HW 3, Physics 501
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University Of Wisconsin, Milwaukee

Nasser M. Abbasi

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Contents

1 Problem set

HW: MATH 601 | Sections 15.2 – 15.5 Eds 9,10 | Sept 24, 201 Tues

[1] Consider the power series below. Give the center and find the radius of convergence for each.

a) $\sum_{n=1}^{\infty} n(z+i\sqrt{2})^n$ b) $\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n (z-\pi i)^n$ c) $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$

d) $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}(z+2-i)^n$

[2] Find the radius of convergence using both

① The formula $R = \dfrac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$ (Hadamard)

② The termwise differentiation/integration properties of power series

for the series: a) $\sum_{n=1}^{\infty} \frac{6^n}{n}(z-i)^n$; b) $\sum_{n=0}^{\infty} \frac{3^n (n+1)n}{5^n} z^{2n}$

[3] Show that $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n$

a) using the Cauchy product

b) differentiating a suitable series

[4] If $f(z)$ is an even function (i.e. $f(-z) = f(z)$) where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, show that $a_n = 0$ when n is odd.

If $f(z)$ is odd function (ie $f(-z) = -f(z)$) show that $a_n = 0$ for n even.

[5] Develop the functions below in a Maclaurin series and determine the radius of convergence R for each

a) $\cos(2z^2)$ b) $\frac{z+2}{1-z^2}$

6 Develop

a) $f(z) = \frac{1}{z}$ in a Taylor series with center $z_0 = i$

b) $g(z) = e^z$ —— " —— $z_0 = a$.

What is the radius of convergence for each?

7 Show that $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ converges uniformly in $|z| \leq 3$

8 Where does the series $\sum_{n=1}^{\infty} \left(\frac{n+2}{5n-3}\right)^n z^n$ converges uniformly?

2 Problem 1

Consider the power series below. Given the center and find radius of convergence for each.

1. $\sum_{n=1}^{\infty} n\left(z + i\sqrt{2}\right)^n$

2. $\sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^n (z - i\pi)^n$

3. $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$

4. $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} (z + 2 - i)^n$

Solution

1) Comparing to form $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ then center is $-i\sqrt{2}$. Now,

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)}{n}\right| \\
&= 1
\end{aligned}$$

Hence $R = 1$

2) Comparing to form $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ then center is $i\pi$. Now,

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^{n}}\right| \\
&= \lim_{n\to\infty} \left|\left(\frac{a}{b}\right)\right| \\
&= \frac{a}{b}
\end{aligned}$$

Hence $R = \frac{b}{a}$

3) Comparing $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$ to $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ then center is 0. Now

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{\frac{(3(n+1))!}{2^{n+1}((n+1)!)^3}}{\frac{(3n)!}{2^n (n!)^3}}\right| \\
&= \lim_{n\to\infty} \left|\frac{(3(n+1))!2^n (n!)^3}{(3n)!2^{n+1}((n+1)!)^3}\right| \\
&= \frac{1}{2}\lim_{n\to\infty} \left|\frac{(3n+3)!(n!)^3}{(3n)!((n+1)!)^3}\right| \\
&= \frac{1}{2}\lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)(3n)!(n!)^3}{(3n)!((n+1)!)^3}\right| \\
&= \frac{1}{2}\lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)(n!)^3}{((n+1)!)^3}\right| \\
&= \frac{1}{2}\lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)(n!)^3}{((n+1)n!)^3}\right| \\
&= \frac{1}{2}\lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)(n!)^3}{(n+1)^3 (n!)^3}\right| \\
&= \frac{1}{2}\lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)}{(n+1)^3}\right| \\
&= \frac{1}{2}\lim_{n\to\infty} \left|\frac{(3n+1)(3n+2)(3n+1)}{(n+1)^3}\right|
\end{aligned}$$

Hence the above becomes

$$\begin{aligned}
L &= \frac{1}{2}\lim_{n\to\infty} \left|\frac{27n^3 + 36n^2 + 15n + 2}{n^3 + 3n^2 + 3n + 1}\right| \\
&= \frac{1}{2}\lim_{n\to\infty} \left|\frac{27 + 36\frac{1}{n} + 15\frac{1}{n^2} + \frac{2}{n^3}}{1 + 3\frac{1}{n} + 3\frac{1}{n^2} + \frac{1}{n^3}}\right| \\
&= \frac{27}{2}
\end{aligned}$$

Hence $R = \frac{2}{27}$.

