Math 320 (Smith): Practice Problems for Exam 2

1. Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 3 \\ -1/3 & -3/2 & -1 \\ -1/2 & -1/4 & -3/2 \end{bmatrix}, \tag{1}$$

for what vectors $\mathbf{b}$ does $\mathbf{Ax} = \mathbf{b}$ have a solution?

2. (a) For what vectors $\mathbf{b}$ does $\mathbf{Ax} = \mathbf{b}$ have a solution, with $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1/2 \\ 3 & 1 & 2 \\ 0 & 6 & 3 \end{bmatrix}. \tag{3}$$

(b) Find all possible solutions (or no solution) for $\mathbf{b}^T = [0\ \ 1\ \ 12/5]$ and for $\mathbf{b}^T = [0\ \ 12/5\ \ 1]$.

3. Consider $\mathbf{Ax} = \mathbf{b}$ for

$$\mathbf{A} = \begin{bmatrix} 2/3 & a_{12} & -2 \\ -1/5 & -1/3 & 3/5 \\ 1/2 & 5/6 & -3/2 \end{bmatrix}. \tag{3}$$

(a) For what values of a_{12} is $\mathbf{A}$ non-singular?
(b) For what values of a_{12} is $\mathbf{A}$ singular?
(c) In all cases of $\mathbf{A}$ singular, analyze the system $\mathbf{Ax} = \mathbf{b}$. What vectors $\mathbf{b}$ lead to solutions $\mathbf{x}$? What are those solutions $\mathbf{x}$?

4. Given that two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, are $3\mathbf{u} - 5\mathbf{v}$ and $\mathbf{v}$ linearly dependent or linearly independent? Prove your answer.

5. Are the following statements TRUE or FALSE? If the statement is false, correct it.

(a) A square matrix with two identical rows is row equivalent to the identity matrix.

(b) The inverse of a square matrix $\mathbf{A}$ exists if $\mathbf{A}$ is row equivalent to the identity matrix $\mathbf{I}$ with the same dimensions.

(c) The determinant of an upper triangular square matrix is the sum of the diagonal elements.

6. Prove Property 4 of the seven properties of determinants.

7. Consider the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 0 & a_{32} & a_{33} \end{bmatrix}, \tag{7}$$

(a) Find a condition on a_{32} and a_{33} such that $\mathbf{A}^{-1}$ exists.

(b) Find the value of the determinant for $a_{32} = 1$ and $a_{33} = -2$. How many columns of $\mathbf{A}$ are independent for $a_{32} = 1$ and $a_{33} = -2$?

(c) For $a_{32} = 5$ and $a_{33} = -4$, can $\mathbf{p}^T = [3\ \ 5\ \ 0]$ be expressed as a linear combination of the columns of $\mathbf{A}$? Support your answer with a calculation (no work, no credit).

(d) Find the value of the determinant for $a_{32} = 5$ and $a_{33} = -4$. How many columns of $\mathbf{A}$ are independent for $a_{32} = 5$ and $a_{33} = -4$?

8. (a) Consider a 3×3 matrix $\mathbf{A}$. Show that $\det(\mathbf{A}^T) = \det(\mathbf{A})$.
[In fact, $\det(\mathbf{A}^T) = \det(\mathbf{A})$ for $\mathbf{A}$ $n \times n$.]

(b) The square matrix $\mathbf{A}$ is called orthogonal if $\mathbf{A}^T = \mathbf{A}^{-1}$. Show that the determinant of an orthogonal matrix is either $+1$ or -1. You may use the fact that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

9. Find the determinant Hint: Use elementary row operations.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 & 5 \\ -1 & 2 & 3 & 4 \\ 1 & 3 & 1 & -2 \\ -1 & -3 & 0 & -4 \end{bmatrix} \tag{9}$$

10. Using elementary row operations, find the inverse of

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}. \tag{10}$$

11. (a) Show that any plane through the origin is a subspace of $\mathbf{R}^3$.
(b) Show that the plane $x + 3y - 2z = 5$ is not a subspace of $\mathbf{R}^3$.