#1.

a) First we find a particular solution of the form $y = x^r$. Plug in, we get.

$$x^2(r(r-1)x^{r-2}) - 3x(rx^{r-1}) + 4x^r = 0.$$

i.e $x^r[r^2 - 4r + 4] = 0.$ $x > 0$

So we get $r^2 - 4r + 4 = 0$, $r_1 = r_2 = 2$.

So we get one particular soln $y_1(x) = x^2$.

To find the other soln, we use reduction of order. Put

$$y_2(x) = x^2 v(x).$$

$y_2' = 2xv + x^2 v'$, $y_2'' = 2v + 4xv' + x^2 v''$.

Plug in: $x^2(2v + 4xv' + x^2 v'') - 3x(2xv + x^2 v') + 4x^2 v = 0.$

Collect terms: $x^3 v' + x^4 v'' = 0.$

i.e $v' + xv'' = 0.$

We solve this using integral factors.

$$v'' + \frac{v'}{x} = 0.$$

$\mu(x) = e^{\int \frac{1}{x}dx} = \cancel{\ln x}\ x.$

multiply $\mu(x)$ to both sides:

$$(xv')' = xv'' + v' = 0 \Rightarrow xv' = C_1$$

$$v = \int \frac{C_1}{x} dx = C_1 \ln x.$$

So another particular soln $y_2(x) = x^2 \ln x.$

general soln: $y(x) = C_1 x^2 + C_2 x^2 \ln x.$

b). $y'(x) = 2C_1 x + C_2 x + 2C_2 x \ln x.$

then the eqn is:

$$4C_1 + 4C_2 \ln 2 = a.$$

$$2(2C_1 + C_2) + 4C_2 \ln 2 = b.$$

or $\begin{pmatrix} 4 & 4\ln 2 \\ 4 & 2+4\ln 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

Next we solve C_1, C_2.

$$\begin{pmatrix} 4 & 4\ln 2 & a \\ 4 & 2+4\ln 2 & b \end{pmatrix} \xrightarrow{r_2 - r_1} \begin{pmatrix} 4 & 4\ln 2 & a \\ 0 & 2 & b-a \end{pmatrix}$$

$$\xrightarrow{r_1 - 2\ln 2\, r_2} \begin{pmatrix} 4 & 0 & (1+2\ln 2)a - 2\ln 2\, b \\ 0 & 2 & b-a \end{pmatrix}$$

i.e. $c_1 = \frac{1}{4}\left((1+2\ln 2)a - (2\ln 2)b\right)$

$c_2 = \frac{1}{2}(b-a)$.

So the unique soln:

$$y(x) = \frac{x^2}{4}\left((1+2\ln 2)a - (2\ln 2)b\right) + \frac{1}{2}(b-a)x^2\ln x.$$

c). To show linear independence, we show

$W(y_1, y_2) \neq 0$.

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & x + 2x\ln x \end{vmatrix}$$

$$= x^3 + 2x^3\ln x - 2x^3\ln x = x^3 \neq 0$$

for $x > 0$.

So y_1, y_2 linearly indep.

#2. We do reduction of order.

Put $y_2(x) = x\,v(x)$.

then $y_2' = v + xv'$, $y_2'' = 2v' + xv''$.

plug in,

$$x^2(2v' + xv'') - x(x+2)(v + xv') + (x+2)xv = 0$$

collect terms:

$$x^3 v'' - x^3 v' = 0 \quad \cancel{\text{since } x>0}$$

$$x > 0 \Rightarrow v'' - v' = 0.$$

integral factor: $\mu(x) = e^{\int -1\,dx} = e^{-x}$.

$$e^{-x} v'' - e^{-x} v' = (e^{-x} v')' = 0.$$

$$\Rightarrow e^{-x} v' = C_1 \quad v' = C_1 e^x \Rightarrow v = C_1 e^x + C_2.$$

