = 4y$ separable eq.

$\frac{1}{y} y' = \frac{4}{x}$ integrate

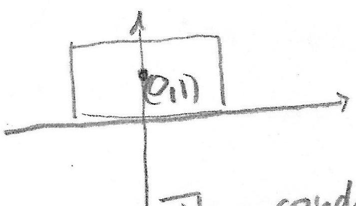
$\ln|y| = 4\ln|x| + c$

$|y| = |x|^4 e^c$

$y = Cx^4$ general solution

$1 = y(0) = C \cdot 0 = 0$
no such solution.

$y' = f(x,y) = \frac{4y}{x}$ not continuous at $x=0$ i.e not continuous in any R rectangle around the $(x,y)=(0,1)$ point in the x-y plane



Thus conditions of the existence theorem are not satisfied.

$\boxed{2}$ $y' = f(x,y) = |\sin y| + x$

Lipschitz continuity for $f(x,y)$: $|f(x,y_2) - f(x,y_1)| \le L|y_2 - y_1|$
for L constant

$|f(x,y_2) - f(x,y_1)| = \big| |\sin y_2| + \cancel{x} - |\sin y_1| - \cancel{x} \big| \le |\sin y_2 - \sin y_1|$
Mean value T.
$\le |\cos z| \cdot |y_2 - y_1|$
$\le 1 \cdot |y_2 - y_1|$
(with z in between y_2 and y_1) $\stackrel{L=1}{=}$

(Note $|a| = |a - b + b| \le |a-b| + |b| \Rightarrow |a| - |b| \le |a-b|$
& $|b| = |b - a + a| \le |b-a| + |a| \Rightarrow |b| - |a| \le |b-a| = |a-b|$)

so $\big||a| - |b|\big| \le |a-b|$

(2)

$\boxed{3}$ Picard iteration:

$$\begin{cases} y' = y = f(x,y) \\ y(0) = 1 \end{cases}$$

$$\Updownarrow$$

$$y(x) = y(0) + \int_0^x f(s,y)\,ds = 1 + \int_0^x y\,ds$$

$$\phi_0(x) = 1$$

$$\phi_1(x) = 1 + \int_0^x 1\,ds = 1 + x$$

$$\phi_2(x) = 1 + \int_0^x 1+s\,ds = 1 + x + \frac{x^2}{2}$$

$$\phi_3(x) = 1 + \int_0^x 1+s+\frac{s^2}{2}\,ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3}$$

$$\phi_4(x) = 1 + \int_0^x \phi_3(s)\,ds = 1 + \int_0^x 1 + s + \frac{s^2}{2} + \frac{s^3}{2\cdot 3}\,ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4}$$

By induction

if $\phi_{n-1} = 1 + x + \cdots + \frac{x^{n-1}}{(n-1)!}$

$$\phi_n(x) = 1 + \int_0^x 1 + s + \frac{s^2}{2} + \cdots \frac{s^{n-1}}{(n-1)!}\,ds = 1 + x + \frac{x^2}{2} + \cdots \frac{x^n}{n\cdot(n-1)!}$$

$\vdots$

$$\phi_n \to 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$$

$\boxed{5}$ $u = y'$ $\quad u' = \sqrt{1-u^2}$ $\quad \frac{u'}{\sqrt{1-u^2}} = 1$ $\quad \sin^{-1} u = x + c$

$u = \sin(x+c)$

$u = y' = \sin(x+c) \Rightarrow$

$y = -\cos(x+c) + d$ solves $y'' = \sqrt{1-(y')^2}$

$y = y(x)$

$$\begin{cases} y'' = \sqrt{1-(y')^2} \\ y(-1) = y(1) = 0 \end{cases}$$

correct problem # 5 $\Rightarrow$ last page

$$\begin{cases} y'' = \sqrt{1+(y')^2} \\ y(-1) = y(1) = 0 \end{cases}$$

$\boxed{4}$ a) $\begin{cases} y'' + 2y' + y = 0 \\ y(0) = 1, \ y'(0) = 0 \end{cases}$ ⟵ constant coeff. homogeneous, lin, second ord. (3)

look for linearly independent solutions in the form

$$y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x}$$
$$y'' = \lambda^2 e^{\lambda x}$$

$$y'' + 2y' + y = e^{\lambda x}(\lambda^2 + 2\lambda + 1) = 0$$

$$\Rightarrow (\lambda^2 + 2\lambda + 1) = (\lambda + 1)^2 = 0$$

$$\Rightarrow \lambda = -1$$

double root

$y_1 = e^{-x}$ is solution, second independent solution can be obtained as

$$y_2 = x e^{-x} = u(x)\, y_1(x)$$

$y_2 = u y_1$

$(u y_1)'' + 2(u y_1)' + u y_1 = 0$

$u(\underbrace{y_1'' + 2y_1' + y_1}_{0}) + u'' y_1 + 2u' y_1' + 2u' y_1 = 0$

$\dfrac{u''}{u'} = \dfrac{U'}{U} = -\dfrac{2y_1'}{y_1} - 2 = 0$ $\quad \left(\dfrac{y_1'}{y_1} = \dfrac{-e^{-x}}{e^{-x}} = -1\right)$

