My solution for final practice exam. Math 320

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Contents

0.1 My solution for final exam practice.

0.1.1 Problem 1

(a) Find general solution to $x^2y'' - 3xy + 4y = 0$ with $x > 0$. (b) For initial conditions $y(2) = a, y'(2) = b$ give a 2×2 matrix-vector equation to determine the coefficients of the unique solution. Solve the system them write the solution to the initial value problem. (c) Show that general solution contains two L.I. solutions y_1, y_2 with $x > 0$

Solution

Part(a) Let $y = x^r$ then $y' = Arx^{r-1}, y'' = Ar(r-1)x^{r-2}$. Substituting these into the ODE gives

$$
\begin{aligned}
x^2 r(r-1)x^{r-2} - 3xrx^{r-1} + 4x^r &= 0\\
r(r-1)x^r - 3rx^r + 4x^r &= 0
\end{aligned}
$$

Since $x > 0$, we can cancel x^r and obtain the characteristic equation

$$
\begin{aligned}
r(r-1) - 3r + 4 &= 0\\
r^2 - 4r + 4 &= 0\\
(r-2)^2 &= 0
\end{aligned}
$$

Hence $r = 2$ double root. Therefore

$$
\begin{aligned}
y_1 &= x^2\\
y_2 &= x^2 \ln x
\end{aligned}
$$

And the homogenous solution is

$$
y_h(x) = c_1 x^2 + c_2 x^2 \ln x \tag{1A}
$$

Part(b) Applying $y(2) = a$ gives

$$
a = 4c_1 + 4c_2 \ln 2 \tag{1}
$$

Taking derivative of $y_h(x)$

$$
y_h'(x) = 2c_1 x + 2c_2 x \ln x + c_2 x
$$

Applying $y'(2) = b$ gives

$$
b = 4c_1 + c_2(4\ln 2 + 2) \tag{2}
$$

Using (1,2), we write them in matrix form to solve for c_1, c_2

$$
\begin{pmatrix} 4 & 4\ln 2\\ 4 & 4\ln 2 + 2 \end{pmatrix} \begin{pmatrix} c_1\\ c_2 \end{pmatrix} = \begin{pmatrix} a\\ b \end{pmatrix}
$$

$R_2 = R_2 - R_1$

$$
\begin{pmatrix} 4 & 4\ln 2\\ 0 & 2 \end{pmatrix} \begin{pmatrix} c_1\\ c_2 \end{pmatrix} = \begin{pmatrix} a\\ b-a \end{pmatrix}
$$

From second row,

$$
\begin{aligned}
2c_2 &= (b-a)\\
c_2 &= \frac{b-a}{2}
\end{aligned}
$$

From first row

$$\begin{aligned}4c_1 + 4\ln 2c_2 &= a\\ c_1 &= \frac{a - 4c_2 \ln 2}{4}\\ &= \frac{a - 4\left(\frac{b-a}{2}\right)\ln 2}{4}\\ &= \frac{a}{4} - \left(\frac{b-a}{2}\right)\ln 2\end{aligned}$$

Therefore

$$\begin{aligned}c_1 &= \frac{a}{4} - \left(\frac{b-a}{2}\right)\ln 2\\ c_2 &= \frac{b-a}{2}\end{aligned}$$

Pugging these into the $y_h(x) = c_1x^2 + c_2x^2 \ln x$ found in part(a) gives

$$y_h(x) = \left(\frac{a}{4} - \left(\frac{b-a}{2}\right)\ln 2\right)x^2 + \left(\frac{b-a}{2}\right)x^2 \ln x$$

Part (c) We found that

$$\begin{aligned}y_1 &= x^2\\ y_2 &= x^2 \ln x\end{aligned}$$

Hence the Wronskian is

$$\begin{aligned}W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x\ln x + x \end{vmatrix}\\ &= 2x^3 \ln x + x^3 - 2x^3 \ln x\\ &= x^3\end{aligned}$$

Since $x > 0$, hence $W(x)$ never zero. Therefore y_1, y_2 are L.I.

