My solution for second mid-term practice exam. Math 320

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December 30, 2019

Contents

0.0.1 Problem 1

Question: Given Matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 3 \\ \frac{-1}{3} & \frac{-3}{2} & -1 \\ \frac{-1}{2} & \frac{-1}{4} & \frac{-3}{2} \end{pmatrix}$$

for what vectors $\bar{b}$ does $A\bar{x} = \bar{b}$ have a solution?

answer Let $\bar{b} = (b_1, b_2, b_3)$. We start by setting up the augmented matrix. The augmented matrix is

$$\begin{pmatrix} 1 & \frac{1}{2} & 3 & b_1 \\ \frac{-1}{3} & \frac{-3}{2} & -1 & b_2 \\ \frac{-1}{2} & \frac{-1}{4} & \frac{-3}{2} & b_3 \end{pmatrix}$$

Applying row operation: $R_2 = R_2 + \frac{1}{3}R_1$ gives

$$\begin{pmatrix} 1 & \frac{1}{2} & 3 & b_1 \\ 0 & -\frac{4}{3} & 0 & b_2 + \frac{b_1}{3} \\ \frac{-1}{2} & \frac{-1}{4} & \frac{-3}{2} & b_3 \end{pmatrix}$$

$R_3 = R_3 + \frac{1}{2}R_1$ gives

$$\begin{pmatrix} 1 & \frac{1}{2} & 3 & b_1 \\ 0 & -\frac{4}{3} & 0 & b_2 + \frac{b_1}{3} \\ 0 & 0 & 0 & b_3 + \frac{1}{2}b_1 \end{pmatrix}$$

The above is Echelon form. Therefore, from last row, we see that $0x_3 = b_3 + \frac{1}{2}b_1$. For solution to exist, we need $b_3 + \frac{1}{2}b_1 = 0$ or $b_3 = -\frac{1}{2}b_1$. Hence any vector b where the third entry is $-\frac{1}{2}$ the first entry, will result in $A\bar{x} = \bar{b}$ having (infinite) solutions. So $\bar{b}$ needs to have this form

$$\begin{aligned} \bar{b} &= \begin{pmatrix} b_1 \\ b_2 \\ \frac{-1}{2}b_1 \end{pmatrix} \\ &= b_1 \begin{pmatrix} 1 \\ 0 \\ \frac{-1}{2} \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

0.0.2 Problem 2

part a For what vector $\bar{b}$ does $A\bar{x} = \bar{b}$ have solution

$$A = \begin{pmatrix} 2 & -1 & \frac{1}{2} \\ 3 & 1 & 2 \\ 0 & 6 & 3 \end{pmatrix}$$

answer

Let $\bar{b} = (b_1, b_2, b_3)$ then the augmented matrix is

$$\begin{pmatrix} 2 & -1 & \frac{1}{2} & b_1 \\ 3 & 1 & 2 & b_2 \\ 0 & 6 & 3 & b_3 \end{pmatrix}$$

Applying row operations: $R_2 = R_2 - \frac{3}{2}R_1$ gives

$$\begin{pmatrix} 2 & -1 & \frac{1}{2} & b_1 \\ 0 & \frac{5}{2} & \frac{5}{4} & b_2 - \frac{3}{2}b_1 \\ 0 & 6 & 3 & b_3 \end{pmatrix}$$

$R_3 = R_3 - \frac{6}{\left(\frac{5}{2}\right)} R_2$ gives

$$\begin{pmatrix} 2 & -1 & \frac{1}{2} & b_1 \\ 0 & \frac{5}{2} & \frac{5}{4} & b_2 - \frac{3}{2} b_1 \\ 0 & 0 & 0 & \frac{18}{5} b_1 - \frac{12}{5} b_2 + b_3 \end{pmatrix}$$

The above is Echelon form. Last row says that $0x_3 = \frac{18}{5} b_1 - \frac{12}{5} b_2 + b_3$. Therefore for solution to exist, we need

$$\frac{18}{5} b_1 - \frac{12}{5} b_2 + b_3 = 0$$

This will generate infinite number of solutions. Any $\bar{b}$ vector of 3 elements where the above constraint is satisfied, will make $A\bar{x} = \bar{b}$ have (infinite) number of solutions. Solving for b_1 in terms of b_2, b_3

$$b_1 = \frac{12}{18} b_2 - \frac{5}{18} b_3$$

Hence $\bar{b}$ can be written as

$$\bar{b} = \begin{pmatrix} \frac{12}{18} b_2 - \frac{5}{18} b_3 \\ b_2 \\ b_3 \end{pmatrix}$$

