HW 4 (First HW on ODE's), Math 601
Fall 2018
University Of Wisconsin, Milwaukee

Nasser M. Abbasi

October 6, 2018 Compiled on October 6, 2018 at 1:06am

Contents

1 problems description

HW5 MATH 601 HW 1 ODEs oct 4, 2018
second ODE HW HW on ODE

1 a, Verify that $y(x) = -\sin x + ax^2 + bx + c$ for any a, b, c constants is a solution to $y''' = \cos x$

b, Show that $y(x) = \tan(x + c)$ solves $y' = 1 + y^2$ for any constant c

2 Show that the given function solves the specified initial value problem

a) $y(x) = Ce^{x/2}$ $\begin{cases} y' = \frac{1}{2}y \\ y(2) = 2 \end{cases}$

b) $y(x) = ce^{-x^2}$ $\begin{cases} y' + 2xy = 0 \\ y(1) = 1/e \end{cases}$

3 $y = cx - c^2$ is a solution to the ODE $(y')^2 - xy' + y = 0$ for any c const.
Find a singular solution to the ODE (not given by $y = cx - c^2$) by rewriting the ODE using the quadratic formula.

4 Find all solutions of the following differential equations.

a) $yy' + 25x = 0$ b) $y' = ky^2$

c) $xy' = x + y$ (Hint $u = y/x$)

5 Solve the IVPs.

a) $\begin{cases} y' = 1 + 4y^2 \\ y(0) = 0 \end{cases}$ b $\begin{cases} y' = -x/y \\ y(1) = \sqrt{3} \end{cases}$

c) $\begin{cases} e^x y' = 2(x+1)y^2 \\ y(0) = 1/6 \end{cases}$

16 a, Show that the solution curves of $y' = -\frac{x}{y}$ lie on circles

b, Consider the hyperbolas $x \cdot y = c$.
Give a differential equation for which all these curves are solutions

c, Find an ODE that has the straight lines throught as solutions (except $x=0$ line).

d, Note for $y' = -\frac{x}{y} = f_1(x,y)$ and $y' = \frac{y}{x} = f_2(x,y)$

$f_1(x,y) \cdot f_2(x,y) = -1$, also the respective solution curves intersect each other at right angle.
Explane why is this always the case if
$f_1(x,y) \cdot f_2(x,y) = -1$

2 Problem 1

2.1 part a

$$
\begin{aligned}
y' &= -\cos x + 2ax + b \\
y'' &= \sin x + 2a \\
y''' &= \cos x
\end{aligned}
$$

Substituting into the ODE $y''' = \cos x$ shows it satsifies it. Hence this is true for any a, b, c.

2.2 part b

Since $\tan(x+c) = \frac{\sin(x+c)}{\cos(x+c)}$ then

$$y' = 1 + \tan^2(x+c)$$

Substituting this into the ode $y' = 1 + y^2$ gives

$$1 + \tan^2(x+c) = 1 + \tan^2(x+c)$$

Which is true for any c

3 Problem 2

see Key.

4 Problem 3

see Key

5 Problem 4

(a) Find all solutions to $yy' + 25x = 0$ (b) $y' = ky^2$ (c) $xy' = x + y$

5.1 Part a

$$\begin{aligned} y\frac{dy}{dx} &= -25x \\ ydy &= -25xdx \\ \frac{y^2}{2} &= -\frac{25}{2}x^2 + C \\ y^2 &= -25x^2 + C_1 \end{aligned}$$

Hence

$$y = \pm\sqrt{C_1 - 25x^2}$$

For real solution, we want $C_1 > 25x^2$.