0.1.2 Problem 2

Given one solution $y_1(x) = x$, find general solution of $x^2y'' - x(x+2)y' + (x+2)y = 0$ for $x > 0$

<u>Solution</u>

Assume solution is $y(x) = vy_1(x)$. Hence

$$\begin{aligned}y' &= v'y_1 + vy_1'\\ y'' &= v''y_1 + v'y_1' + v'y_1' + vy_1''\\ &= v''y_1 + 2v'y_1' + vy_1''\end{aligned}$$

Plugging the second solution into the original ODE gives

$$\begin{aligned}x^2y'' - x(x+2)y' + (x+2)y &= 0\\ x^2\left(v''y_1 + 2v'y_1' + vy_1''\right) - x(x+2)\left(v'y_1 + vy_1'\right) + (x+2)\left(vy_1\right) &= 0\end{aligned}$$

Collecting terms on v, v', v'' gives

$$v''\left(x^2y_1\right) + v'\left(2x^2y_1' - x(x+2)y_1\right) + v\overbrace{\left(x^2y_1'' - x(x+2)y_1' + (x+2)y_1\right)}^{0} = 0$$

Hence

$$v''\left(x^2y_1\right) + v'\left(2x^2y_1' - x(x+2)y_1\right) = 0$$

But $y_1 = x$, hence $y_1' = 1$ and the above becomes

$$\begin{aligned} x^3v'' + v'\left(2x^2 - x^2(x+2)\right) &= 0 \\ x^3v'' - x^3v' &= 0 \end{aligned}$$

Since $x > 0$ then above reduces to

$$v'' - v' = 0$$

Let $v' = z$ then the above becomes $z' - z = 0$ or $\frac{dz}{dx} = z$ which is separable. Hence the solution is $\ln|z| = x + c_1$ or $z = c_1e^x$. Therefore

$$v' = c_1e^x$$

Integrating

$$v = c_1e^x + c_2$$

Hence, since

$$\begin{aligned} y &= vy_1 \\ &= (c_1e^x + c_2)x \end{aligned}$$

Therefore the solution is

$$y(x) = c_1xe^x + c_2x$$

0.1.3 Problem 3

Find general solution to $x^2y'' - 3xy' + 4y = x^2\ln(x)$ with $x > 0$

Solution We first solve the homogenous part

$$x^2y'' - 3xy' + 4y = 0$$

We solved this in problem 1, the solution is

$$y_h(x) = c_1x^2 + c_2x^2\ln x$$

To find particular solution, we will use variation of parameters since $\ln(x)$ is not one of the good functions to guess for. Writing the ODE in standard form

$$y'' - 3\frac{1}{x}y' + \frac{4}{x^2}y = \ln(x)$$

We see from the homogeneous solution that $y_1(x) = x^2, y_2 = x^2\ln(x)$. Hence we assume the complete solution (including particular solution) is

$$y = y_1u_1 + y_2u_2 \tag{1A}$$

Where

$$u_1 = -\int \frac{y_2 f(x)}{W} dx \tag{1}$$

$$u_2 = \int \frac{y_1 f(x)}{W} dx \tag{2}$$

Where in the above, $f(x) = \ln(x)$ and not $x^2 \ln(x)$ since we divide by x^2 in order to make the ODE standard form. W is the Wronskian. We found the Wronskian for this ODE in part(c) problem 1, which is $W = x^3$, Hence (1) becomes

$$u_1 = -\int \frac{x^2 \ln^2(x)}{x^3} dx = -\int \frac{\ln^2(x)}{x} dx$$

Let $z = \ln(x)$ hence $\frac{dz}{dx} = \frac{1}{x}$ or $dx = xdz$., Hence the integral becomes

$$u_1 = -\int \frac{z^2}{x} x dz = -\int z^2 dz = -\frac{z^3}{3} + c_1$$

Replacing back gives

$$u_1 = -\frac{1}{3} \ln^3(x) + c_1$$

And from (2)

$$u_2 = \int \frac{x^2 \ln(x)}{x^3} dx = \int \frac{\ln(x)}{x} dx$$

Let $z = \ln(x)$ hence $\frac{dz}{dx} = \frac{1}{x}$ or $dx = xdz$., Hence the integral becomes

$$u_2 = \int \frac{z}{x} x dz = \int z dz = \frac{z^2}{2} + c_2$$

Replacing back gives

$$u_2 = \frac{1}{2} \ln^2(x) + c_2$$

Hence from (1A)

$$\begin{aligned} y &= y_1 u_1 + y_2 u_2 \\ &= x^2 \left(-\frac{1}{3} \ln^3(x) + c_1 \right) + x^2 \ln(x) \left(\frac{1}{2} \ln^2(x) + c_2 \right) \\ &= -\frac{1}{3} x^2 \ln^3(x) + c_1 x^2 + \frac{1}{2} x^2 \ln^3(x) + c_2 x^2 \ln(x) \end{aligned}$$

Hence

$$\boxed{y(x) = c_1 x^2 + c_2 x^2 \ln(x) + \frac{1}{6} x^2 \ln^3(x)}$$

0.1.4 Problem 4

Consider the equation $2y'' - 5y' + cy = 0$ with $-\infty < x < \infty$ for c real and constant. (a) For what values of c does characteristic equation have 2 different real roots? (b) for what values of c does the characteristic equation have 1 real repeated root? (c) Find general solution for $c = 2$. (d) for $c = 2$ and initial conditions $y(x_0) = p, y'(x_0) = q$ write a 2×2 matrix equation to determined the coefficients of general solutions.