4) Comparing $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} (z - (-2+i))^n$ to $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ shows that center is $z_0 = -2 + i$. Now

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{1}{(1+i)^{n+1}}}{\frac{1}{(1+i)^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(1+i)^n}{(1+i)^{n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{(1+i)}\right| \\
&= \left|\frac{1}{(1+i)}\right| \\
&= \frac{1}{\sqrt{2}}
\end{aligned}
$$

Hence $R = \sqrt{2}$

3 Problem 2

Find radius of convergence using both 1) $R = \frac{1}{L}$ where $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ and 2) the termwise differentiation/integration properties of power series. Do this for

1. $\sum_{n=1}^{\infty}\frac{6^n}{n}(z-i)^n$
2. $\sum_{n=0}^{\infty}\frac{3^n(n+1)n}{5^n}z^{2n}$

Solution
1) First method. The center is i. And

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{6^{n+1}}{n+1}}{\frac{6^n}{n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{\left(6^{n+1}\right)n}{6^n(n+1)}\right| \\
&= \lim_{n\to\infty}\left|\frac{6n}{(n+1)}\right| \\
&= 6\lim_{n\to\infty}\left|\frac{n}{(n+1)}\right| \\
&= 6
\end{aligned}
$$

Hence $R = \frac{1}{6}$.
Second method: Taking termwise differentiation gives

$$
\begin{aligned}
f'(z) &= \sum_{n=1}^{\infty}\frac{6^n}{n}n(z-i)^{n-1} \\
&= 6\sum_{n=1}^{\infty}6^{n-1}(z-i)^{n-1}
\end{aligned}
$$

Changing the indexing gives

$$
\begin{aligned}
f'(z) &= 6\sum_{n=0}^{\infty}6^n(z-i)^n \\
&= 6\sum_{n=0}^{\infty}(6(z-i))^n
\end{aligned}
$$

Comparing to Binomial series $\sum_{n=0}^{\infty}r^n$, the above is $6\frac{1}{1-r}$ where $r = 6(z-i)$. Hence this converges for $|r| < 1$ or $|6(z-i)| < 1$ or $|(z-i)| < \frac{1}{6}$ and diverges for $|z-i| > \frac{1}{6}$. Since termwise differentiated series has same radius of convergence, then $R = \frac{1}{6}$ as using first method.
2) First method. Comparing $\sum_{n=0}^{\infty}\frac{3^n(n+1)n}{5^n}\left(z^2\right)^n$ to $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ then center is zero. And
TODO

4 Problem 3

Show that $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n$, using (a) the Cauchy product. (b) By differentiating a suitable series.
Solution (a)

$$\begin{aligned}
\frac{1}{(1-z)^2} &= \frac{1}{(1-z)} \frac{1}{(1-z)} \\
&= \left(1+z+z^2+z^3+\cdots\right)\left(1+z+z^2+z^3+\cdots\right) \\
&= \left(1+z+z^2+z^3+\cdots\right) + z\left(1+z+z^2+z^3+\cdots\right) + z^2\left(1+z+z^2+z^3+\cdots\right) + \cdots \\
&= \left(1+z+z^2+z^3+\cdots\right) + \left(z+z^2+z^3+\cdots\right) + \left(z^2++z^3+z^4+\cdots\right) + \cdots \\
&= 1+2z+3z^2+4z^4+\cdots \qquad |z|<1
\end{aligned}$$

But $\sum_{n=0}^{\infty} (n+1) z^n = 1+2z+3z^2+4z^4+\cdots$. Hence the same.
Solution (b) Observing that

$$(n+1) z^n = \frac{d}{dz} z^{n+1}$$

Then

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+1) z^n &= \sum_{n=0}^{\infty} \frac{d}{dz} z^{n+1} \\
&= \frac{d}{dz} \sum_{n=0}^{\infty} z^{n+1} \\
&= \frac{d}{dz} \sum_{n=1}^{\infty} z^{n} \\
&= \frac{d}{dz} \left(z+z^2+z^3+\cdots\right) \\
&= \frac{d}{dz} \left(z\left(1+z+z^2+\cdots\right)\right) \\
&= \frac{d}{dz} \left(\frac{z}{1-z}\right)
\end{aligned}$$