5.2 Part b

$$\begin{aligned} \frac{1}{y^2}\frac{dy}{dx} &= k \\ \frac{1}{y^2}dy &= kdx \\ \frac{-1}{y} &= kx + C \\ y &= \frac{-1}{kx + C} \end{aligned}$$

5.3 Part c

$$\frac{dy}{dx} = 1 + \frac{y}{x} \qquad x \neq 0$$

Let $u = \frac{y}{x}$ or $y = ux$. Hence $\frac{dy}{dx} = u'x + u$ and the above ODE becomes

$$\begin{aligned} u'x + u &= 1 + u \\ u' &= \frac{1}{x} \\ du &= \frac{1}{x}dx \\ u &= \ln|x| + C \end{aligned}$$

Hence

$$y = x(\ln|x| + C)$$

6 Problem 5

(a) Solve the IVP $y'(x) = 1 + 4y^2$ with $y(0) = 0$. (b) $y' = -\frac{x}{y}$ with $y(1) = \sqrt{3}$ (c) $e^x y' = 2(x+1)y^2$ with $y(0) = \frac{1}{6}$

6.1 Part a

$$\begin{aligned} y'(x) &= 1 + 4y^2 \\ \frac{dy}{1 + 4y^2} &= dx \\ \frac{1}{2}\arctan(2y) &= x + C \\ \arctan(2y) &= 2x + C_1 \\ y &= \frac{\tan(2x + C_1)}{2} \end{aligned}$$

Applying IC gives

$$0 = \frac{1}{2}\tan(C_1)$$

Hence $C_1 = 0$. Therefore the solution is

$$y = \frac{1}{2}\tan(2x)$$

6.2 Part b

$$
\begin{aligned}
y' &= -\frac{x}{y} \\
ydy &= -xdx \\
\frac{1}{2}y^2 &= -\frac{1}{2}x^2 + C \\
y^2 &= -x^2 + C_1
\end{aligned}
$$

Applying IC gives

$$
\begin{aligned}
3 &= -1 + C_1 \\
C_1 &= 4
\end{aligned}
$$

Hence solution is

$$
\begin{aligned}
y^2 &= -x^2 + 4 \\
y &= \pm\sqrt{4 - x^2}
\end{aligned}
$$

For real solution $4 - x^2 > 0$.

6.3 Part c

$$
\begin{aligned}
e^x y' &= 2\left(x+1\right)y^2 \\
\frac{y'}{y^2} &= 2\left(x+1\right)e^{-x} \\
y^{-2}dy &= 2\left(x+1\right)e^{-x} \\
-\frac{1}{y} &= \int 2\left(x+1\right)e^{-x}dx \\
&= -2\left(x+2\right)e^{-x} + C
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= \frac{1}{2\left(x+2\right)e^{-x} + C_1} \\
&= \frac{1}{2xe^{-x} + 4e^{-x} + C_1}
\end{aligned}
$$

Applying IC gives

$$
\begin{aligned}
\frac{1}{6} &= \frac{1}{4 + C_1} \\
4 + C_1 &= 6 \\
C_1 &= 2
\end{aligned}
$$

Hence solution is

$$
y = \frac{1}{2xe^{-x} + 4e^{-x} + 2}
$$

7 Key solution

Math 601 Solutions HW1 ODEs (1)

1. a) $y = -\sin x + ax^2 + bx + c$ $\quad a, b, c$ constants

$y' = -\cos x + a\cdot 2x + b\cdot 1 + 0$

$y'' = -(-\sin x) + 2a + 0$

$y''' = \cos x$

b) $y(x) = \tan(x+C)$ $\quad C$ const.

$y' = \dfrac{1}{\cos^2(x+C)} = \dfrac{\sin^2(x+C) + \cos^2(x+C)}{\cos^2(x+C)} = \tan^2(x+C) + 1$

$= 1 + y^2$

2. (a) $\begin{cases} y' = \frac{1}{2}y \\ y(2) = 2 \end{cases}$ $\quad y = Ce^{x/2}$

$y' = C\frac{1}{2}e^{x/2} = \frac{1}{2}y$ solves ODE

initial condition $2 = y(2) = Ce^{2/2} = C\cdot e$

only if $C = 2/e$

$\underline{y = \frac{2}{e}e^{x/2} \text{ is solution}}$

(b) $\begin{cases} y' + 2xy = 0 \\ y(1) = 1/e \end{cases}$ $\quad y = Ce^{-x^2}$

$\underbrace{-2xCe^{-x^2}}_{y'(x)} + 2xCe^{-x^2} = 0$

for all x

solves ODE for all C

initial condition

$\frac{1}{e} = y(1) = Ce^{-1} = \frac{C}{e} \Rightarrow C = 1$ solution to IVP only if $C = 1$ $\quad y = e^{-x^2}$