Solution

Part (a) Assuming $y = Ae^r$ and substituting into the ODE gives the characteristic equation is

$$2r^2 - 5r + cr = 0$$

The roots are

$$\begin{aligned} r &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{5}{4} \pm \frac{1}{4}\sqrt{25 - 8c} \end{aligned}$$

For two different real roots we want $25 - 8c > 0$. Therefore $25 > 8c$ or

$$\boxed{c < \frac{25}{8}}$$

Part (b) For repeated real root, we want $r = \frac{5}{4}$. Which means we want $25 - 8c = 0$ or

$$\boxed{c = \frac{25}{8}}$$

Part (c) When $c = 2$ the ODE becomes

$$2y'' - 5y' + 2y = 0$$

The characteristic equation is

$$\begin{aligned} 2r^2 - 5r + 2 &= 0 \\ (r-2)\left(r - \frac{1}{2}\right) &= 0 \end{aligned}$$

Hence the solution is

$$y_h(x) = c_1 e^{2x} + c_2 e^{\frac{x}{2}}$$

Part(d) From above,

$$y_h' = c_1 2e^{2x} + \frac{1}{2} c_2 e^{\frac{x}{2}}$$

Applying first initial conditions gives the equation

$$p = c_1 e^{2x_0} + c_2 e^{\frac{x_0}{2}} \tag{1}$$

Applying second initial conditions gives the equation

$$q = c_1 2e^{2x_0} + \frac{1}{2} c_2 e^{\frac{x_0}{2}} \tag{2}$$

Writing (1,2) in matrix form gives

$$\begin{pmatrix} e^{2x_0} & e^{\frac{x_0}{2}} \\ 2e^{2x_0} & \frac{1}{2}e^{\frac{x_0}{2}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

We are asked not to solve if. Solution of the above gives c_1, c_2 which completes the solution.

0.1.5 Problem 5

Consider

$$x' = \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix} x$$

(a) find general solution. (b) Write the solution in terms of real functions only. (c) using method of undetermined coefficients, write the particular solution for

$$x' = \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix} x + \begin{pmatrix} te^{4t} \\ e^{4t} \end{pmatrix}$$

(d) Find the algebraic equation that given the undetermined coefficients. Do not solve.

Solution

Part (a) The first step is to determine the eigenvalues from

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} -3-\lambda & 5 \\ -5 & 3-\lambda \end{vmatrix} &= 0 \\ \lambda^2 + 16 &= 0 \\ \lambda &= \pm 4i \end{aligned}$$

For $\underline{\lambda_1 = 4i}$ we solve $(A - \lambda I)\, v_1 = 0$

$$\begin{pmatrix} -3-\lambda_1 & 5 \\ -5 & 3-\lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3-4i & 5 \\ -5 & 3-4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation we find $(-3-4i)\, v_1 + 5v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{3+4i}{5}$, hence

$$v_1 = \begin{pmatrix} 1 \\ \frac{3+4i}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 3+4i \end{pmatrix}$$

For $\underline{\lambda_2 = -4i}$ we solve $(A - \lambda I)\, v_2 = 0$

$$\begin{pmatrix} -3-\lambda_2 & 5 \\ -5 & 3-\lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3+4i & 5 \\ -5 & 3+4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation we find $(-3+4i)\, v_1 + 5v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{3-4i}{5}$, hence

$$v_2 = \begin{pmatrix} 1 \\ \frac{3-4i}{5} \end{pmatrix} = \begin{pmatrix} 5 \\ 3-4i \end{pmatrix}$$

Therefore, the homogenous solution is

$$
\begin{aligned}
\boldsymbol{x}_h(t) &= c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t)\\
&= c_1\boldsymbol{v}_1 e^{\lambda_1 t} + c_2\boldsymbol{v}_2 e^{\lambda_2 t}\\
&= c_1\begin{pmatrix}5\\3+4i\end{pmatrix}e^{4it} + c_2\begin{pmatrix}5\\3-4i\end{pmatrix}e^{-4it}
\end{aligned}
$$

Convert to new basis.