One such example of $\bar{b}$ can be

$$\bar{b} = \begin{pmatrix} \frac{7}{18} \\ 1 \\ 1 \end{pmatrix}$$

<u>part b</u> Find all possible solutions (or no solution) for

$$\bar{b} = \begin{pmatrix} 0 \\ \frac{12}{5} \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{12}{5} \end{pmatrix}$$

We need to first check if these vectors meet the constraint found in part (a), which is $b_1 = \frac{12}{18} b_2 - \frac{5}{18} b_3$. For the first vector given, we get

$$0 \stackrel{?}{=} \frac{12}{18} \left(\frac{12}{5}\right) - \frac{5}{18} (1)$$
$$0 \stackrel{?}{=} \frac{119}{90}$$

Which is not valid. Therefore, $\bar{b} = \begin{pmatrix} 0 \\ \frac{12}{5} \\ 1 \end{pmatrix}$ will produce <u>no solution</u> for when used in $A\bar{x} = \bar{b}$. Now we check the second vector to see if it meets the constraint or not.

$$0 \stackrel{?}{=} \frac{12}{18} (1) - \frac{5}{18} \left(\frac{12}{5}\right)$$
$$0 \stackrel{?}{=} 0$$

Yes. It satisfies the constraint. Hence this vector will produce <u>solution</u> for $A\bar{x} = \bar{b}$. To find the solution, we plugin this $\bar{b}$ vector and solve for x

$$\begin{pmatrix} 1 & \frac{1}{2} & 3 \\ \frac{-1}{3} & \frac{-3}{2} & -1 \\ \frac{-1}{2} & \frac{-1}{4} & \frac{-3}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \frac{12}{5} \end{pmatrix}$$

Following the row operation we did above, the output is

$$\begin{pmatrix} 2 & -1 & \frac{1}{2} & b_1 \\ 0 & \frac{5}{2} & \frac{5}{4} & b_2 - \frac{3}{2}b_1 \\ 0 & 0 & 0 & \frac{18}{5}b_1 - \frac{12}{5}b_2 + b_3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & \frac{5}{4} & 1 \\ 0 & 0 & 0 & -\frac{12}{5}(1) + \frac{12}{5} \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & \frac{5}{4} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence from last row $x_3 = t$, and from second row $\frac{5}{2}x_2 + \frac{5}{4}t = 1$ or $x_2 = \frac{2}{5} - \frac{1}{2}t$ and from first row $2x_1 - x_2 + \frac{1}{2}x_3 = 0$ or $2x_1 = \left(\frac{2}{5} - \frac{1}{2}t\right) - \frac{1}{2}t$ hence solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} - \frac{t}{2} \\ \frac{2}{5} - \frac{1}{2}t \\ t \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2}t \\ 1 \end{pmatrix}$$

0.0.3 Problem 3

Consider $A\bar{x} = \bar{b}$ for

$$A = \begin{pmatrix} \frac{2}{3} & a_{12} & -2 \\ -\frac{1}{5} & \frac{-1}{3} & \frac{3}{5} \\ \frac{1}{2} & \frac{5}{6} & \frac{-3}{2} \end{pmatrix}$$

(a) for what values of a_{12} is A non-singular? (b) For what values of a_{12} is A singular? (c) In all cases of A singular, analyze the system $A\bar{x} = \bar{b}$. For what vectors $\bar{b}$ lead to solution $\bar{x}$? What are those solutions?

Answer (a). Expanding along first row gives

$$\begin{aligned} |A| &= \frac{2}{3}A_{11} + a_{12}A_{12} - 2A_{13} \\ &= \frac{2}{3}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} - 2(-1)^{1+3}M_{13} \\ &= \frac{2}{3}M_{11} - a_{12}M_{12} - 2M_{13} \\ &= \frac{2}{3}\begin{vmatrix} \frac{-1}{3} & \frac{3}{5} \\ \frac{5}{6} & \frac{-3}{2} \end{vmatrix} - a_{12}\begin{vmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{1}{2} & \frac{-3}{2} \end{vmatrix} - 2\begin{vmatrix} -\frac{1}{5} & \frac{-1}{3} \\ \frac{1}{2} & \frac{5}{6} \end{vmatrix} \\ &= \frac{2}{3}(0) - a_{12}(0) - 2(0) \\ &= 0a_{12} \end{aligned}$$

Therefore, there are no values of a_{12} will make A non-singular, since anything times zero is zero.

