PROBLEM SET 13.1

1. **(Powers of i)** Show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$ and $1/i = -i$, $1/i^2 = -1$, $1/i^3 = i, \cdots$.
2. **(Rotation)** Multiplication by i is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing z and iz and the angle of rotation for $z = 2 + 2i$, $z = -1 - 5i$, $z = 4 - 3i$.
3. **(Division)** Verify the calculation in (7).
4. **(Multiplication)** If the product of two complex numbers is zero, show that at least one factor must be zero.
5. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.
6. **(Laws for conjugates)** Verify (9) for $z_1 = 24 + 10i$, $z_2 = 4 + 6i$.

7–15 **COMPLEX ARITHMETIC**

Let $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$. Showing the details of your work, find (in the form $x + iy$):

7. $(5z_1 + 3z_2)^2$
8. $\bar{z}_1\bar{z}_2$
9. $\text{Re}\,(1/z_1^{\,2})$
10. $\text{Re}\,(z_2^{\,2})$, $(\text{Re}\,z_2)^2$
11. z_2/z_1
12. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$
13. $(4z_1 - z_2)^2$
14. $\bar{z}_1/z_1$, $z_1/\bar{z}_1$
15. $(z_1 + z_2)/(z_1 - z_2)$

16–19 Let $z = x + iy$. Find:

16. $\text{Im}\,z^3$, $(\text{Im}\,z)^3$
17. $\text{Re}\,(1/\bar{z})$
18. $\text{Im}\,[(1 + i)^8 z^2]$
19. $\text{Re}\,(1/\bar{z}^2)$

20. **(Laws of addition and multiplication)** Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1, \quad z_1z_2 = z_2z_1 \quad \textit{(Commutative laws)}$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3),$$

$$(z_1z_2)z_3 = z_1(z_2z_3) \quad \textit{(Associative laws)}$$

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \quad \textit{(Distributive law)}$$

$$0 + z = z + 0 = z,$$

$$z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

Problem set 13.2

7. $\dfrac{-6+5i}{3i}$

8. $\dfrac{2+3i}{5+4i}$

9–15 PRINCIPAL ARGUMENT

Determine the principal value of the argument.

9. $-1-i$

10. $-20+i,\ -20-i$

11. $4 \pm 3i$

12. $-\pi^2$

13. $7 \pm 7i$

14. $(1+i)^{12}$

15. $(9+9i)^3$

16–20 CONVERSION TO $x + iy$

Represent in the form $x + iy$ and graph it in the complex plane.

16. $\cos \frac{1}{2}\pi + i \sin (\pm \frac{1}{2}\pi)$

17. $3(\cos 0.2 + i \sin 0.2)$

18. $4(\cos \frac{1}{3}\pi \pm i \sin \frac{1}{3}\pi)$

19. $\cos(-1) + i \sin(-1)$

20. $12(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi)$

21–25 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt{-i}$

22. $\sqrt[8]{1}$

23. $\sqrt[4]{-1}$

24. $\sqrt[3]{3+4i}$

25. $\sqrt[5]{-1}$

26. **TEAM PROJECT. Square Root.** **(a)** Show that $w = \sqrt{z}$ has the values

$$w_1 = \sqrt{r}\left[\cos\frac{\theta}{2} + i \sin\frac{\theta}{2}\right],$$

$$(18)\quad w_2 = \sqrt{r}\left[\cos\left(\frac{\theta}{2}+\pi\right) + i \sin\left(\frac{\theta}{2}+\pi\right)\right] = -w_1.$$

(b) Obtain from (18) the often more practical formula

$$(19)\quad \sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z|+x)} + (\operatorname{sign} y)\, i \sqrt{\tfrac{1}{2}(|z|-x)}\right]$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $4i$, $16 - 30i$, and $9 + 8\sqrt{7}\,i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

27–30 EQUATIONS

Solve and graph all solutions, showing the details:

27. $z^2 - (8-5i)z + 40 - 20i = 0$ (Use (19).)