$$
\begin{aligned}
\boldsymbol{x}_1(t) &= \operatorname{Re}(\boldsymbol{x}_1(t))\\
&= \operatorname{Re}\begin{pmatrix}5\\3+4i\end{pmatrix}e^{4it}\\
&= \operatorname{Re}\begin{pmatrix}5(\cos 4t + i\sin 4t)\\(3+4i)(\cos 4t + i\sin 4t)\end{pmatrix}\\
&= \operatorname{Re}\begin{pmatrix}5(\cos 4t + i\sin 4t)\\3\cos 4t + 3i\sin 4t + 4i\cos 4t - 4\sin 4t\end{pmatrix}\\
&= \operatorname{Re}\begin{pmatrix}5(\cos 4t + i\sin 4t)\\(3\cos 4t - 4\sin 4t) + i(3\sin 4t + 4\cos 4t)\end{pmatrix}\\
&= \begin{pmatrix}5\cos 4t\\(3\cos 4t - 4\sin 4t)\end{pmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\boldsymbol{x}_2(t) &= \operatorname{Im}\begin{pmatrix}5(\cos 4t + i\sin 4t)\\(3\cos 4t - 4\sin 4t) + i(3\sin 4t + 4\cos 4t)\end{pmatrix}\\
&= \begin{pmatrix}5\sin 4t\\3\sin 4t + 4\cos 4t\end{pmatrix}
\end{aligned}
$$

Hence the solution using the new basis is

$$
\boldsymbol{x}_h(t) = C_1\begin{pmatrix}5\cos 4t\\(3\cos 4t - 4\sin 4t)\end{pmatrix} + C_2\begin{pmatrix}5\sin 4t\\3\sin 4t + 4\cos 4t\end{pmatrix}
$$

part (c) Since the RHS is $\begin{pmatrix}te^{4t}\\e^{4t}\end{pmatrix}$ then we try to see what we would do in the scalar case and then convert it to vector form. In scalar case, when RHS is te^{4t}, then the guess for t is $(a+bt)$ and the guess for e^{4t} is ce^{4t}. Therefore for the product, it will be $(a+bt)\left(ce^{4t}\right) = ace^{4t} + cbte^{4t}$. Let $ac = A, cb = B$, then the guess will becomes $Ae^{4t} + Bte^{4t}$ or $(A+Bt)e^{4t}$. We convert this to vector form now

$$
\begin{aligned}
\boldsymbol{x}_p &= \begin{pmatrix}a_1\\a_2\end{pmatrix}e^{4t} + \begin{pmatrix}b_1\\b_2\end{pmatrix}te^{4t}\\
&= \begin{pmatrix}a_1 + b_1 t\\a_2 + b_2 t\end{pmatrix}e^{4t}
\end{aligned}
$$

Therefore

$$x_p' = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{4t} + 4 \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} e^{4t}$$

Plugging this into the ODE

$$\begin{aligned}
x_p' &= \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix} x_p + \begin{pmatrix} te^{4t} \\ e^{4t} \end{pmatrix} \\
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{4t} + 4 \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} e^{4t} &= \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} e^{4t} + \begin{pmatrix} te^{4t} \\ e^{4t} \end{pmatrix} \\
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + 4 \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} &= \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix} \\
\begin{pmatrix} 4a_1 + b_1 + 4b_1 t \\ 4a_2 + b_2 + 4b_2 t \end{pmatrix} &= \begin{pmatrix} 5a_2 - 3a_1 - 3tb_1 + 5tb_2 \\ 3a_2 - 5a_1 - 5tb_1 + 3tb_2 \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix} \\
\begin{pmatrix} 4a_1 + b_1 + 4b_1 t \\ 4a_2 + b_2 + 4b_2 t \end{pmatrix} &= \begin{pmatrix} 5a_2 - 3a_1 - 3tb_1 + 5tb_2 + t \\ 3a_2 - 5a_1 - 5tb_1 + 3tb_2 + 1 \end{pmatrix} \\
\begin{pmatrix} 4a_1 + b_1 + 4b_1 t \\ 4a_2 + b_2 + 4b_2 t \end{pmatrix} &= \begin{pmatrix} 5a_2 - 3a_1 + t\,(5b_2 + 1 - 3b_1) \\ 3a_2 - 5a_1 + 1 + t\,(3b_2 - 5b_1) \end{pmatrix}
\end{aligned}$$

From first row in the above, we get two equations. And from the second row in the above, we get two equations. These are

$$\begin{aligned}
4a_1 + b_1 &= 5a_2 - 3a_1 \\
4b_1 &= 5b_2 + 1 - 3b_1 \\
4a_2 + b_2 &= 3a_2 - 5a_1 + 1 \\
4b_2 &= 3b_2 - 5b_1
\end{aligned}$$

Or

$$\begin{aligned}
7a_1 - 5a_2 + b_1 &= 0 \\
7b_1 - 5b_2 &= 1 \\
5a_1 + a_2 + b_2 &= 1 \\
b_2 + 5b_1 &= 0
\end{aligned}$$