3. $y = cx - c^2$ $\quad (y')^2 - xy' + y = c^2 - x(c) + cx - c^2 = 0$

$y' = c$

$y' = \dfrac{x \pm \sqrt{x^2 - 4y}}{2} \Rightarrow y' = \frac{x}{2}$ if $4y = x^2$

$\Downarrow$

$y = \frac{x^2}{4} + c$ $\quad \boxed{y = \frac{x^2}{4}}$

singular solution

$\boxed{4}$ a) $yy' = -25x$ separable

$$\frac{d}{dx}\left(\frac{y^2}{2}\right) = -25x$$

$$\frac{y^2}{2} = -\frac{25x^2}{2} + C \qquad D = 2C$$

$$y = \pm\sqrt{D - 25x^2} \quad \text{where } D - 25x^2 \geq 0 \text{ only if } D \geq 0$$

and $|5x| \leq \sqrt{D}$

$$-\frac{1}{5}\sqrt{D} \leq x \leq \frac{1}{5}\sqrt{D}$$

b) $y' = ky^2$ if $y \neq 0$ on I then separable $\to$ $y = 0 \Rightarrow y' = 0$

$y \equiv 0$ solution for all x for any k

$$\frac{y'}{y^2} = k$$

$$-y^{-1} = kx + C$$

$$y = \frac{-1}{kx + C} \quad \text{if } x \neq -\frac{C}{k}$$

sol. on $\left(-\infty, -\frac{C}{k}\right)$ & on $\left(-\frac{C}{k}, \infty\right)$

c) $xy' = x + y$ if $x \neq 0$ $\quad u = y/x \quad y = ux$

$y' = u'x + u$

$$y' = 1 + y/x$$

$$xu' + \cancel{u} = 1 + \cancel{u}$$

$$u' \cdot x = 1$$

$$u' = \frac{1}{x} \qquad u = \ln|x| + C \qquad y(x) = \frac{\ln|x|}{x} + \frac{C}{x}$$

$\boxed{5}$ $\begin{cases} y' = 1 + 4y^2 \\ y(0) = 0 \end{cases}$ $\quad 1 + 4y^2 > 0 \Rightarrow y' = 1 + 4y^2$ and $\frac{y'}{1+4y^2} = 1$ have same sols.

separable

$$\int \frac{1}{1+4y^2}\,dy = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1} u = \frac{\tan^{-1} 2y}{2}$$

$u = 2y$

$dy = du/2$

$\tan^{-1} 2y = 2t + C$

$$\boxed{y = \frac{\tan(2t + C)}{2}} \text{ for all } t$$

I.C.: $0 = y(0) = \frac{\tan(C)}{2} \Rightarrow \boxed{C = 0}$

③

b) $\begin{cases} y' = -\dfrac{x}{y} \\ y(1) = \sqrt{3} \end{cases}$ $\overset{y\neq 0:}{\Longrightarrow}$ $yy' = -x$ $\Rightarrow$ $\dfrac{y^2}{2} = -\dfrac{x^2}{2} + c$ $D = 2C$

Solution: $\boxed{y = \pm\sqrt{D - x^2}}$

$\boxed{y \neq 0} \Rightarrow x \neq \pm\sqrt{D}$

$D - x^2 \geq 0 \Rightarrow$

on: $\boxed{-\sqrt{D} \leq x \leq \sqrt{D}}$

c) $e^x y' = 2(x+1)y^2$ Separable $\begin{cases} e^x y' = 2(x+1)y^2 \\ y(0) = 1/6 \end{cases}$

$\dfrac{y'}{y^2} = 2e^{-x}(x+1) = 2\underbrace{xe^{-x}}_{\text{in by parts}} + 2e^{-x}$