So general solution

$$y(x) = x(C_1 e^x + C_2).$$

#3. we have found in problem 1 the general soln to homogeneous eqn:

$$y_c(x) = C_1 x^2 + C_2 x^2 \ln x.$$

can write the eqn as:

$$y'' - \frac{3y'}{x} + \frac{4y}{x^2} = \ln x.$$

we use variation of parameters:

$$Y_p(x) = \left[-\int \frac{Y_2 f}{W}\right] Y_1 + \left[\int \frac{Y_1 f}{W}\right] Y_2.$$

Also we found in problem 1 $W(x) = x^3$.

$$Y_p(x) = \left[-\int \frac{x^2 \ln x \cdot \ln x}{x^3} dx\right] \cdot x^2 + \left[\int \frac{x^2 \cdot \ln x}{x^3} dx\right] x^2 \ln x.$$

$$= \left[-\int \frac{(\ln x)^2}{x} dx\right] x^2 + \left[\int \frac{\ln x}{x} dx\right] x^2 \ln x.$$

$$\int \frac{(\ln x)^2}{x} dx \overset{u=\ln x}{=\!=} \int u^2 du = \frac{1}{3}u^3 = \frac{1}{3}(\ln x)^3.$$

$$\int \frac{\ln x}{x} dx \overset{u=\ln x}{=\!=} \int u\, du = \frac{1}{2}u^2 = \frac{1}{2}(\ln x)^2.$$

so $Y_p(x) = -\frac{1}{3}(\ln x)^3 \cdot x^2 + \frac{1}{2} x^2 (\ln x)^3 = \frac{1}{6} x^2 (\ln x)^3.$

so general soln

$$y(x) = Y_p(x) + Y_c(x)$$

$$= C_1 x^2 + C_2 x^2 \ln x + \frac{1}{6} x^2 (\ln x)^3$$

#4.

characteristic eqn:

$$2r^2 - 5r + c = 0.$$

a). this means $25 - 4 \times 2 \times c > 0$.

i.e $c < \frac{25}{8}$.

b). this means $25 - 8c = 0$. i.e $c = 25/8$.

c). $2r^2 - 5r + 2 = 0$ $r_1 = 2$, $r_2 = \frac{1}{2}$.

So general soln $y(x) = C_1 e^{2x} + C_2 e^{\frac{1}{2}x}$.

d). $y'(x) = 2C_1 e^{2x} + \frac{1}{2} C_2 e^{\frac{1}{2}x}$.

So

$$C_1 e^{2x_0} + C_2 e^{\frac{1}{2}x_0} = p.$$

$$2C_1 e^{2x_0} + \frac{1}{2} C_2 e^{\frac{1}{2}x_0} = q.$$

or $\begin{pmatrix} e^{2x_0} & e^{\frac{1}{2}x_0} \\ 2e^{2x_0} & \frac{1}{2} e^{\frac{1}{2}x_0} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

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$$\frac{d\underline{x}}{dt} = \begin{bmatrix} -3 & 5 \\ -5 & 3 \end{bmatrix} \underline{x}$$

$$(-3-\lambda)(3-\lambda) + 25 = -9 - \cancel{3\lambda} + \cancel{3\lambda} + \lambda^2 + 25 = 0$$

$$\lambda^2 + 16 = 0, \quad \lambda^2 = -16, \quad \lambda = \pm 4i$$

$$\boxed{\lambda_1 = 4i} \qquad \begin{bmatrix} -3-4i & 5 \\ -5 & 3-4i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Top Row $(-3-4i)\xi_1 + 5\xi_2 = 0$

Choose, e.g. $\xi_1 = 5 \quad \xi_2 = 3+4i \implies$

$$(-3-4i)5 + 5(3+4i) = -15 - 20i + 15 + 20i = 0$$

$$\underline{\xi}^{(1)} = \begin{bmatrix} 5 \\ 3+4i \end{bmatrix} \qquad \underline{x}^{(1)} = \begin{bmatrix} 5 \\ 3+4i \end{bmatrix} e^{4it}$$

$$\underline{x} = C_1 \begin{bmatrix} 5 \\ 3+4i \end{bmatrix} e^{4it} + C_2 \begin{bmatrix} 5 \\ 3-4i \end{bmatrix} e^{-4it}$$