(b) This follows from part (a). For any value a_{12}, the matrix A remains singular.

(c) Let $\bar{b} = (b_1, b_2, b_3)$, then the augmented matrix is

$$\begin{pmatrix} \frac{2}{3} & a_{12} & -2 & b_1 \\ -\frac{1}{5} & \frac{-1}{3} & \frac{3}{5} & b_2 \\ \frac{1}{2} & \frac{5}{6} & \frac{-3}{2} & b_3 \end{pmatrix}$$

$R_2 = R_2 - \left(\frac{-\frac{1}{5}}{\frac{2}{3}}\right)R_1$ gives

$$\begin{pmatrix} \frac{2}{3} & a_{12} & -2 & b_1 \\ 0 & \frac{3}{10}a_{12} - \frac{1}{3} & 0 & \frac{3}{10}b_1 + b_2 \\ \frac{1}{2} & \frac{5}{6} & \frac{-3}{2} & b_3 \end{pmatrix}$$

$R_3 = R_3 - \frac{\frac{1}{2}}{\frac{2}{3}} R_1$ gives

$$\begin{pmatrix} \frac{2}{3} & a_{12} & -2 & b_1 \\ 0 & \frac{3}{10}a_{12} - \frac{1}{3} & 0 & \frac{3}{10}b_1 + b_2 \\ 0 & \frac{5}{6} - \frac{3}{4}a_{12} & 0 & b_3 - \frac{3}{4}b_1 \end{pmatrix}$$

$R_3 = R_3 - \frac{\frac{5}{6} - \frac{3}{4}a_{12}}{\frac{3}{10}a_{12} - \frac{1}{3}} R_2$ gives

$$\begin{pmatrix} \frac{2}{3} & a_{12} & -2 & b_1 \\ 0 & \frac{3}{10}a_{12} - \frac{1}{3} & 0 & \frac{3}{10}b_1 + b_2 \\ 0 & 0 & 0 & \frac{5}{2}b_2 + b_3 \end{pmatrix}$$

From last row, we see that $0x_3 = \frac{5}{2}b_2 + b_3$. Hence we need (for infinite solutions) to have the constraint

$$\frac{5}{2}b_2 + b_3 = 0$$
$$b_2 = -\frac{2}{5}b_3$$

In which case we assume $x_3 = t$ in this case (parameter). The second row says that

$$\left(\frac{3}{10}a_{12} - \frac{1}{3}\right)x_2 = \frac{3}{10}b_1 + b_2$$

Here we have to consider the case where $a_{12} = \frac{10}{9}$ (which can happen, since a_{12} can be any value for A singular). In this case, we end up with $0x_2 = \frac{3}{10}b_1 + b_2$. Then now, for solution to exist, we need $\frac{3}{10}b_1 + b_2 = 0$ or $b_1 = -\frac{10}{3}b_2$ and now we set $x_2 = s$, second parameter.

On the other hand, if $a_{12} \neq \frac{10}{9}$ then this leads to $\left(\frac{3}{10}a_{12} - \frac{1}{3}\right)x_2 = \frac{3}{10}b_1 + b_2$ and now $x_2 = \frac{\frac{3}{10}b_1 + b_2}{\frac{3}{10}a_{12} - \frac{1}{3}}$. Therefore in summary

$$x_2 = \begin{cases} s & a_{12} = \frac{10}{9} \text{ and} \frac{3}{10}b_1 + b_2 = 0 \\ \frac{\frac{3}{10}b_1 + b_2}{\frac{3}{10}a_{12} - \frac{1}{3}} & a_{12} \neq \frac{10}{9} \end{cases}$$

Finally, first row gives

$$\frac{2}{3}x_1 + a_{12}x_2 - 2x_3 = b_1$$
$$x_1 = b_1 - a_{12}x_2 + 2x_3$$
$$= \frac{3}{2}b_1 - \frac{3}{2}a_{12}x_2 + 3t$$