But $\frac{d}{dz}\frac{A(z)}{B(z)} = \frac{A'B-AB'}{B^2}$, hence the above becomes, where $A = z, B = 1-z$

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+1) z^n &= \frac{(1-z) - z(-1)}{(1-z)^2} \\
&= \frac{1-z+z}{(1-z)^2} \\
&= \frac{1}{(1-z)^2}
\end{aligned}$$

5 Problem 4

If $f(z)$ is an even function, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, show that $a_n = 0$ when n is odd. And if $f(z)$ is odd function, show that $a_n = 0$ when n is even.
Solution
If $f(z)$ is even, then $f(-z) = f(z)$. Therefore

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n (-z)^n &= \sum_{n=0}^{\infty} a_n z^n \\
a_0 - a_1 z + a_2 z^2 - a_3 z^3 + a_4 z^4 - \cdots &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots
\end{aligned}$$

Since power series is unique, then we must have $a_1 = -a_1$ which means $a_1 = 0$, the same for $a_3 = -a_3$, which gives $a_3 = 0$ and so on for all odd a_n.
If $f(z)$ is odd, then $f(-z) = -f(z)$. Therefore

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n (-z)^n &= -\sum_{n=0}^{\infty} a_n z^n \\
a_0 - a_1 z + a_2 z^2 - a_3 z^3 + a_4 z^4 - \cdots &= -\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots\right) \\
&= -a_0 - a_1 z - a_2 z^2 - a_3 z^3 - a_4 z^4 + \cdots
\end{aligned}$$

Since power series is unique, then we must have $a_0 = -a_0$ which means $a_0 = 0$, the same for $a_2 = -a_2$, which gives $a_2 = 0$ and so on for all even a_n.

6 Problem 5

Develop the functions below in Maclaurin's series and determine the radius of convergence R for each. (a) $\cos\left(2z^2\right)$, (b) $\frac{z+2}{1-z^2}$
Solution (a)

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Replacing $x = 2z^2$ gives

$$\begin{aligned}
\cos\left(2z^2\right) &= 1 - \frac{\left(2z^2\right)^2}{2!} + \frac{\left(2z^2\right)^4}{4!} - \frac{\left(2z^2\right)^6}{6!} + \cdots \\
&= 1 - \frac{2^2 z^4}{2!} + \frac{2^4 z^8}{4!} - \frac{2^6 z^{12}}{6!} + \cdots \\
&= 1 - \frac{4z^4}{2!} + \frac{4^2 z^8}{4!} - \frac{4^3 z^{12}}{6!} + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{4^n z^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\left(4z^2\right)^n}{(2n)!}
\end{aligned}$$

Hence

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{\frac{\left(4z^2\right)^{n+1}}{(2(n+1))!}}{\frac{\left(4z^2\right)^n}{(2n)!}}\right| \\
&= \lim_{n\to\infty} \left|\frac{\left(4z^2\right)^{n+1}(2n)!}{\left(4z^2\right)^n(2(n+1))!}\right| \\
&= \lim_{n\to\infty} \left|\frac{4z^2(2n)!}{(2n+2)!}\right| \\
&= \lim_{n\to\infty} \left|\frac{4z^2(2n)!}{(2n+2)(2n+1)(2n)!}\right| \\
&= \lim_{n\to\infty} \left|\frac{4z^2}{(2n+2)(2n+1)}\right| \\
&= \lim_{n\to\infty} \left|\frac{4z^2}{4n^2+6n+2}\right| \\
&= \lim_{n\to\infty} \left|\frac{\frac{4}{n^2}z^2}{4+6\frac{1}{n}+\frac{2}{n}}\right| \\
&= \left|z^2\right| \lim_{n\to\infty} \left|\frac{0}{4}\right| \\
&= 0
\end{aligned}$$

Hence $R = \frac{1}{L} = \infty$
(b) $\frac{z+2}{1-z^2}$. Apply partial fractions. Obtain two binomial series and combine.