(b) $$Re\left(\underline{X}^{(1)}\right) = Re\left\{ \begin{bmatrix} 5 \\ 3+4i \end{bmatrix} (\cos 4t + i\sin 4t) \right\}$$

$$= \begin{bmatrix} 5\cos 4t \\ 3\cos 4t - 4\sin 4t \end{bmatrix}$$

$$Im\left(\underline{X}^{(1)}\right) = \begin{bmatrix} 5\sin 4t \\ 3\sin 4t + 4\cos 4t \end{bmatrix}$$

$$\underline{x} = C_3 \begin{bmatrix} 5\cos 4t \\ 3\cos 4t - 4\sin 4t \end{bmatrix} + C_4 \begin{bmatrix} 5\sin 4t \\ 3\sin 4t + 4\cos 4t \end{bmatrix}$$

(c) $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{g}_1 + \underline{g}_2$

$\underline{g}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} te^{4t} \qquad \underline{g}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}$

Let $\underline{x}_p = \underline{a}te^{4t} + \underline{b}e^{4t}$

$\underline{x}_p' = \underline{a}e^{4t} + 4\underline{a}te^{4t} + 4\underline{b}e^{4t}$

$\underline{a}e^{4t} + 4\underline{a}te^{4t} + 4\underline{b}e^{4t} = \underline{\underline{A}}\left(\underline{a}te^{4t} + \underline{b}e^{4t}\right)$
$+ \underline{g}_1 + \underline{g}_2$

e^{4t} : $\underline{a} + 4\underline{b} = \underline{\underline{A}}\,\underline{b} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

te^{4t} : $4\underline{a} = \underline{\underline{A}}\,\underline{a} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

⑥

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{g}(t) \qquad \underline{\underline{A}} = \begin{bmatrix} 5 & 3 \\ -\frac{1}{3} & 3 \end{bmatrix}$$

$$\begin{vmatrix} 5-\lambda & 3 \\ -\frac{1}{3} & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda) + 1$$
$$= 15 - 3\lambda - 5\lambda + \lambda^2 + 1$$
$$= \lambda^2 - 8\lambda + 16 = 0$$
$$= (\lambda - 4)^2$$

$\lambda = 4$ repeated

$$\begin{bmatrix} 1 & 3 \\ -\frac{1}{3} & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \xi_1 + 3\xi_2 = 0$$

$$\underline{\xi} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\underline{x}_h = c_1 \underline{\xi} e^{4t} + c_2 \left\{ \underline{\xi} t e^{4t} + \underline{\chi} e^{4t} \right\}$$

$$\begin{bmatrix} 1 & 3 \\ -\frac{1}{3} & -1 \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$x_1 + 3x_2 = -3 \quad$ e.g. $\quad x_1 = 0 \quad x_2 = -1$

$$\underline{x}_h = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} + c_2 \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{4t} \right\}$$

(b) Let $\underline{\underline{\Psi}}$ be the fundamental matrix. Each column of $\underline{\underline{\Psi}}$ satisfies $\frac{dx}{dt} = \underline{\underline{A}}\,\underline{x}$; there are n linearly independent columns.

Let $\underline{x} = \underline{\underline{\Psi}}\,\underline{u}$, $\underline{u}$ unknown

$$\underline{x}' = \underline{\underline{\Psi}}'\underline{u} + \underline{\underline{\Psi}}\,\underline{u}' = \underline{\underline{A}}\,\underline{\underline{\Psi}}\,\underline{u} + \underline{g}$$

Then $\underline{\underline{\Psi}}'\underline{u} = \underline{\underline{A}}\,\underline{\underline{\Psi}}\,\underline{u}$ because