If $a_{12} = \frac{10}{9}$ and $\frac{3}{10}b_1 + b_2 = 0$ then $x_2 = s$ and above becomes

$$x_1 = \frac{3}{2}b_1 - \frac{3}{2}\left(\frac{10}{9}\right)s + 3t$$
$$= 3t - \frac{5}{3}s + \frac{3}{2}b_1$$

If $a_{12} \neq \frac{10}{9}$ then $x_2 = \frac{\frac{3}{10}b_1 + b_2}{\frac{3}{10}a_{12} - \frac{1}{3}}$ and x_1 becomes

$$x_1 = \frac{3}{2}b_1 - \frac{3}{2}a_{12}\left(\frac{\frac{3}{10}b_1 + b_2}{\frac{3}{10}a_{12} - \frac{1}{3}}\right) + 3t$$

Therefore in summary

$$x_1 = \begin{cases} 3t - \frac{5}{3}s + \frac{3}{2}b_1 & a_{12} = \frac{10}{9} \text{ and} \frac{3}{10}b_1 + b_2 = 0 \\ \frac{3}{2}b_1 - \frac{3}{2}a_{12}\left(\frac{\frac{3}{10}b_1 + b_2}{\frac{3}{10}a_{12} - \frac{1}{3}}\right) + 3t & a_{12} \neq \frac{10}{9} \end{cases}$$

Hence solution vector is,

for case $a_{12}=\frac{10}{9}$ and $\frac{3}{10}b_1+b_2=0$ and $\frac{5}{2}b_2+b_3=0$ then solution is

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}3t-\frac{5}{3}s+\frac{3}{2}b_1\\s\\t\end{pmatrix}=\begin{pmatrix}3t-\frac{5}{3}s+2b_3\\s\\t\end{pmatrix}$$
$$=t\begin{pmatrix}3\\0\\1\end{pmatrix}+s\begin{pmatrix}-\frac{5}{3}\\1\\0\end{pmatrix}+\begin{pmatrix}2b_3\\0\\0\end{pmatrix}$$

And the $\bar{b}$ vector now is

$$\bar{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}-\frac{10}{3}b_2\\-\frac{2}{5}b_3\\b_3\end{pmatrix}=\begin{pmatrix}\frac{4}{3}b_3\\-\frac{2}{5}b_3\\b_3\end{pmatrix}$$
$$=b_3\begin{pmatrix}\frac{4}{3}\\-\frac{2}{5}\\1\end{pmatrix}$$

For case $a_{12}\neq\frac{10}{9}$ and $\frac{5}{2}b_2+b_3=0$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}\frac{3}{2}b_1-\frac{3}{2}a_{12}\left(\frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}}\right)+3t\\\frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}}\\t\end{pmatrix}$$
$$=t\begin{pmatrix}3\\0\\t\end{pmatrix}+\begin{pmatrix}\frac{3}{2}b_1-\frac{3}{2}a_{12}\left(\frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}}\right)\\\frac{\frac{3}{10}b_1+b_2}{\frac{3}{10}a_{12}-\frac{1}{3}}\\0\end{pmatrix}$$

And the $\bar{b}$ vector now is

$$\bar{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}b_1\\-\frac{2}{5}b_3\\b_3\end{pmatrix}$$
$$=b_1\begin{pmatrix}1\\0\\0\end{pmatrix}+b_3\begin{pmatrix}1\\-\frac{2}{5}\\1\end{pmatrix}$$

0.0.4 Problem 4

Given that two vectors $\bar{u},\bar{v}$ are L.I., are $3\bar{u}-5\bar{v}$ and $\bar{v}$ L.I. or L.D.? prove your answer.

answer

The two vectors are L.I. if the only solution to

$$c_1(3\bar{u}-5\bar{v})+c_2(\bar{v})=\bar{0}$$

is $c_1=0,c_2=0$. Therefore

$$\begin{aligned}c_1(3\bar{u}-5\bar{v})+c_2(\bar{v})&=3c_1\bar{u}-5c_1\bar{v}+c_2\bar{v}\\&=3c_1\bar{u}+\bar{v}(c_2-5c_1)\end{aligned}\tag{1}$$

Let

$$\begin{aligned}3c_1&=k_1\\c_2-5c_1&=k_2\end{aligned}\tag{2}$$

And (1) becomes

$$c_1(3\bar{u}-5\bar{v})+c_2(\bar{v})=k_1\bar{u}+k_2\bar{v}$$

But $\bar{u},\bar{v}$ are L.I., hence $k_1\bar{u}+k_2\bar{v}=\bar{0}$ implies that $k_1=k_2=0$. This means (from (2)) that

$$\begin{aligned} 3c_1 &= 0 \\ c_2 - 5c_1 &= 0 \end{aligned}$$

First equation gives $c_1=0$. The second equation now gives $c_2=0$. Hence this shows that $3\bar{u}-5\bar{v}$ and $\bar{v}$ are L.I.