7 Problem 6

Develop (a) $f(z) = \frac{1}{z}$ in Taylor series around $z_0 = i$. (b) $g(z) = e^z$ around $z_0 = a$. What is radius of convergence?
Solution (a)

$$f(z) = f(i) + (z-i)f'(i) + \frac{(z-i)^2 f''(i)}{2!} + \frac{(z-i)^3 f'''(i)}{3!} + \cdots$$

But $f'(z) = -\frac{1}{z^2}, f''(z) = \frac{2}{z^3}, f'''(z) = -\frac{(2)(3)}{z^4}, \cdots$, hence the above becomes

$$
\begin{aligned}
f(z) &= \frac{1}{i} - (z-i)\frac{1}{i^2} + \frac{(z-i)^2}{2!}\frac{2}{i^3} + \frac{(z-i)^3}{3!}\left(-\frac{2(3)}{i^4}\right) + \cdots \\
&= -i + (z-i) + 2i\frac{(z-i)^2}{2!} - 2(3)\frac{(z-i)^3}{3!} + \cdots \\
&= -i + (z-i) + i(z-i)^2 - (z-i)^3 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{i^{n+1}}(z-i)^n
\end{aligned}
$$

Hence this convergence for $|z-i| < 1$.
Solution (b)

$$
g(z) = g(a) + (z-a)g'(a) + \frac{(z-a)^2 g''(a)}{2!} + \frac{(z-a)^3 g'''(a)}{3!} + \cdots
$$

But $g'(z) = e^z, g''(z) = e^z, g'''(z) = e^z, \cdots$, hence the above becomes

$$
\begin{aligned}
g(z) &= e^a + (z-a)e^a + \frac{(z-a)^2 e^a}{2!} + \frac{(z-a)^3 e^a}{3!} + \cdots \\
&= e^a\left(1 + (z-a) + \frac{(z-a)^2}{2!} + \frac{(z-a)^3}{3!} + \cdots\right) \\
&= e^a \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!}
\end{aligned}
$$

Where $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty}\left|\frac{n!}{(1+n)!}\right| = \lim_{n\to\infty}\frac{n!}{n!(1+n)} = \lim_{n\to\infty}\frac{1}{1+n} = 0.$
Hence $R = \frac{1}{L} = \infty$. Converges everywhere.

8 Problem 7

Show that $\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}z^n$ converges uniformly in $|z| \leq 3$
Solution:
To find if it converges uniformly for $|z| \leq 3$, we need to find R, the radius of converges using normal method, then it $R > 3$, then it will converge uniformly for $|z| \leq 3$.

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}}\right| \\
&= \lim_{n\to\infty}\left|\frac{((n+1)!)^2(2n)!}{(n!)^2(2(n+1))!}\right| \\
&= \lim_{n\to\infty}\left|\frac{((n+1)n!)^2(2n)!}{(n!)^2(2(n+1))!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^2(2n)!}{(2n+2)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^2(2n)!}{(2n+2)(2n+1)(2n)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^2+2n+1}{4n^2+6n+2}\right| \\
&= \lim_{n\to\infty}\left|\frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{6}{n^2}+\frac{2}{n^2}}\right| \\
&= \frac{1}{4}
\end{aligned}
$$

Hence Radius of convergence $R = 4$. Since $3 < 4$, then it converges uniformly for $R < 3$.

9 Problem 8

Where does $\sum_{n=1}^{\infty}\left(\frac{n+2}{5n-3}\right)^n z^n$ converges uniformly?
<u>Solution</u> We first find R. Since the series of the form $\sum_{n=1}^{\infty} A^n z^n$ then it is easier to use

$$\begin{aligned} L &= \lim_{n\to\infty} \sqrt[\frac{1}{n}]{|A^n|} \\ &= \lim_{n\to\infty} \sqrt[\frac{1}{n}]{|A^n|} \\ &= \lim_{n\to\infty} \sqrt[\frac{1}{n}]{\left|\frac{n+2}{5n-3}\right|^n} \\ &= \lim_{n\to\infty} \sqrt[\frac{1}{n}]{\left|\frac{1+\frac{2}{n}}{5-\frac{3}{n}}\right|^n} \\ &= \frac{1}{5} \end{aligned}$$

Hence $R = 5$. Therefore it converges uniformly for $|z| \leq r < 5$

-- Solutions:

1

(a) $\sum_{n=1}^{\infty} n\,(z+i\sqrt{2})^n = \sum_{n=1}^{\infty} n\,(z-(-i\sqrt{2}))^n \qquad a_n = n$

center $z_0 = -i\sqrt{2} \qquad R = \frac{1}{L^*} = 1$

$$L^* = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{n} = 1$$

(d) $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}\,(z-(i-2))^n \qquad a_n = \frac{1}{(1+i)^n} \qquad z_0 = i-2$

$$L^* = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(1+i)^n}{(1+i)^{n+1}}\right| = \frac{1}{|1+i|} = \frac{1}{\sqrt{2}}\ ; \quad R = \sqrt{2}$$

(c) $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n \qquad$ center $z_0 = 0 \qquad a_n = \frac{(3n)!}{2^n (n!)^3}$

$$L^* = \lim_{n\to\infty} \frac{(3n+3)!}{2^{n+1}(n+1)!^3} \Big/ \frac{3n!}{2^n (n!)^3} = \lim \frac{\cancel{1\ldots 3n}\,(3n+1)(3n+2)(3n+3)\cancel{2^n}(n!)^3}{\cancel{1\ldots 3n}\ 2^{n+1}\,[(n+1)!]^3}$$

$$= \lim_{n\to\infty} \frac{(3n+1)(3n+2)(3n+3)}{2\cdot(n+1)(n+1)(n+1)} = \frac{3^3}{2} = \frac{27}{2}$$

$$R = \frac{1}{L^*} = \frac{2}{27}$$

(b) $$L^* = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^n} = \frac{a}{b}\ ; \quad R = \frac{1}{L^*} = \frac{b}{a}$$

$\boxed{2}$ (a) $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{n}(z-i)^n \qquad a_n = 6^n/n$

1, $\displaystyle L^* = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{6^{n+1}}{6^n}\frac{n}{n+1} = 6\cdot\lim_{n\to\infty}\frac{n}{n+1} = 6\ ; \quad R = \frac{1}{6}$

2, $\displaystyle\sum_{n=1}^{\infty}\frac{6^n}{n}(z-i)^n$ termwise differentiated gives

$$6\sum_{n=1}^{\infty} 6^{n-1}(z-i)^{n-1} = 6\sum_{m=0}^{\infty} 6^m (z-i)^m = 6\cdot\frac{1}{1-6(z-i)}$$

for $|6(z-i)|<1$ or $|z-i|<\frac{1}{6}$ convergent

for $|z-i|>\frac{1}{6}$ divergent.

Termwise differentiated series has same rad. of conv. $R=\frac{1}{6}$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{3^n(n+1)n}{5^n}z^{2n}$

1.) the coefficient of series are

$$a_0 = 0 \qquad a_1 = 0$$
$$a_2 = \frac{3^1(1+1)1}{5^1} \qquad a_3 = 0$$
$$a_4 = \frac{3^2(2+1)2}{5^2} \qquad a_5 = 0$$
$$\vdots \qquad\qquad \vdots$$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ or $\displaystyle\lim_{n\to\infty}\sqrt[k]{|a_k|}$ does not exist.

z^k k odd: $\displaystyle\lim_{n\to\infty}\sqrt[k]{|a_k|} = 0$

k even: $\displaystyle\lim_{n\to\infty}\sqrt[k]{|a_k|} = \lim\sqrt[2n]{\frac{3^n(n+1)n}{5^n}} = \lim\sqrt[2]{\sqrt[n]{\frac{3^n}{5^n}(n+1)n}} =$
$k=2n$

$$\left(\lim_{n\to\infty}\frac{3}{5}\sqrt[n]{n}\sqrt[n]{n+1}\right)^{1/2} = \sqrt{3/5}$$

$\sqrt[k]{|a_k|}$ has two cluster points 0 and $\sqrt{3/5}$

$\displaystyle\overline{\lim}\sqrt[k]{|a_k|} = \sqrt{3/5} \qquad R = \frac{1}{\overline{\lim}\sqrt[k]{|a_k|}} = \sqrt{5/3}$

2 cont.