In system form, these equations are

$$\begin{pmatrix} 7 & -5 & 1 & 0 \\ 0 & 0 & 7 & -5 \\ 5 & 1 & 0 & 1 \\ 0 & 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

We are asked not to solve this. But to verify the solution with computer solution, here is the complete solution. Solving the above gives

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{23}{128} \\ \frac{33}{128} \\ \frac{1}{32} \\ \frac{-5}{32} \end{pmatrix}$$

Using these values, the particular solution becomes

$$\begin{aligned} \boldsymbol{x}_p &= \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix} e^{4t} \\ &= \begin{pmatrix} \frac{23}{128} + \frac{1}{32} t \\ \frac{33}{128} - \frac{5}{32} t \end{pmatrix} e^{4t} \end{aligned}$$

And the full solution is

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t) \\ \boldsymbol{x}(t) &= C_1 \begin{pmatrix} 5\cos 4t \\ 3\cos 4t - 4\sin 4t \end{pmatrix} + C_2 \begin{pmatrix} 5\sin 4t \\ 3\sin 4t + 4\cos 4t \end{pmatrix} + \begin{pmatrix} \frac{23}{128} + \frac{1}{32} t \\ \frac{33}{128} - \frac{5}{32} t \end{pmatrix} e^{4t} \end{aligned}$$

Or

$$\begin{aligned} x_1(t) &= 5C_1 \cos 4t + 5C_2 \sin 4t + \left(\frac{23}{128} + \frac{1}{32} t \right) e^{4t} \\ x_2(t) &= C_1 (3\cos 4t - 4\sin 4t) + C_2 (3\sin 4t + 4\cos 4t) + \left(\frac{33}{128} - \frac{5}{32} t \right) e^{4t} \end{aligned}$$

0.1.6 Problem 6

Consider

$$x' = \begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} x + \begin{pmatrix} 48 \\ 9t \end{pmatrix}$$

(a) find homogeneous solution. (b) using undetermined coefficients, find particular solution. (c) find Wronskian (fundamental matrix). (d) Derive the variation of parameters formula for the solution $x(t)$.

Solution

Part (a) The first step is to determine the eigenvalues from

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -4 \\ \frac{1}{4} & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0$$

Hence $\lambda = 3$ repeated. Let us see if complete eigenvalue or defective. We solve $(A - \lambda I) v_1 = 0$

$$\begin{pmatrix} 2-\lambda & -4 \\ \frac{1}{4} & 4-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -4 \\ \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

First row gives $-v_1 - 4v_2 = 0$. Assuming $v_1 = -1$ gives $v_2 = \frac{1}{4}$, hence

$$\boxed{\mathbf{v}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}} \tag{1A}$$

We see that we can only get one eigenvector since the second row gives same result. Therefore we need a way to find the second eigenvector v_2. We start by assuming

$$x_2(t) = v_1 t e^{\lambda t} + v_2 e^{\lambda t}$$

We plug this back into the ODE and by comparing terms we find that

$$(A - \lambda I) v_2 = v_1$$

And now we solve for v_2 from the above equation (since we know v_1 already)

$$\begin{pmatrix} -1 & -4 \\ \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} -1 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

First row gives $-v_1 - 4v_2 = -4$. let $v_1 = 0$, then $v_2 = 1$ Hence

$$\boxed{\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Which is the same as book method. Now that we found $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ (using either method), then $\boldsymbol{x}_h(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t)$. Or

$$\begin{aligned} \boldsymbol{x}_h(t) &= c_1\boldsymbol{v}_1 e^{\lambda t} + c_2(\boldsymbol{v}_1 t + \boldsymbol{v}_2)e^{\lambda t} \\ &= c_1\begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{3t} + c_2\left(\begin{pmatrix} -4 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) e^{3t} \\ &= c_1\begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{3t} + c_2\begin{pmatrix} -4t \\ t+1 \end{pmatrix} e^{3t} \end{aligned} \tag{2}$$

part (b) The RHS is $\begin{pmatrix} 48 \\ 9t \end{pmatrix} = \begin{pmatrix} 48 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 9 \end{pmatrix} t$. Hence the guess is

$$\begin{aligned} \boldsymbol{x}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t \\ &= \boldsymbol{a} + \boldsymbol{b}t \end{aligned}$$

Therefore

$$\boldsymbol{x}_p' = \boldsymbol{b}$$

Substituting into ODE and balancing terms, we solve for $\boldsymbol{a}, \boldsymbol{b}$ as follows

$$\begin{aligned} \boldsymbol{x}_p' &= \begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} \boldsymbol{x}_p + \begin{pmatrix} 48 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 9 \end{pmatrix} t \\ \boldsymbol{b} &= \begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} (\boldsymbol{a} + \boldsymbol{b}t) + \begin{pmatrix} 48 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 9 \end{pmatrix} t \end{aligned}$$

