**PROOF** The proof parallels that of Theorem 8.

**(a)** Let $L = 1 - a^* < 1$. Then by the definition of a limit we have $\sqrt[n]{|z_n|} < q = 1 - \frac{1}{2}a^* < 1$ for all $n$ greater than some (sufficiently large) $N^*$. Hence $|z_n| < q^n < 1$ for all $n > N^*$. Absolute convergence of the series $z_1 + z_2 + \cdots$ now follows by the comparison with the geometric series.

**(b)** If $L > 1$, then we also have $\sqrt[n]{|z_n|} > 1$ for all sufficiently large $n$. Hence $|z_n| > 1$ for those $n$. Theorem 3 now implies that $z_1 + z_2 + \cdots$ diverges.

**(c)** Both the *divergent* harmonic series and the *convergent* series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$ give $L = 1$. This can be seen from $(\ln n)/n \to 0$ and

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$$\sqrt[n]{\frac{1}{n}} = \frac{1}{n^{1/n}} = \frac{1}{e^{(1/n)\ln n}} \to \frac{1}{e^0}, \qquad \sqrt[n]{\frac{1}{n^2}} = \frac{1}{n^{2/n}} = \frac{1}{e^{(2/n)\ln n}} \to \frac{1}{e^0}. \quad \blacksquare$$

## PROBLEM SET 15.1

### 1–10 SEQUENCES

Are the following sequences $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find their limit points. (Show the details of your work.)

1. $z_n = (-1)^n + i/2^n$
2. $z_n = e^{-n\pi i/4}$
3. $z_n = (-1)^n/(n + i)$
4. $z_n = (1 + i)^n$
5. $z_n = \mathrm{Ln}\,((2 + i)^n)$
6. $z_n = (3 + 4i)^n/n!$
7. $z_n = \sin(n\pi/4) + i^n$
8. $z_n = [(1 + 3i)/\sqrt{10}]^n$
9. $z_n = (0.9 + 0.1i)^{2n}$
10. $z_n = (5 + 5i)^{-n}$

11. Illustrate Theorem 1 by an example of your own.
12. **(Uniqueness of limit)** Show that if a sequence converges, its limit is unique.
13. **(Addition)** If $z_1, z_2, \cdots$ converges with the limit $l$ and $z_1^*, z_2^*, \cdots$ converges with the limit $l^*$, show that $z_1 + z_1^*, z_2 + z_2^*, \cdots$ converges with the limit $l + l^*$.
14. **(Multiplication)** Show that under the assumptions of Prob. 13 the sequence $z_1z_1^*, z_2z_2^*, \cdots$ converges with the limit $ll^*$.
15. **(Boundedness)** Show that a complex sequence is bounded if and only if the two corresponding sequences of the real parts and of the imaginary parts are bounded.

### 16–24 SERIES

Are the following series convergent or divergent? (Give a reason.)

16. $\sum_{n=0}^{\infty} \frac{(10 - 15i)^n}{n!}$
17. $\sum_{n=0}^{\infty} \frac{(-1)^n(1 + 2i)^{2n+1}}{(2n + 1)!}$
18. $\sum_{n=0}^{\infty} \frac{i^n}{n^2 - 2i}$
19. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
20. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$
21. $\sum_{n=1}^{\infty} \frac{i^n}{n}$

compare to $\frac{1}{n^2}$.

22. $\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}(1 + i)^n$
23. $\sum_{n=0}^{\infty} \frac{n - i}{3n + 2i}$
24. $\sum_{n=1}^{\infty} n^2\left(\frac{i}{3}\right)^n$

25. What is the difference between (7) and just stating $|z_{n+1}/z_n| < 1$?
26. Illustrate Theorem 2 by an example of your choice.
27. For what $n$ do we obtain the term of greatest absolute value of the series in Example 4? About how big is it? First guess, then calculate it by the Stirling formula in Sec. 24.4.
28. Give another example showing that Theorem 7 is more general than Theorem 8.
29. **CAS PROJECT. Sequences and Series. (a)** Write a program for graphing complex **sequences.** Apply it to sequences of your choice that have interesting "geometrical" properties (e.g., lying on an ellipse, spiraling toward its limit, etc.).

    **(b)** Write a program for computing and graphing numeric values of the first $n$ partial sums of a **series** of complex numbers. Use the program to experiment with the rapidity of convergence of series of your choice.
30. **TEAM PROJECT. Series. (a) Absolute convergence.** Show that if a series converges absolutely, it is convergent.

    **(b)** Write a short report on the basic concepts and properties of series of numbers, explaining in each case whether or not they carry over from real series (discussed in calculus) to complex series, with reasons given.

Formula (6) will not help if $L^*$ does not exist, but extensions of Theorem 2 are still possible, as we discuss in Example 6 below.

**EXAMPLE 5 Radius of Convergence**

By (6) the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n$ is

$$R = \lim_{n\to\infty}\left[\frac{(2n)!}{(n!)^2}\bigg/\frac{(2n+2)!}{((n+1)!)^2}\right] = \lim_{n\to\infty}\left[\frac{(2n)!}{(2n+2)!}\cdot\frac{((n+1)!)^2}{(n!)^2}\right] = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}.$$

The series converges in the open disk $|z-3i| < \frac{1}{4}$ of radius $\frac{1}{4}$ and center $3i$. ∎

**EXAMPLE 6 Extension of Theorem 2**

Find the radius of convergence $R$ of the power series

$$\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n = 3+\frac{1}{2}z+\left(2+\frac{1}{4}\right)z^2+\frac{1}{8}z^3+\left(2+\frac{1}{16}\right)z^4+\cdots.$$