0.0.5 Problem 5

Are the following statements true or false? If false, correct it.

1. Square matrix with two identical rows is row equivalent to identity matrix
2. Inverse of square matrix A exists if A is row equivalent to identity matrix I with the same dimension.
3. Determinant of upper triangle square matrix is sum of diagonal elements.

Answer

1. False. Since two rows are identical, the matrix is singular which means there are no row operations which leads to reduced Echelon form.
2. True.
3. False. Determinant of upper triangle square matrix is product (not sum) of diagonal elements.

0.0.6 Problem 6

Prove property 4 of the seven properties of determinants.

Answer

Property 4 says that if A,B,C are identical except for one row i, and that row is such that $A(i)+B(i)=C(i)$ then $|A|+|B|=|C|$

Let the three matrices be

$$A=\begin{pmatrix}\times & \times & \times & \times\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ \times & \times & \times & \times\\ \times & \times & \times & \times\end{pmatrix},B=\begin{pmatrix}\times & \times & \times & \times\\ b_{i1} & b_{i2} & \cdots & b_{in}\\ \times & \times & \times & \times\\ \times & \times & \times & \times\end{pmatrix},C=\begin{pmatrix}\times & \times & \times & \times\\ c_{i1} & c_{i2} & \cdots & c_{in}\\ \times & \times & \times & \times\\ \times & \times & \times & \times\end{pmatrix}$$

Where in the above, the i^{th} is shown. We are also told that $A(i)+B(i)=C(i)$ which implies

$$\begin{aligned} a_{i1}+b_{i1} &= c_{i1} \\ a_{i2}+b_{i2} &= c_{i2} \\ \vdots &= \vdots \\ a_{in}+b_{in} &= c_{in} \end{aligned} \tag{1}$$

Taking the determinant of each matrix, and expanding along the i^{th} row gives

$$|A| = a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in}$$

Similarly for B and C

$$|B| = b_{i1}B_{i1}+b_{i2}B_{i2}+\cdots+b_{in}B_{in}$$

And

$$|C| = c_{i1}C_{i1}+c_{i2}C_{i2}+\cdots+c_{in}C_{in}$$

Where But since $A_{ij}=B_{ij}=C_{ij}$ is the submatrix for all matrices, we are told the matrices are identical in all other rows (and columns) except for the i^{th} row. Then we can just use any one of them. Lets use C_{ij} for each case. Therefore from above, we can write

$$\begin{aligned} |A|+|B| &= (a_{i1}C_{i1}+a_{i2}C_{i2}+\cdots+a_{in}C_{in})+(b_{i1}C_{i1}+b_{i2}C_{i2}+\cdots+b_{in}C_{in}) \\ &= (a_{i1}+b_{i1})C_{i1}+(a_{i2}+b_{i2})C_{i2}+\cdots+(a_{in}+b_{in})C_{in} \end{aligned} \tag{2}$$

Substituting (1) into (2) gives

$$\begin{aligned} |A|+|B| &= c_{i1}C_{i1}+c_{i2}C_{i2}+\cdots+c_{in}C_{in} \\ &= |C| \end{aligned}$$

QED.

0.0.7 Problem 7

Consider matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

1. Find condition on a_{32}, a_{33} such that A^{-1} exist.
2. Find value of determinant for $a_{32} = 1$ and $a_{33} = -2$. How many columns of A are independent for $a_{32} = 1, a_{33} = -2$?
3. For $a_{32} = 5, a_{33} = -4$, can $p^T = (3, 5, 0)$ be expressed as linear combination of columns of A?
4. Find value of the determinant for $a_{32} = 5, a_{33} = -4$. How many columns of A are independent?