$\ln|U| = c$

$U = C = \pm e^c$

$u = \int u = Cx + D$

take $C = 1, D = 0 \Rightarrow u = x$

general solution: $y = c_1 y_1 + c_2 y_2$

Initial conditions:

$$\begin{cases} y = c_1 e^{-x} + c_2 x e^{-x} \\ y' = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x} \end{cases}$$

$1 = y(0) = c_1$

$0 = y'(0) = -e^0 + c_2 e^0 + 0$

$\Rightarrow c_2 = e^0 = 1$

thus $\underline{y(x) = e^{-x} + x e^{-x}}$

---

b) $\begin{cases} y'' + 9y = 0 \\ y(0) = 4, \ y'(0) = -6 \end{cases}$

if $y = e^{\lambda x}$ then $y'' + 9y = e^{\lambda x}(\lambda^2 + 9) = 0$

$\lambda^2 + 9 = 0 \quad \lambda = \pm 3i$

general real solution

$c_1 = y(0) = 4 \Leftarrow y(x) = e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$

$3c_2 = y'(0) = -6 \Leftarrow y'(x) = -3c_1 \sin 3x + 3c_2 \cos 3x$

$\Rightarrow c_2 = -2$ thus, $\underline{y(x) = 4\cos 3x - 2\cos 3x}$

$\boxed{5}$ (4)

$$\begin{cases} y'' = \sqrt{1+(y')^2} \ (>0) \\ y(-1) = y(1) = 0 \end{cases} \qquad u = y'$$

„Second order boundary value problem"

$$\frac{u'}{\sqrt{1+u^2}} = 1$$

$$\int \frac{1}{\sqrt{1+u^2}} \boxed{\overset{du}{u' dx}} = x + c$$

integrate change of var

$(\theta \neq x)$ $u = \tan\theta, \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$du = \sec^2\theta \, d\theta;$

$\sqrt{1+u^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta}$

$= |\sec\theta| = \sec\theta$

$$\int \frac{1}{\sqrt{1+u^2}} du = \int \frac{1}{\sec\theta} \cdot \sec^2\theta \, d\theta =$$

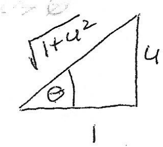
$$= \int \sec\theta \, d\theta = \int \sec\theta \, \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} d\theta =$$

$$= \int \frac{\sec^2\theta + \tan\theta \cdot \sec\theta}{\tan\theta + \sec\theta} d\theta = \int \frac{dv}{v} = \ln v + \tilde{c}$$

$$= \ln|\sec\theta + \tan\theta| + \tilde{c}$$

$v = \tan\theta + \sec\theta$

$dv = \sec^2\theta + \sec\theta \cdot \tan\theta \, d\theta$



$\tan\theta = \frac{u}{1}$

$\sec\theta = \sqrt{1+u^2}$

$$\int \frac{1}{\sqrt{1+u^2}} du = \ln|u + \sqrt{1+u^2}| + \tilde{c} = x + c$$

$$u + \sqrt{1+u^2} = c e^x$$

$$\sqrt{1+u^2} = c e^x - u$$

$$1 + \cancel{u^2} = c^2 e^{2x} - 2 c e^x u + \cancel{u^2}$$

$$\Rightarrow u = \frac{c^2 e^{2x} - 1}{2 c e^x} = \frac{c e^x}{2} - \frac{1}{2c} e^{-x}$$

$$y = \int u \, dx = \frac{c e^x}{2} + \frac{e^{-x}}{2c} + d$$

general sol for y.

(*) Note also, that

$u' = y'' = \sqrt{1+(y')^2} > 0$

$u' = \frac{c e^x}{2} + \frac{1}{2c} \cdot e^{-x} > 0$

$\Rightarrow c > 0$

$$\left.\begin{aligned} y(1) &= \frac{ce}{2} + \frac{1}{2}\frac{1}{ce} + d = 0 \\ y(-1) &= \frac{c}{2e} + \frac{e}{2c} + d = 0 \end{aligned}\right\} \quad \frac{ce}{2} + \frac{1}{2}\frac{1}{ce} - \frac{c}{2e} - \frac{e}{2c} = 0 \quad \cdot / 2e$$

$$e^2 c^2 + 1 - c^2 - e^2 = 0$$

$$\Rightarrow c^2(e^2 - 1) = e^2 - 1 \Rightarrow$$

$$\Rightarrow c = \pm 1 \quad \text{(*) implies } c = 1$$

If $c = 1 \Rightarrow d = -\left(\frac{e + 1/e}{2}\right)$ $= \cosh 1$

thus $y(x) := \cosh x - \cosh 1$