28. $z^4 + (5-14i)z^2 - (24+10i) = 0$

29. $8z^2 - (36-6i)z + 42 - 11i = 0$

30. $z^4 + 16 = 0$. Then use the solutions to factor $z^4 + 16$ into quadratic factors with *real* coefficients.

31. **CAS PROJECT. Roots of Unity and Their Graphs.** Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

32–35 INEQUALITIES AND AN EQUATION

Verify or prove as indicated.

32. **(Re and Im)** Prove $|\text{Re}\, z| \leqq |z|$, $|\text{Im}\, z| \leqq |z|$.

33. **(Parallelogram equality)** Prove

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Explain the name.

34. **(Triangle inequality)** Verify (6) for $z_1 = 4 + 7i$, $z_2 = 5 + 2i$.

35. **(Triangle inequality)** Prove (6).

PROBLEM SET 13.3

1–10 CURVES AND REGIONS OF PRACTICAL INTEREST

Find and sketch or graph the sets in the complex plane given by

1. $|z - 3 - 2i| = \frac{4}{3}$
2. $1 \leqq |z - 1 + 4i| \leqq 5$
3. $0 < |z - 1| < 1$
4. $-\pi < \text{Re}\, z < \pi$
5. $\text{Im}\, z^2 = 2$
6. $\text{Re}\, z > -1$
7. $|z + 1| = |z - 1|$
8. $|\text{Arg}\, z| \leqq \frac{1}{4}\pi$
9. $\text{Re}\, z \leqq \text{Im}\, z$
10. $\text{Re}\, (1/z) < 1$

PROBLEM SET 13.5

1. Using the Cauchy–Riemann equations, show that e^z is entire.

2–8 **Values of e^z.** Compute e^z in the form $u + iv$ and $|e^z|$, where z equals:

2. $3 + \pi i$
3. $1 + 2i$
4. $\sqrt{2} - \frac{1}{2}\pi i$
5. $7\pi i/2$
6. $(1 + i)\pi$
7. $0.8 - 5i$
8. $9\pi i/2$

9–12 **Real and Imaginary Parts.** Find Re and Im of:

9. e^{-2z}
10. e^{z^3}
11. e^{z^2}
12. $e^{1/z}$

13–17 **Polar Form.** Write in polar form:

13. $\sqrt{i}$
14. $1 + i$
15. $\sqrt[n]{z}$
16. $3 + 4i$
17. -9

18–21 **Equations.** Find all solutions and graph some of them in the complex plane.

18. $e^{3z} = 4$
19. $e^z = -2$
20. $e^z = 0$
21. $e^z = 4 - 3i$

22. **TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity.** Show that e^z is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos ky + i \sin ky)$? (Use the Cauchy–Riemann equations.)

(b) Special values. Find all z such that (i) e^z is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

(c) Harmonic function. Show that $u = e^{xy} \cos(x^2/2 - y^2/2)$ is harmonic and find a conjugate.

(d) Uniqueness. It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this using the Cauchy–Riemann equations.

PROBLEM SET 13.6

1. Prove that $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are entire functions.
2. Verify by differentiation that $\operatorname{Re} \cos z$ and $\operatorname{Im} \sin z$ are harmonic.

3–6 FORMULAS FOR HYPERBOLIC FUNCTIONS

Show that

3. $$\cosh z = \cosh x \cos y + i \sinh x \sin y$$
$$\sinh z = \sinh x \cos y + i \cosh x \sin y.$$

4. $$\cosh (z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$
$$\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$$

5. $\cosh^2 z - \sinh^2 z = 1$
6. $\cosh^2 z + \sinh^2 z = \cosh 2z$

7–15 Function Values.

Compute (in the form $u + iv$)

7. $\cos (1 + i)$
8. $\sin (1 + i)$
9. $\sin 5i$, $\cos 5i$
10. $\cos 3\pi i$
11. $\cosh (-2 + 3i)$, $\cos (-3 - 2i)$
12. $-i \sinh (-\pi + 2i)$, $\sin (2 + \pi i)$
13. $\cosh (2n + 1)\pi i$, $n = 1, 2, \cdots$
14. $\sinh (4 - 3i)$
15. $\cosh (4 - 6\pi i)$
16. **(Real and imaginary parts)** Show that

$$\operatorname{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$

$$\operatorname{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

17–21 Equations.