<u>Balance constants</u>

$$\boldsymbol{b} = \begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} \boldsymbol{a} + \begin{pmatrix} 48 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 - 48 \\ b_2 \end{pmatrix} \tag{1}$$

Balance t

$$0 = \begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} \boldsymbol{b} + \begin{pmatrix} 0 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \end{pmatrix}$$

Solve for $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ by elimination. $R_2 = R_2 - \frac{1}{8} R_1$

$$\begin{pmatrix} 2 & -4 \\ 0 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \end{pmatrix}$$

Hence $\frac{9}{2} b_2 = -9$, or $b_2 = -2$, from first row $2b_1 - 4b_2 = 0$ or $b_1 = -4$. hence

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

Substituting this in (1) above gives equation to solve for $\boldsymbol{a}$

$$\begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -4 - 48 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 \\ \frac{1}{4} & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -52 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 \\ 0 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -52 \\ \frac{9}{2} \end{pmatrix}$$

From second row $a_2 = 1$ and from first row $2a_1 - 4a_2 = -52$ or $a_1 = \frac{-52+4}{2} = -24$, hence

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -24 \\ 1 \end{pmatrix}$$

Hence the particular solution is

$$\begin{aligned} \boldsymbol{x}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t \\ &= \begin{pmatrix} -24 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \end{pmatrix} t \\ &= \begin{pmatrix} -24 - 4t \\ 1 - 2t \end{pmatrix} \end{aligned}$$

And the complete solution is

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{x}_h + \boldsymbol{x}_p \\ &= c_1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -4t \\ t+1 \end{pmatrix} e^{3t} + \begin{pmatrix} -24 - 4t \\ 1 - 2t \end{pmatrix} \end{aligned}$$

Or

$$\begin{aligned} x_1(t) &= -4c_1 e^{3t} - 4c_2 t e^{3t} - 24 - 4t \\ x_2(t) &= c_1 e^{3t} + c_2 (1+t) e^{3t} + 1 - 2t \end{aligned}$$

part (c) The fundamental matrix $\Phi(t)$ is

$$\begin{aligned} \Phi(t) &= \begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{pmatrix} \\ &= \begin{pmatrix} -4e^{3t} & -4te^{3t} \\ e^{3t} & (t+1) e^{3t} \end{pmatrix} \end{aligned}$$

part (d) The derivation is given in textbook at 498.

0.1.7 Problem 7

Write down the form of the solution including homogenous and particular parts to the following ODE's

$$
\begin{aligned}
y'' - 2y' + 4y &= e^{x}\left(x\sin\left(\sqrt{3}x\right) + e^{\sqrt{3}x}\right)\\
y'' - 6y' &= x^{2} + x\cosh(6x)
\end{aligned}
$$

solution

For the first ODE

$$
y'' - 2y' + 4y = e^{x}x\sin\left(\sqrt{3}x\right) + e^{x\left(1+\sqrt{3}\right)}
$$

We start by finding the homogenous solution for $y'' - 2y' + 4y = 0$. The characteristic equation is $r^{2} - 2r + 4 = 0$, which has roots

$$
\begin{aligned}
r_1 &= 1 + i\sqrt{3}\\
r_2 &= 1 - i\sqrt{3}
\end{aligned}
$$

Hence

$$
y_h(x) = c_1\overbrace{e^{x}\cos\left(\sqrt{3}x\right)}^{y_1} + c_2\overbrace{e^{x}\sin\left(\sqrt{3}x\right)}^{y_2}
$$

To find particular solution, we need to find a guess. since the RHS is $e^{x}x\sin\left(\sqrt{3}x\right) + e^{x\left(1+\sqrt{3}\right)}$, the guess for $e^{x\left(1+\sqrt{3}\right)}$ is $C_0e^{x\left(1+\sqrt{3}\right)}$ and the guess for x is $c_3 + c_4x$ and the guess for $\sin\left(\sqrt{3}x\right)$ is $c_5\sin\left(\sqrt{3}x\right) + c_6\cos\left(\sqrt{3}x\right)$. Hence the guess for $e^{x}x\sin\left(\sqrt{3}x\right)$ term only is

$$
\begin{aligned}
e^{x}x\sin\left(\sqrt{3}x\right) &\to e^{x}\left(c_3 + c_4x\right)\left(c_5\sin\left(\sqrt{3}x\right) + c_6\cos\left(\sqrt{3}x\right)\right)\\
&\to \left(c_3e^{x} + c_4xe^{x}\right)\left(c_5\sin\left(\sqrt{3}x\right) + c_6\cos\left(\sqrt{3}x\right)\right)\\
&= c_3c_5e^{x}\sin\left(\sqrt{3}x\right) + c_3c_6e^{x}\cos\left(\sqrt{3}x\right) + c_4c_5xe^{x}\sin\left(\sqrt{3}x\right) + c_4c_6xe^{x}\cos\left(\sqrt{3}x\right)
\end{aligned}
$$

