practice Ax=b, Math 320, Spring 2017

Nasser M. Abbasi (Discussion section 383, 8:50 AM - 9:40 AM Monday)

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Contents

0.1 Problem 1

1. Write the following system as $\mathbf{Ax} = \mathbf{b}$ and determine for what values of k the system has (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions. In the case of (i) or (iii), find the solution(s).

$$
\begin{aligned}
2x_1 + 2x_2 - x_3 &= 1 \\
3x_2 + 3x_3 &= 3 \\
4x_1 + x_2 + kx_3 &= -1
\end{aligned} \tag{1}
$$

solution

$$\begin{pmatrix} 2 & 2 & -1 \\ 0 & 3 & 3 \\ 4 & 1 & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & 3 & 3 & 3 \\ 4 & 1 & k & -1 \end{pmatrix}$$

We start by converting the above to Echelon form

$$\begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & 3 & 3 & 3 \\ 4 & 1 & k & -1 \end{pmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & -3 & k+2 & -3 \end{pmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & k+5 & 0 \end{pmatrix}$$

We see that the last equation now has the form

$$(k+5)\, x_3 = 0$$

If $k + 5 = n \neq 0$ then the equation becomes $nx_3 = 0$, which means $x_3 = 0$ is only choice, since $n \neq 0$. This means, from the second equation, $3x_2 + 3x_3 = 3$ or $x_2 = 1$ and from the first equation, $2x_1 + 2x_2 - x_3 = 1$ or $2x_1 + 2 = 1$ or $x_1 = \frac{-1}{2}$. Hence a unique solution. But if $k + 5 = 0$ then last equation gives $0x_3 = 0$, which means any x_3 will do the job. Hence infinite number of solutions.

Therefore, (i) $k \neq -5$ gives unique solution. (ii) Not possible. (iii) $k = -5$ gives infinite solutions.

0.2 Problem 2

2. For what values of k does $\mathbf{Ax} = \mathbf{b}$ have (i) no solution, (ii) a unique solution, or (iii) an infinite number of solutions? In the case of (ii) or (iii), find the solution(s).

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -2 \\ -1 & 1 & k \\ 3 & 1 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ 4 \\ 20 \end{bmatrix} \tag{2}$$

<u>solution</u>

$$\begin{pmatrix} 2 & 0 & -2 \\ -1 & 1 & k \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 20 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 2 & 0 & -2 & 8 \\ -1 & 1 & k & 4 \\ 3 & 1 & 4 & 20 \end{pmatrix}$$

We start by converting the above to Echelon form. Swap the second and third row

$$\begin{pmatrix} 2 & 0 & -2 & 8 \\ 3 & 1 & 4 & 20 \\ -1 & 1 & k & 4 \end{pmatrix}$$

Now

$$\begin{pmatrix} 2 & 0 & -2 & 8 \\ 3 & 1 & 4 & 20 \\ -1 & 1 & k & 4 \end{pmatrix} \xrightarrow[R_3 = R_3 + \frac{1}{2} R_1]{R_2 = R_2 - \frac{3}{2} R_1} \begin{pmatrix} 2 & 0 & -2 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 1 & k-1 & 8 \end{pmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 2 & 0 & -2 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & k-8 & 0 \end{pmatrix}$$

From last equation, we obtain $(k-8)\,x_3 = 0$.

(i) No solution case is not possible.

(ii) When $k \neq 8$, then unique solution. Hence $x_3 = 0$. Which means from second equation that $x_2 = 8$ and from first equation, $2x_1 = 8$ or $x_1 = 4$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 0 \end{pmatrix}$$

(iii) infinite number of solutions when $k = 8$. This gives $0\,(x_3) = 0$, hence any x_3 will do the job. Let $x_3 = t$, the second equation gives $x_2 + 7t = 8$ or $x_2 = 8 - 7t$. and the first equation gives $2x_1 - 2t = 8$ or $x_1 = 4 - t$. Hence solution is

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 4 - t \\ 8 - 7t \\ t \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 8 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -7 \\ 1 \end{pmatrix} \end{aligned}$$

0.3 Problem 3

3. In the following exercises, we write the augmented coefficient matrix for $\mathbf{Ax} = \mathbf{b}$. Determine for what values of the parameter p the system has (i) an unique solution, (ii) no solution, (iii) an infinite number of solutions. In case (i), find the unique solution. In case (iii), determine if there is a one-parameter family of solutions, or a two-parameter family of solutions, and find an expression for the solutions $\mathbf{x}$ in terms of the parameter(s).