Find all solutions of the following equations.

17. $\cosh z = 0$
18. $\sin z = 100$
19. $\cos z = 2i$
20. $\cosh z = -1$
21. $\sinh z = 0$
22. Find all z for which (a) $\cos z$, (b) $\sin z$ has real values.

23–25 Equations and Inequalities.

Using the definitions, prove:

23. $\cos z$ is even, $\cos (-z) = \cos z$, and $\sin z$ is odd, $\sin (-z) = -\sin z$.
24. $|\sinh y| \leqq |\cos z| \leqq \cosh y$, $|\sinh y| \leqq |\sin z| \leqq \cosh y$. Conclude that the complex cosine and sine are not bounded in the whole complex plane.
25. $\sin z_1 \cos z_2 = \frac{1}{2}[\sin (z_1 + z_2) + \sin (z_1 - z_2)]$

PROBLEM SET 13.7

1–9 **Principal Value Ln z.** Find Ln z when z equals:

1. -10
2. $2 + 2i$
3. $2 - 2i$
4. $-5 \pm 0.1i$
5. $-3 - 4i$
6. -100
7. $0.6 + 0.8i$
8. $-ei$
9. $1 - i$

10–16 **All Values of ln z.** Find all values and graph some of them in the complex plane.

10. $\ln 1$
11. $\ln(-1)$
12. $\ln e$
13. $\ln(-6)$
14. $\ln(4 + 3i)$
15. $\ln(-e^{-i})$
16. $\ln(e^{3i})$

17. Show that the set of values of $\ln(i^2)$ differs from the set of values of $2 \ln i$.

18–21 **Equations.** Solve for z:

18. $\ln z = (2 - \frac{1}{2}i)\pi$
19. $\ln z = 0.3 + 0.7i$
20. $\ln z = e - \pi i$
21. $\ln z = 2 + \frac{1}{4}\pi i$

22–28 **General Powers.** Showing the details of your work, find the principal value of:

22. i^{2i}, $(2i)^i$
23. 4^{3+i}
24. $(1 - i)^{1+i}$
25. $(1 + i)^{1-i}$
26. $(-1)^{1-2i}$
27. $i^{1/2}$
28. $(3 - 4i)^{1/3}$

29. How can you find the answer to Prob. 24 from the answer to Prob. 25?

30. **TEAM PROJECT. Inverse Trigonometric and Hyperbolic Functions.** By definition, the **inverse sine** $w = \arcsin z$ is the relation such that $\sin w = z$. The **inverse cosine** $w = \arccos z$ is the relation such that $\cos w = z$. The **inverse tangent, inverse cotangent, inverse hyperbolic sine,** etc., are defined and denoted in a similar fashion. (Note that all these relations are ***multivalued.***) Using $\sin w = (e^{iw} - e^{-iw})/(2i)$ and similar representations of $\cos w$, etc., show that

(a) $\arccos z = -i \ln (z + \sqrt{z^2 - 1})$

(b) $\arcsin z = -i \ln (iz + \sqrt{1 - z^2})$

(c) $\operatorname{arccosh} z = \ln (z + \sqrt{z^2 - 1})$

(d) $\operatorname{arcsinh} z = \ln (z + \sqrt{z^2 + 1})$

(e) $\arctan z = \dfrac{i}{2} \ln \dfrac{i + z}{i - z}$

(f) $\operatorname{arctanh} z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$

(g) Show that $w = \arcsin z$ is infinitely many-valued, and if w_1 is one of these values, the others are of the form $w_1 \pm 2n\pi$ and $\pi - w_1 \pm 2n\pi$, $n = 0, 1, \cdots$. (The *principal value of* $w = u + iv = \arcsin z$ is defined to be the value for which $-\pi/2 \leqq u \leqq \pi/2$ if $v \geqq 0$ and $-\pi/2 < u < \pi/2$ if $v < 0$.)