Rename the constants, and the hence the guess for $e^{x}x\sin\left(\sqrt{3}x\right)$ term only

$$
e^{x}x\sin\left(\sqrt{3}x\right) \to C_1e^{x}\sin\left(\sqrt{3}x\right) + C_2e^{x}\cos\left(\sqrt{3}x\right) + C_3xe^{x}\sin\left(\sqrt{3}x\right) + C_4xe^{x}\cos\left(\sqrt{3}x\right)
$$

Now that we found the initial guess, we have to look at it again and see if y_1 or y_2 are in the guess just made. If so, we add x. We see that since $e^{x}x\sin\left(\sqrt{3}x\right)$ is y_1, and $xe^{x}\cos\left(\sqrt{3}x\right)$ is y_2 so we need to multiply these terms in the guess by x, therefore the above becomes

$$
e^{x}x\sin\left(\sqrt{3}x\right) \to C_1xe^{x}\sin\left(\sqrt{3}x\right) + C_2xe^{x}\cos\left(\sqrt{3}x\right) + C_3x^{2}e^{x}\sin\left(\sqrt{3}x\right) + C_4x^{2}e^{x}\cos\left(\sqrt{3}x\right)
$$

Therefore the final guess is

$$
\begin{aligned}
y_p &= C_0e^{x\left(1+\sqrt{3}\right)} + C_1xe^{x}\sin\left(\sqrt{3}x\right) + C_2xe^{x}\cos\left(\sqrt{3}x\right) + C_3x^{2}e^{x}\sin\left(\sqrt{3}x\right) + C_4x^{2}e^{x}\cos\left(\sqrt{3}x\right)\\
&= C_0e^{x\left(1+\sqrt{3}\right)} + \left(C_1 + C_3x\right)xe^{x}\sin\left(\sqrt{3}x\right) + \left(C_2 + C_4x\right)xe^{x}\cos\left(\sqrt{3}x\right)
\end{aligned}
$$

We are asked to stop here and not solve for the coefficients. (good, since this is hard). Another option to find y_p is to use the Wronskian. But this generates hard to evaluate integral. The Wronskian is

$$\begin{aligned}
W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos\left(\sqrt{3}x\right) & e^x \sin\left(\sqrt{3}x\right) \\ e^x \cos\left(\sqrt{3}x\right) - \sqrt{3}\sin\left(\sqrt{3}x\right) e^x & e^x \sin\left(\sqrt{3}x\right) + \sqrt{3}\cos\left(\sqrt{3}x\right) e^x \end{vmatrix} \\
&= e^x \cos\left(\sqrt{3}x\right)\left(e^x \sin\left(\sqrt{3}x\right) + \sqrt{3}\cos\left(\sqrt{3}x\right) e^x\right) - e^x \sin\left(\sqrt{3}x\right)\left(e^x \cos\left(\sqrt{3}x\right) - \sqrt{3}\sin\left(\sqrt{3}x\right) e^x\right) \\
&= e^{2x} \sin\left(\sqrt{3}x\right)\cos\left(\sqrt{3}x\right) + \sqrt{3}e^{2x}\cos^2\left(\sqrt{3}x\right) - \left(e^{2x}\cos\left(\sqrt{3}x\right)\sin\left(\sqrt{3}x\right) - \sqrt{3}e^{2x}\sin^2\left(\sqrt{3}x\right)\right) \\
&= \sqrt{3}e^{2x}\cos^2\left(\sqrt{3}x\right) + \sqrt{3}e^{2x}\sin^2\left(\sqrt{3}x\right) \\
&= \sqrt{3}e^x
\end{aligned}$$

Assume now y_p is

$$y_p = y_1 u_1 + y_2 u_2$$

Where

$$\begin{aligned}
u_1 &= -\int \frac{y_2 f(x)}{W} dx \\
u_2 &= \int \frac{y_1 f(x)}{W} dx
\end{aligned}$$

Where $f(x) = e^x\left(x\sin\left(\sqrt{3}x\right) + \exp\left(\sqrt{3}x\right)\right)$. Hence

$$\begin{aligned}
u_1 &= -\int \frac{e^x \sin\left(\sqrt{3}x\right) e^x \left(x \sin\left(\sqrt{3}x\right) + e^{\sqrt{3}x}\right)}{\sqrt{3}e^x} dx \\
&= -\frac{1}{\sqrt{3}} \int x e^x \sin^2\left(\sqrt{3}x\right) + \sin\left(\sqrt{3}x\right) e^{1+\sqrt{3}x} dx
\end{aligned}$$