Answer

(1) Expanding along last row gives

$$\begin{aligned} |A| &= a_{32}A_{32} + a_{33}A_{33} \\ &= a_{32}(-1)^{3+2} M_{32} + a_{33}(-1)^{3+3} M_{33} \\ &= -a_{32}M_{32} + a_{33}M_{33} \\ &= -a_{32}\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} + a_{33}\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \\ &= -4a_{32} - 5a_{33} \end{aligned}$$

Hence for A^{-1} to exist, we want $|A| \neq 0$, which means we want $-4a_{32} - 5a_{33} \neq 0$ or

$$4a_{32} + 5a_{33} \neq 0$$

(2) When $a_{32} = 1$ and $a_{33} = -2$, the matrix becomes

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix}$$

Expanding along last row gives

$$\begin{aligned} |A| &= a_{32}A_{32} + a_{33}A_{33} \\ &= (-1)^{3+2} M_{32} - 2(-1)^{3+3} M_{33} \\ &= -M_{32} - 2M_{33} \\ &= -\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \\ &= -4 + 10 \\ &= 6 \end{aligned}$$

Since $|A| \neq 0$, Hence all columns are L.I. (Matrix is full rank).

(3) For $a_{32} = 5, a_{33} = -4$ the matrix becomes

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & 5 & -4 \end{pmatrix}$$

To find if $p^T = (3, 5, 0)$ can be expressed are linear combinations of columns of A, implies

$$c_1\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

Has solution in c. The above can be written as

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & 5 & -4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

Setting up the augmented matrix gives

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 6 \\ 0 & 5 & -4 & 0 \end{pmatrix}$$

$R_2 = R_2 - 2R_1$ gives

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 4 & 0 \\ 0 & 5 & -4 & 0 \end{pmatrix}$$

$R_3 = R_3 + R_2$ gives

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, last row gives $0c_3 = 0$. Hence c_3 can be any value, say t. Second row gives

$$\begin{aligned} -5c_2 + 4c_3 &= 0 \\ c_2 &= \frac{4}{5}t \end{aligned}$$

And from first row

$$\begin{aligned} c_1 + 2c_2 - c_3 &= 3 \\ c_1 &= 3 - 2c_2 + c_3 \\ &= 3 - 2\frac{4}{5}t + t \\ &= 3 - \frac{9}{5}t \end{aligned}$$

Hence there are infinite solutions.

$$\begin{aligned} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} &= \begin{pmatrix} 3 - \frac{9}{5}t \\ \frac{4}{5}t \\ t \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{9}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix} \end{aligned}$$

For any t we can find linear combination of columns of A which gives p^T. For example, using $t = 0$ results in solution $c_1 = 3, c_2 = 0, c_3 = 0$. To verify

$$\begin{aligned} c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} &= 3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \\ &= p \end{aligned}$$

(4). For $a_{32} = 5, a_{33} = -4$ the matrix becomes

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & 5 & -4 \end{pmatrix}$$

The determinant is zero, this is because from part (3), we ended up with one zero pivot in Echelon form, which implies $|A| = 0$. Since solution has one parameter family, and matrix is 3×3, then there are now 2 L.I. columns in A. This is the same as saying rank of A is 2.

0.0.8 Problem 8

Consider 3×3 matrix A. Show that $|A|^T = |A|$

Answer Let A be

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

Expanding along first row gives

$$\begin{aligned} |A| &= a_{11}(-1)^{1+1}M_{11}+a_{12}(-1)^{1+2}M_{12}+a_{13}(-1)^{1+3}M_{13}\\ &= a_{11}\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}\\ &= a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31})\\ &= a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31} \qquad (1)\end{aligned}$$

Now

$$A^T=\begin{pmatrix} a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{pmatrix}$$

Expanding along first row gives

$$\begin{aligned} \left|A^T\right| &= a_{11}(-1)^{1+1}M_{11}+a_{21}(-1)^{1+2}M_{12}+a_{31}(-1)^{1+3}M_{13}\\ &= a_{11}\begin{vmatrix} a_{22} & a_{32}\\ a_{23} & a_{33}\end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{32}\\ a_{13} & a_{33}\end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{22}\\ a_{13} & a_{23}\end{vmatrix}\\ &= a_{11}(a_{22}a_{33}-a_{32}a_{23}) - a_{21}(a_{12}a_{33}-a_{32}a_{13}) + a_{31}(a_{12}a_{23}-a_{22}a_{13})\\ &= a_{11}a_{22}a_{33}-a_{11}a_{32}a_{23}-a_{21}a_{12}a_{33}+a_{21}a_{32}a_{13}+a_{31}a_{12}a_{23}-a_{31}a_{22}a_{13} \qquad (2)\end{aligned}$$