***Solution.*** The sequence of the ratios $1/6$, $2(2+\frac{1}{4})$, $1/(8(2+\frac{1}{4}))$, $\cdots$ does not converge, so that Theorem 2 is of no help. It can be shown that

$$R = 1/\tilde{L}, \qquad \tilde{L} = \lim_{n\to\infty}\sqrt[n]{|a_n|}. \tag{6*}$$

This still does not help here, since $\{\sqrt[n]{|a_n|}\}$ does not converge because $\sqrt[n]{|a_n|} = \sqrt[n]{1/2^n} = 1/2$ for odd $n$, whereas for even $n$ we have

$$\sqrt[n]{|a_n|} = \sqrt[n]{2+1/2^n} \to 1 \qquad \text{as} \qquad n\to\infty,$$

so that $\sqrt[n]{|a_n|}$ has the two limit points $1/2$ and $1$. It can further be shown that

$$R = 1/\tilde{l}, \qquad \tilde{l} \text{ the greatest limit point of the sequence } \left\{\sqrt[n]{|a_n|}\right\}. \tag{6**}$$

Here $\tilde{l} = 1$, so that $R = 1$. *Answer.* The series converges for $|z| < 1$. ∎

**Summary.** Power series converge in an open circular disk or some even for every $z$ (or some only at the center, but they are useless); for the radius of convergence, see (6) or Example 6.

Except for the useless ones, power series have sums that are analytic functions (as we show in the next section); this accounts for their importance in complex analysis.

## PROBLEM SET 15.2

1. **(Powers missing)** Show that if $\Sigma\, a_n z^n$ has radius of convergence $R$ (assumed finite), then $\Sigma\, a_n z^{2n}$ has radius of convergence $\sqrt{R}$. Give examples.
2. **(Convergence behavior)** Illustrate the facts shown by Examples 1–3 by further examples of your own.

**3–18 RADIUS OF CONVERGENCE**

Find the center and the radius of convergence of the following power series. (Show the details.)

3. $\sum_{n=1}^{\infty}\frac{(z+i)^n}{n^2}$
4. $\sum_{n=0}^{\infty}\frac{n^n}{n!}(z+2i)^n$
5. $\sum_{n=0}^{\infty}\frac{n!}{n^n}(z+1)^n$
6. $\sum_{n=0}^{\infty}\frac{2^{100n}}{n!}z^n$
7. $\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^n z^n$
8. $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}z^{2n}$
9. $\sum_{n=0}^{\infty}(n-i)^n z^n$
10. $\sum_{n=0}^{\infty}\frac{(2z)^{2n}}{(2n)!}$
11. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}z^n$
12. $\sum_{n=0}^{\infty}\frac{4^n}{(1+i)^n}(z-5)^n$

13. $\sum_{n=2}^{\infty} n(n-1)(z-3+2i)^n$

14. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$

15. $\sum_{n=0}^{\infty} 2^n(z-i)^{4n}$

16. $\sum_{n=0}^{\infty} \left(\frac{2+3i}{5-i}\right)^n (z-\pi)^n$

17. $\sum_{n=0}^{\infty} \frac{n^4}{2^n} z^{2n}$

18. $\sum_{n=0}^{\infty} \frac{(4n)!}{2^n(n!)^4} (z+\pi i)^n$

19. **CAS PROJECT. Radius of Convergence.** Write a program for computing $R$ from (6), (6*), or (6**), in this order, depending on the existence of the limits needed. Test the program on series of your choice and such that all three formulas (6), (6*), and (6**) will come up.

20. **TEAM PROJECT. Radius of Convergence. (a)** Formula (6) for $R$ contains $|a_n/a_{n+1}|$, not $|a_{n+1}/a_n|$. How could you memorize this by using a qualitative argument?

**(b) Change of coefficients.** What happens to $R$ $(0 < R < \infty)$ if you (i) multiply all $a_n$ by $k \neq 0$, (ii) multiply $a_n$ by $k^n \neq 0$, (iii) replace $a_n$ by $1/a_n$?

**(c) Example 6** extends Theorem 2 to nonconvergent cases of $a_n/a_{n+1}$. Do you understand the principle of "mixing" by which Example 6 was obtained? Use this principle for making up further examples.

**(d)** Does there exist a power series in powers of $z$ that converges at $z = 30 + 10i$ and diverges at $z = 31 - 6i$? (Give reason.)

# 15.3 Functions Given by Power Series

The main goal of this section is to show that power series represent analytic functions (Theorem 5). Along our way we shall see that power series behave nicely under addition, multiplication, differentiation, and integration, which makes these series very useful in complex analysis.

To simplify the formulas in this section, we take $z_0 = 0$ and write

$$\sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

This is no restriction because a series in powers of $\hat{z} - z_0$ with any $z_0$ can always be reduced to the form (1) if we set $\hat{z} - z_0 = z$.

**Terminology and Notation.** If any given power series (1) has a nonzero radius of convergence $R$ (thus $R > 0$), its sum is a function of $z$, say $f(z)$. Then we write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z| < R). \tag{2}$$

We say that $f(z)$ *is* **represented** *by the power series* or that *it is* **developed** *in the power series.* For instance, the geometric series *represents* the function $f(z) = 1/(1-z)$ in the interior of the unit circle $|z| = 1$. (See Theorem 6 in Sec. 15.1.)

**Uniqueness of a Power Series Representation.** This is our next goal. It means that *a function $f(z)$ cannot be represented by two different power series with the same center.* We claim that if $f(z)$ can at all be developed in a power series with center $z_0$, the development is unique. This important fact is frequently used in complex analysis (as well as in calculus). We shall prove it in Theorem 2. The proof will follow from