$\int \underset{u}{x}\, \underset{v'}{e^{-x}} dx = -xe^{-x} - \int -e^{-x} dx$

$v = -e^{-x}$ $= -xe^{-x} - e^{-x}$

$-\dfrac{1}{y} = 2[-xe^{-x} - e^{-x}] - 2e^{-x} + c$

$y = \dfrac{1}{e^{-x}(2x+4) + c}$ $D = -c$

I.C.: $\dfrac{1}{6} = y(0) = \dfrac{1}{4 + D}$ $\Rightarrow D = 2$

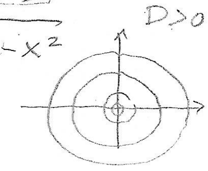
solution $y = \dfrac{1}{2 + e^{-x}(2x+4)}$

solution on (x^*, ∞)

for all $x \in \mathbb{R}$ except $(2x+4)e^{-x} = -2$ at $x = x^* < -2$

6 a) $y' = -\dfrac{x}{y}$ $\underset{y\neq 0}{\Rightarrow}$ $yy' = -x$ $\dfrac{y^2}{2} = -\dfrac{x^2}{2} + c$ $\boxed{y^2 + x^2 = D}$ circles

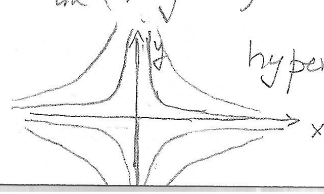
$D = 2C$ $y = \pm\sqrt{D - x^2}$ $D > 0$



b) $x \cdot y = c$ $y = y(x)$

$\dfrac{d}{dx}(x \cdot y = c) \Rightarrow y + xy' = 0$ $\underset{\text{if } y\neq 0}{\Rightarrow}$ $\boxed{y' = -\dfrac{y}{x}}$

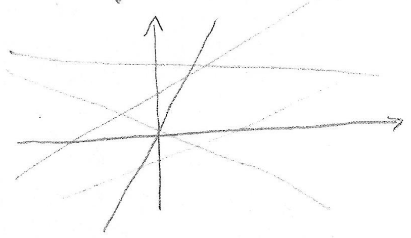
hyperbolas

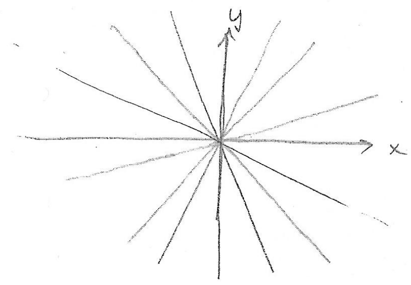


c) straight lines: linear function graphs

$y = mx + b$ $\quad y' = m$ constant

straight lines through origin $\quad y = mx$, $b = 0$ $\quad \begin{cases} y' = m \\ y(0) = 0 \end{cases}$ or $y' = \dots = y/x$, $x \neq 0$





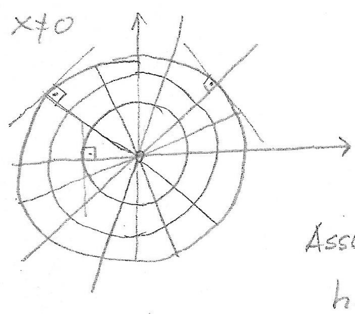
d) $y' = -\frac{x}{y}$ circles

$y' = y/x \Rightarrow \frac{1}{y} y' = \frac{1}{x} \Rightarrow \ln|y| = \ln|x| + c$

$y = \pm x \cdot e^c = Cx \quad C \neq 0$

$x \neq 0$

straight lines through origin



Assume that $y' = f_1(x,y)$ and $y' = f_2(x,y)$ have two solutions $y_1(x)$ & $y_2(x)$ respectively intersecting at (x_0, y_0) point.

$y_1'(x_0) = f_1(x_0, y_1(x_0)) = f_1(x_0, y_0)$ is the slope of tangent to y_1 at (x_0, y_0)

$y_2'(x_0) = f_2(x_0, y_2(x_0)) = f_2(x_0, y_0)$ is the slope of tangent to y_2 at (x_0, y_0)

intersecting tangent lines $\quad y = f_1(x_0, y_0)x + y_0$

$y = f_2(x_0, y_0)x + y_0$

with finite slopes $m_1 = f_1(x_0, y_0)$ and $m_2 = f_2(x_0, y_0)$ are intersecting exactly when

$m_1 = -\frac{1}{m_2}$ or $f_1(x_0, y_0) f_2(x_0, y_0) = -1$