Examining (1) and (2), we see they are the same. Hence $|A|=\left|A^T\right|$

0.0.9 Problem 9

Find $|A|$

$$A=\begin{pmatrix} 1 & 2 & -2 & 5\\ -1 & 2 & 3 & 4\\ 1 & 3 & 1 & -2\\ -1 & -3 & 0 & -4\end{pmatrix}$$

<u>Answer</u>

$R_2=R_2+R_1$ gives

$$A=\begin{pmatrix} 1 & 2 & -2 & 5\\ 0 & 4 & 1 & 9\\ 1 & 3 & 1 & -2\\ -1 & -3 & 0 & -4\end{pmatrix}$$

$R_3=R_3-R_1$ gives

$$A=\begin{pmatrix} 1 & 2 & -2 & 5\\ 0 & 4 & 1 & 9\\ 0 & 1 & 3 & -7\\ -1 & -3 & 0 & -4\end{pmatrix}$$

$R_4=R_4+R_1$ gives

$$A=\begin{pmatrix} 1 & 2 & -2 & 5\\ 0 & 4 & 1 & 9\\ 0 & 1 & 3 & -7\\ 0 & -1 & -2 & 1\end{pmatrix}$$

$R_3=R_3-\frac{1}{4}R_2$ gives

$$A=\begin{pmatrix} 1 & 2 & -2 & 5\\ 0 & 4 & 1 & 9\\ 0 & 0 & \frac{11}{4} & -\frac{37}{4}\\ 0 & -1 & -2 & 1\end{pmatrix}$$

$R_4 = R_4 + \frac{1}{4}R_2$ gives

$$A = \begin{pmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 1 & 9 \\ 0 & 0 & \frac{11}{4} & -\frac{37}{4} \\ 0 & 0 & -\frac{7}{4} & \frac{13}{4} \end{pmatrix}$$

$R_4 = R_4 - \frac{-\frac{7}{4}}{\frac{11}{4}}R_3$ gives

$$A = \begin{pmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 1 & 9 \\ 0 & 0 & \frac{11}{4} & -\frac{37}{4} \\ 0 & 0 & 0 & -\frac{29}{11} \end{pmatrix}$$

Hence

$$\begin{aligned} |A| &= 1 \times 4 \times \frac{11}{4} \times -\frac{29}{11} \\ &= -29 \end{aligned}$$

0.0.10 Problem 10

Using elementary row operations, find the inverse of

$$A = \begin{pmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix}$$

Answer

Set up augmented matrix

$$C = \begin{pmatrix} 3 & 5 & 6 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{pmatrix}$$

$R_2 = R_2 - \frac{2}{3}R_1$ gives

$$C = \begin{pmatrix} 3 & 5 & 6 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & -1 & -\frac{2}{3} & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{pmatrix}$$

$R_3 = R_3 - \frac{2}{3}R_1$ gives

$$C = \begin{pmatrix} 3 & 5 & 6 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & -1 & -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{1}{3} & 1 & -\frac{2}{3} & 0 & 1 \end{pmatrix}$$

$R_3 = R_3 - \frac{-\frac{1}{3}}{\frac{2}{3}}R_2$ gives

$$C = \begin{pmatrix} 3 & 5 & 6 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & -1 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{pmatrix}$$

Start backward elimination now. $R_1 = R_1 - \frac{5}{\frac{2}{3}}R_2$ gives

$$C = \begin{pmatrix} 3 & 0 & \frac{27}{2} & 6 & -\frac{15}{2} & 0 \\ 0 & \frac{2}{3} & -1 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{pmatrix}$$

$R_1 = R_1 - \frac{\frac{27}{2}}{\frac{1}{2}} R_3$ gives

$$C = \begin{pmatrix} 3 & 0 & 0 & 33 & -21 & -27 \\ 0 & \frac{2}{3} & -1 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{pmatrix}$$

$R_2 = R_2 - \frac{-1}{\frac{1}{2}} R_3$ gives

$$C = \begin{pmatrix} 3 & 0 & 0 & 33 & -21 & -27 \\ 0 & \frac{2}{3} & 0 & -\frac{8}{3} & 2 & 2 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{pmatrix}$$