2) Let $v = z^2$ new complex variable

in terms of v the series can be written as

$$\sum_{n=0}^{\infty} \frac{3^n (n+1) n}{5^n} v^n = \underbrace{v}_{\text{everywhere convergent}} \underbrace{\sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1) n \, v^{n-1}}_{} \qquad R_v$$

integrate termwise w.r.t. v

(same radius of conv)

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1) v^n \qquad R_v$$

integrate w.r.t. v

(same R)

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^{n+1} = \underbrace{v}_{\text{convegent everywhere}} \underbrace{\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^n}_{\text{geom series}}$$

$\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^n$ convegent when

$\left|\frac{3}{5} v\right| < 1 \qquad |v| < \frac{5}{3}$

$R_v = \frac{5}{3}$

thus $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1) n \, z^{2n} = \sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1) n \cdot v^n$

converges when $|z^2| = |v| < R_v = \frac{5}{3}$.

or $|z|^2 < \frac{5}{3} \qquad |z| < \sqrt{5/3} \qquad \boxed{R = \sqrt{\frac{5}{3}}}$

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3 a) $\displaystyle\frac{1}{(1-z)^2} = \frac{1}{1-z} \cdot \frac{1}{1-z} = \left(\sum_{n=0}^{\infty} z^n\right) \cdot \left(\sum_{n=0}^{\infty} z^n\right) \qquad$ if $|z| < 1$

$\displaystyle = \underbrace{(1 + z + z^2 + \dots)}_{a_n = 1} \underbrace{(1 + z + z^2 + \dots)}_{b_n = 1} = \sum_{n=0}^{\infty} (\underset{1}{a_0} b_n + \underset{1}{a_1} b_{n-1} + \dots + \underset{1}{a_{n-1}} b_1 + \underset{1}{a_n} b_0) z^n$

$\displaystyle = \sum_{0}^{\infty} (n+1) z^n \qquad (\text{if } |z| < 1)$

b) $$\frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) \qquad \text{if } |z|<1$$

termwise derivative of series converges to derivative:

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^{n-1} = \underline{\sum_{\ell=0}^{\infty} (\ell+1) z^{\ell}}$$

$$\ell = n-1$$
$$n = \ell+1$$

$\boxed{4}$ Let $f(z)$ be even: $f(-z) = f(z)$ for all z in $|z|<R$
where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that $a_n = 0$ when n is odd.

$$f(-z) = \sum_{n=0}^{\infty} a_n (-z)^n = \sum_{n=0}^{\infty} a_n (-1)^n z^n$$

$$0 = f(z) - f(-z) = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n (-1)^n z^n = \sum_{n=0}^{\infty} a_n \underbrace{(1-(-1)^n)}_{\substack{0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even}}} z^n$$

thus

$$0 = \sum_{n=0}^{\infty} 0 \, z^n \qquad\qquad 0 = 2a_0 + 2a_2 z^2 + 2a_4 z^4 + \dots$$

for all z $\qquad\qquad$ for $|z|<R$

By theorem 2 in $|z|<R$ /where both series converge/ the coefficients of the two series have to coincide.

Thus $2a_0 = 0$, $2a_2 = 0$, $2a_4 = 0, \dots$

or $a_0 = 0$, $a_2 = 0$, $a_4 = 0, \dots$

[5] $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$ for any $z \in \mathbb{C}$

(a) $\cos(2z^2) = 1 - \frac{(2z^2)^2}{2!} + \frac{(2z^2)^4}{4!} - \frac{(2z^2)^6}{6!} + \dots$

$= 1 - \frac{2^2 z^4}{2!} + \frac{2^4 z^8}{4!} - \frac{2^6 z^{12}}{6!} + \dots$ for any $2z^2 \in \mathbb{C}$

or $z \in \mathbb{C}$

$= \sum_{n=0}^{\infty} \frac{4^n z^{4n}}{2n!} \cdot (-1)^n$

radius of conv. $R = \infty$

(b) $\frac{z+2}{1-z^2} = \frac{z+2}{(1-z)(z+1)} = \frac{A}{1-z} + \frac{B}{z+1} = \frac{Az + A - Bz + B}{(1-z)(z+1)}$

$\Rightarrow \begin{array}{l} A-B=1 \\ A+B=2 \end{array} \Rightarrow \begin{array}{l} A = 3/2 \\ B = 1/2 \end{array}$ $= \frac{3}{2}\frac{1}{1-z} + \frac{1}{2}\frac{1}{z+1}$

$\frac{3}{2}\frac{1}{1-z} = \frac{3}{2}\left[1 + z + z^2 + z^3 + \dots\right]$ for $|z|<1$