$\underline{\underline{\Psi}}' = \underline{\underline{A}}\,\underline{\underline{\Psi}}$ by assumption [column by column]

$$\Rightarrow \quad \underline{\underline{\Psi}}\,\underline{u}' = \underline{g} \quad \Rightarrow \quad \underline{u}' = \underline{\underline{\Psi}}^{-1}\underline{g}$$

$$\Rightarrow \quad \underline{u} = \int \underline{\underline{\Psi}}^{-1}\underline{g}\, dt + \underline{C}$$

$$\underline{x} = \underline{\underline{\psi}} \int \underline{\underline{\psi}}^{-1} \underline{g}\, dt + \underline{\underline{\psi}}\, \underline{c} \quad \text{or}$$

$$\underline{x}(t) = \underline{\underline{\psi}}(t) \int \underline{\underline{\psi}}^{-1}(t')\, \underline{g}(t')\, dt' + \underline{\underline{\psi}}(t)\, \underline{c} \quad \text{or}$$

$$\underline{x}(t) = \underline{\underline{\psi}}(t) \int_{t_0}^{t} \underline{\underline{\psi}}^{-1}(t')\, \underline{g}(t')\, dt' + \underline{\underline{\psi}}(t)\, \underline{\underline{\psi}}^{-1}(t_0)\, \underline{x}(t_0)$$

(c) $$\underline{\underline{\psi}} = \begin{bmatrix} -3e^{4t} & -3te^{4t} \\ e^{4t} & te^{4t} - e^{4t} \end{bmatrix}$$

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & t \\ 1 & t-1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3}e^{-4t} & 0 \\ 0 & e^{-4t} \end{bmatrix} R_2 - R_1$$

$$\begin{bmatrix} 1 & t \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3}e^{-4t} & 0 \\ \frac{1}{3}e^{-4t} & e^{-4t} \end{bmatrix} R_2(-1)$$

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3}e^{-4t} & 0 \\ -\frac{1}{3}e^{-4t} & -e^{-4t} \end{bmatrix} \quad R_1 - tR_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \left(-\frac{1}{3} + \frac{t}{3}\right)e^{-4t} & te^{-4t} \\ -\frac{1}{3}e^{-4t} & -e^{-4t} \end{bmatrix} = \underline{\Psi}^{-1}$$

Check $\underline{\Psi}^{-1}\underline{\Psi}$

$$\begin{bmatrix} \left(-\frac{1}{3} + \frac{t}{3}\right)e^{-4t} & te^{-4t} \\ -\frac{1}{3}e^{-4t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -3e^{4t} & -3te^{4t} \\ e^{4t} & (t-1)e^{4t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

(7) (a) $y'' - 2y' + 4y = e^{x}\left[x\sin\sqrt{3}x + e^{\sqrt{3}x}\right]$ (34)

$= xe^{x}\sin\sqrt{3}x + e^{(1+\sqrt{3})x}$

$y'' - 2y' + 4y = 0 \Rightarrow r^2 - 2r + 4 = 0$

$r = \dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$

$y_h = C_1 e^{x}\cos\sqrt{3}x + C_2 e^{x}\sin\sqrt{3}x$

$y_p = (Ax+B)xe^{x}\cos\sqrt{3}x + (Cx+D)xe^{x}\cos\sqrt{3}x$

$+ Ee^{(1+\sqrt{3})x}$

7 (b) $y'' - 6y' = x^2 + x\cosh 6x$

$$= x^2 + x\left(\frac{e^{6x} + e^{-6x}}{2}\right)$$

$y'' - 6y' = 0 \Rightarrow r^2 - 6r = 0 = r(r-6)$

$y_h(x) = C_1 + C_2 e^{6x}$

$y_p = (Ax^3 + Bx^2 + Cx)$

$+ (Dx + E)xe^{6x} + (Fx + G)e^{-6x}$

(8) $(r+2)^2(r^2+2)^2 = 0$

$r = -2$ repeated

$r^2 = -2 \implies r = \pm i\sqrt{2}$ repeated

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

$$+ C_3 \cos\sqrt{2}x + C_4 \sin\sqrt{2}x$$

$$+ C_5 x \cos\sqrt{2}x + C_6 x \sin\sqrt{2}x$$

$$-\infty < x < \infty$$