Divide each row by diagonal element to make LHS identity matrix. $R_1 = \frac{R_1}{3}$ gives

$$C = \begin{pmatrix} 1 & 0 & 0 & 11 & -7 & -9 \\ 0 & \frac{2}{3} & 0 & -\frac{8}{3} & 2 & 2 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{pmatrix}$$

$R_2 = \frac{R_2}{\frac{2}{3}}$ gives

$$C = \begin{pmatrix} 1 & 0 & 0 & 11 & -7 & -9 \\ 0 & 1 & 0 & -4 & 3 & 3 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{pmatrix}$$

$R_3 = \frac{R_3}{\frac{1}{2}}$ gives

$$C = \begin{pmatrix} 1 & 0 & 0 & 11 & -7 & -9 \\ 0 & 1 & 0 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{pmatrix}$$

Hence

$$A^{-1} = \begin{pmatrix} 11 & -7 & -9 \\ -4 & 3 & 3 \\ -2 & 1 & 2 \end{pmatrix}$$

0.0.11 Problem 11

(a) Show that any plane through the origin is subspace of $\mathbb{R}^3$

(b) Show that the plane $x + 3y - 2z = 5$ is not subspace of $\mathbb{R}^3$

Answer

part(a) The plane through the origin is the set W of all vectors $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, such that $ax + by + cz = 0$, where x, y, z are the coordinates of the vector v and a, b, c are any arbitrary constants not all zero. To show that W is subspace of $\mathbb{R}^3$, we need to show that additions of *any* two vectors $u, v \in W$ gives vector $w \in W$ (closed under addition) and multiplying any vector $u \in W$ by *any* scalar k gives vector $ku \in W$ (closed under scalar multiplication). We are told the zero vector $0 \in W$ already, so we do not have to show this. (since the plane passes though origin).

To show closure under addition, consider any two vectors $v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $u = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$. Since these two vectors are taken from W, then we know they satisfy the equation of the plane already. i.e.

$$\begin{aligned} ax_1 + by_1 + cz_1 &= 0 \\ ax_2 + by_2 + cz_2 &= 0 \end{aligned} \tag{1}$$

Now lets add these two vectors

$$
\begin{aligned}
\boldsymbol{v}+\boldsymbol{u} &= \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \qquad (2)
\end{aligned}
$$

We now need to check if the above vector in still in W (i.e. in the plane passing through the origin). To do so, we take the original equation of the plane $ax+by+cz=0$ and replace x,y,z in this equation by the coordinates in (2) and see if we still get zero in the RHS. This results in

$$
\begin{aligned}
a(x_1+x_2)+b(y_1+y_2)+c(z_1+z_2) &\stackrel{?}{=} 0 \\
ax_1+ax_2+by_1+by_2+cz_1+cz_2 &\stackrel{?}{=} 0 \\
(ax_1+by_1+cz_1)+(ax_2+by_2+cz_2) &\stackrel{?}{=} 0
\end{aligned}
$$

Substituting (1) into the above gives

$$
0+0 \stackrel{?}{=} 0
$$

Yes. Therefore $\boldsymbol{v}+\boldsymbol{u} \in W$. To check closure under scalar multiplication.

$$
\begin{aligned}
k\boldsymbol{v} &= k\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \\
&= \begin{pmatrix} kx_1 \\ ky_1 \\ kz_1 \end{pmatrix} \qquad (3)
\end{aligned}
$$

We now need to check if the above vector in still in W (i.e. in the plane passing through the origin). To do so, we take the original equation of the plane $ax+by+cz=0$ and replace x,y,z in this equation by the coordinates in (3) and see if we still get zero in the RHS. This results in

$$
\begin{aligned}
a(kx_1)+b(ky_1)+c(kz_1) &\stackrel{?}{=} 0 \\
k(ax_1+by_1+cz_1) &\stackrel{?}{=} 0
\end{aligned}
$$

But since $ax_1+by_1+cz_1=0$ from (1). Therefore $k(ax_1+by_1+cz_1)=0$. So closed under scalar multiplication.

Part b A subspace must include the zero vector $0=(0,0,0)$. Replacing the coordinates of this vector into LHS of $x+3y-2z=5$ gives

$$
\begin{aligned}
0+3(0)-2(0) &\stackrel{?}{=} 5 \\
0 &\stackrel{?}{=} 5
\end{aligned}
$$

No. Hence not satisfied. Therefore not subspace of $\mathbb{R}^3$.