$\frac{1}{2}\frac{1}{1+z} = \frac{1}{2}\left[\frac{1}{1-(-z)}\right] = \frac{1}{2}\left[1 - z + z^2 - z^3 + \dots\right]$ if $|-z|=|z|<$

$\frac{z+2}{1-z^2} = \left[\frac{3}{2}+\frac{1}{2}\right] + \left(\frac{3}{2}-\frac{1}{2}\right)z + \left(\frac{3}{2}+\frac{1}{2}\right)z^2 + \left(\frac{3}{2}-\frac{1}{2}\right)z^3 + \dots$

$= 2 + z + 2z^2 + z^3 + 2z^4 + z^5$ $(R=1)$

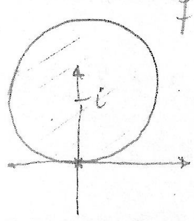
$\sum_{n=0}^{\infty} \left(\frac{3}{2} + \frac{1}{2}(-1)^n\right) z^n$ if $|z|<1$

$\boxed{6}$

(a) $f(z) = \frac{1}{z}$ $\quad z_0 = i$ $\qquad f'(z) = -\frac{1}{z^2}$, $f''(z) = \frac{2\cdot 1}{z^3}$,

$f'''(z) = -\frac{3\cdot 2\cdot 1}{z^4}$ $\qquad f^{(n)}(z) = \frac{(-1)^n\, n!}{z^{n+1}}$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n =$$



$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(i)^{n+1}}(z-i)^n = -i + (z-i) - i(z-i)^2 + (z-i)^3$$

$$-i(z-i)^4 + (z-i)^5 - \dots$$

$|z-i| < 1$

(b) $f(z) = e^z$ $\quad z_0 = a$

$f^{(n)} = e^z$

$$f(z) = \sum_{n=0}^{\infty} \frac{e^a}{n!}(z-a)^n = e^a \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!}$$

$\sqrt[n]{\frac{e^a}{n!}} \to 0$

$R = \infty$

$$= e^a + e^a(z-a) + \frac{e^a}{2!}(z-a)^2 + \frac{e^a}{3!}(z-a)^3 + \dots$$

converges for all z

$\boxed{7}$ Power series $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ is uniformly convergent in $|z| < r < R$ where R is radius of convergence

we need to show that $R > 3$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)! \cdot (n+1)!}{(2n+2)!} \cdot \frac{2n!}{n! \cdot n!} = \frac{(n+1)^2}{(2n+1)(2n+2)} \longrightarrow \frac{1}{4} = L^*$$

$$R = \frac{1}{L^*} = 4$$

$\boxed{8}$ $\sum_{n=1}^{\infty} \underbrace{\left[\frac{n+2}{5n-3}\right]^n}_{a_n} z^n$ $\qquad z_0 = 0$

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty} \frac{1}{\frac{n+2}{5n-3}} = 5$$

Series converges uniformly in any disk $|z| < r < 5$

HW 3 MATH 601

$\boxed{1}$ a) Verify that $y(x) = -\sin x + ax^2 + bx + c$ for any a, b, c constants is a solution to $y'' = \cos x$

b) Show that $y(x) = \tan(x+c)$ solves $y' = 1 + y^2$ for any constant c

$\boxed{2}$ Show that the given function solves the specified initial value problem

a) $y(x) = Ce^{x/2}$ $\quad \begin{cases} y' = \frac{1}{2} y \\ y(2) = 2 \end{cases}$

b) $y(x) = ce^{-x^2}$ $\quad \begin{cases} y' + 2xy = 0 \\ y(1) = 1/e \end{cases}$

$\boxed{3}$ $y = cx - c^2$ is a solution to the ODE $(y')^2 - xy' + y = 0$ for any c const.
Find a singular solution to the ODE (not given by $y = cx - c^2$) by rewriting the ODE using the quadratic formula.

$\boxed{4}$ Find all solutions of the following differential equations.

a) $yy' + 25x = 0$ b) $y' = ky^2$

c) $xy' = x + y$ (Hint $u = y/x$)

$\boxed{5}$ Solve the IVPs:

a) $\begin{cases} y' = 1 + 4y^2 \\ y(0) = 0 \end{cases}$ b $\begin{cases} y' = -x/y \\ y(1) = \sqrt{3} \end{cases}$

c) $\begin{cases} e^x y' = 2(x+1)y^2 \\ y(0) = 1/6 \end{cases}$