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & p & 0 & 1 \\ -1 & -2 & 4 & 3 \end{bmatrix} \tag{3a}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & p & 4 & 2 \\ 3 & p+1 & 6 & p+1 \end{bmatrix} \tag{3b}$$

$$\begin{bmatrix} -2 & 3 & p & 1 \\ 4 & 3/2 & 2 & 2 \\ 3 & 3 & 5/2 & 5/2 \end{bmatrix} \tag{3c}$$

<u>solution</u>

0.3.1 Part a

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & p & 0 \\ -1 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & p & 0 & 1 \\ -1 & -2 & 4+\frac{3}{2} & 3 \end{pmatrix} \xrightarrow{R_3=R_3+\frac{1}{2}R_1} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & p & 0 & 1 \\ 0 & -\frac{3}{2} & \frac{11}{2} & \frac{7}{2} \end{pmatrix} \xrightarrow{R_3=R_3+\frac{3}{2p}R_2} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & p & 0 & 1 \\ 0 & 0 & \frac{11}{2} & \frac{1}{2p}\left(7p+3\right) \end{pmatrix}$$

We now convert the above to reduced Echelon form. First we make each leading entry 1

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{p} \\ 0 & 0 & 1 & \frac{1}{p}\frac{\left(7p+3\right)}{11} \end{pmatrix}$$

Now we zero out all entries in column above leading entries

$$\begin{pmatrix}1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2}\\ 0 & 1 & 0 & \frac{1}{p}\\ 0 & 0 & 1 & \frac{1}{p}\frac{(7p+3)}{11}\end{pmatrix} \xrightarrow{R_1=R_1-\frac{1}{2}R_2} \begin{pmatrix}1 & 0 & \frac{3}{2} & \frac{1}{2p}(p-1)\\ 0 & 1 & 0 & \frac{1}{p}\\ 0 & 0 & 1 & \frac{1}{p}\frac{(7p+3)}{11}\end{pmatrix} \xrightarrow{R_1=R_1-\frac{3}{2}R_3} \begin{pmatrix}1 & 0 & 0 & -\frac{5}{11p}(p+2)\\ 0 & 1 & 0 & \frac{1}{p}\\ 0 & 0 & 1 & \frac{1}{p}\frac{(7p+3)}{11}\end{pmatrix}$$

Hence, the last equation says

$$x_3 = \frac{1}{p}\frac{(7p+3)}{11}$$

Therefore, if $7p+3 \neq 0$ then x_3 is parameterized by p and we have infinite number of solutions. In this case the solution vector is

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}-\frac{5}{11p}(p+2)\\ \frac{1}{p}\\ \frac{1}{p}\frac{(7p+3)}{11}\end{pmatrix}$$

But if $7p+3=0$ then $x_3 = 0$, and this means $p = -\frac{3}{7}$. Then from second equation we obtain $x_2 = \frac{-7}{3}$ and from first equation $x_1 = -\frac{5}{11\left(-\frac{3}{7}\right)}\left(-\frac{3}{7}+2\right) = \frac{5}{3}$. Hence in this case the solution is unique

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}\frac{5}{3}\\ \frac{-7}{3}\\ 0\end{pmatrix}$$

In both cases, we assumed $p \neq 0$. It is no possible to obtain the case (ii) which is no solution.