And

$$\begin{aligned}
u_2 &= \int \frac{e^x \cos\left(\sqrt{3}x\right) e^x \left(x \sin\left(\sqrt{3}x\right) + e^{\sqrt{3}x}\right)}{\sqrt{3}e^x} dx \\
&= \frac{1}{\sqrt{3}} \int x e^x \sin\left(\sqrt{3}x\right)\cos\left(\sqrt{3}x\right) + e^{1+\sqrt{3}x}\cos\left(\sqrt{3}x\right) dx
\end{aligned}$$

<u>For the second ODE</u>

$$y'' - 6y' = x^2 + x\cosh(6x)$$

We start by finding the homogenous solution for $y'' - 6y' = 0$. The characteristic equation is $r^2 - 6r = 0$, or $r(r-6) = 0$ which has roots $r_1 = 0, r_2 = 6$ hence

$$y_h = c_1 + c_2 e^{6x}$$

Therefore $y_1 = 1, y_2 = e^{6x}$. Since RHS is

$$\begin{aligned}
f(x) &= x^2 + x\cosh(6x) \\
&= x^2 + x\frac{e^{6x} + e^{-6x}}{2} \\
&= x^2 + \frac{1}{2}xe^{6x} + \frac{1}{2}xe^{-6x}
\end{aligned}$$

Then we see that y_2 which is solution of the homogenous solution is part of the forcing function. Let us find y_p now.

For x^2 we guess $c_1 + c_2x + c_3x^2$. But now we see that c_1 which is constant, is just $y_1 = 1$ (scalar multiple of). So we have to multiply the whole guess by x, resulting in $\left(c_1x + c_2x^2 + c_3x^3\right)$.

For xe^{6x} we guess $(c_4 + c_5x)e^{6x}$ but since $y_2 = e^{6x}$ we have to multiply the guess by x giving $\left(c_4x + c_5x^2\right)e^{6x}$.

For xe^{-6x} the guess is $(c_6 + c_7x)e^{-6x}$. Hence we collect all these and obtain the guess y_p as

$$y_p = \left(c_1x + c_2x^2 + c_3x^3\right) + (c_4 + c_5x)\,xe^{6x} + (c_6 + c_7x)\,e^{-6x}$$

Another option to find y_p is to use the Wronskian. The Wronskian is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{6x} \\ 0 & 6e^{x} \end{vmatrix} = 6e^{x}$$

Assume now y_p is

$$y_p = y_1u_1 + y_2u_2$$

Where

$$u_1 = -\int \frac{y_2 f(x)}{W}dx$$
$$u_2 = \int \frac{y_1 f(x)}{W}dx$$

Where $f(x) = x^2 + x\cosh(6x)$. Hence

$$u_1 = -\frac{1}{6}\int \frac{x^2 + x\cosh(6x)}{e^x}dx$$

And

$$u_2 = \int \frac{e^{6x}\left(x^2 + x\cosh(6x)\right)}{6e^x}dx = \frac{1}{6}\int e^{6x}\left(x^2 + x\cosh(6x)\right)dx$$

0.1.8 Problem 8

Find general solution to

$$y^{(6)} + 4y^{(5)} + 8y^{(4)} + 16y''' + 20y'' + 16y' + 16y = 0$$

The characteristic equation is

$$r^6 + 4y^5 + 8r^4 + 16r^3 + 20r^2 + 16r + 16 = 0$$

Using the hint

$$(r+2)^2\left(r^2+2\right)^2 = 0$$
$$(r+2)(r+2)\left(r^2+2\right)\left(r^2+2\right) = 0$$

Hence the roots are $r_1 = -2$ multiplicity 2 and $r_2 = \pm i\sqrt{2}$ multiplicity 2. hence the solution is

$$y_h = c_1e^{-2x} + c_2xe^{-2x} + c_3e^{i\sqrt{2}x} + c_4e^{-i\sqrt{2}x} + x\left(c_5e^{i\sqrt{2}x} + c_6e^{-i\sqrt{2}x}\right)$$

Or as real functions, using Euler relation

$$y_h = c_1e^{-2x} + c_2xe^{-2x} + c_3\cos\left(\sqrt{2}x\right) + c_4\sin\left(\sqrt{2}x\right) + c_5x\cos\left(\sqrt{2}x\right) + c_6x\sin\left(\sqrt{2}x\right)$$

Where constants labels kept the same for simplicity (in practice these are not the same). The solution is analytic everywhere, hence range of solution is $-\infty < x < \infty$