0.3.2 Part b

$$\begin{pmatrix}1 & 1 & 2\\ 2 & p & 4\\ 3 & p+1 & 6\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}1\\ 2\\ p+1\end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix}1 & 1 & 2 & 1\\ 2 & p & 4 & 2\\ 3 & p+1 & 6 & p+1\end{pmatrix} \xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1} \begin{pmatrix}1 & 1 & 2 & 1\\ 0 & p-2 & 0 & 0\\ 0 & p-2 & 0 & p-2\end{pmatrix} \xrightarrow{R_3=R_3-R_2} \begin{pmatrix}1 & 1 & 2 & 1\\ 0 & p-2 & 0 & 0\\ 0 & 0 & 0 & p-2\end{pmatrix}$$

We see from last equation that $0(x_3) = p-2$. This means that if $p-2 \neq 0$ then there is no solution. This means if $p \neq 2$ then no solution. On the other hand, if $p = 2$ then last equation becomes $0(x_3) = 0$, which means any x_3 will do. Let $x_3 = t$. From second equation, we have

$$\left(p-2\right)x_2 = 0$$
$$0\left(x_2\right) = 0$$

So any x_2 will do. Let $x_2 = s$. Then the first equation becomes $x_1 + s + 2t = 1$ or $x_1 = 1 - s - 2t$. Hence solution vector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1-s-2t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Case (ii) do not apply. This is two family solution.

0.3.3 Part c

$$\begin{pmatrix} -2 & 3 & p \\ 4 & \frac{3}{2} & 2 \\ 3 & 3 & \frac{5}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \frac{5}{2} \end{pmatrix}$$

The augmented matrix is

$$\begin{pmatrix} -2 & 3 & p & 1 \\ 4 & \frac{3}{2} & 2 & 2 \\ 3 & 3 & \frac{5}{2} & \frac{5}{2} \end{pmatrix} \underset{R_3=R_3+\frac{3}{2}R_1}{\overset{R_2=R_2+2R_1}{\longrightarrow}} \begin{pmatrix} -2 & 3 & p & 1 \\ 0 & \frac{15}{2} & 2+2p & 4 \\ 0 & \frac{15}{2} & \frac{5}{2}+\frac{3}{2}p & 4 \end{pmatrix} \overset{R_3=R_3-R_2}{\longrightarrow} \begin{pmatrix} -2 & 3 & p & 1 \\ 0 & \frac{15}{2} & 2+2p & 4 \\ 0 & 0 & \frac{1}{2}-\frac{1}{2}p & 0 \end{pmatrix}$$

Last equation gives $\left(\frac{1}{2}-\frac{1}{2}p\right)x_3 = 0$. If $\frac{1}{2}-\frac{1}{2}p = 0$ or $p = 1$, then there are infinite number of solutions. Let $x_3 = t$. From second equation, $\frac{15}{2}x_2+\left(2+2p\right)x_3 = 4$ or $\frac{15}{2}x_2+4t = 4$, which gives $x_2 = \frac{8}{15}-\frac{8}{15}t$ and from first equation $-2x_1 + 3x_2 + x_3 = 1$ or $-2x_1 + 3\left(\frac{8}{15}-\frac{8}{15}t\right) + t = 1$, hence $x_1 = \frac{3}{10} - \frac{3}{10}t$. The solution vector is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}-\frac{3}{10}t \\ \frac{8}{15}-\frac{8}{15}t \\ t \end{pmatrix}$$

If $\frac{1}{2}-\frac{1}{2}p \neq 0$, then last equation gives $nx_3 = 0$ which is only possible if $x_3 = 0$. This means if $p \neq 1$, then $x_3 = 0$. Second equation gives $\frac{15}{2}x_2 = 4$ or $x_2 = \frac{8}{15}$ and first equation gives $-2x_1 + 3x_2 + x_3 = 1$ or $-2x_1 + 3\left(\frac{8}{15}\right) = 1$, or $x_1 = \frac{3}{10}$, hence solution vector is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10} \\ \frac{8}{15} \\ 0 \end{pmatrix}$$

case (ii) is not possible.