HW 7, Math 320, Spring 2017

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0.1 Section 3.6 problem 4 (page 216)

Problem Use cofactor expansion along row of column which minimize the amount of computation to find determinant of

$$A = \begin{pmatrix} 5 & 11 & 8 & 7 \\ 3 & -2 & 6 & 23 \\ 0 & 0 & 0 & -3 \\ 0 & 4 & 0 & 17 \end{pmatrix}$$

solution Since the 3^{rd} row has most zeros(as well as first column), expansion is carried on the last row. Therefore

$$\det(A) = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} + a_{34}A_{34}$$

But $a_{31} = a_{32} = a_{33} = 0$. Hence the above simplifies to

$$\begin{aligned} \det(A) &= a_{34}A_{34} \\ &= a_{34}(-1)^{3+4} M_{34} \\ &= -3(-1)^{7} M_{34} \\ &= 3M_{34} \end{aligned} \tag{1}$$

Now we need to find M_{34}, which is determinant of the matrix obtained from A by removing the third row and fourth column. Let this new matrix be called B

$$B = \begin{pmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{pmatrix}$$

$$M_{34} = \det(B)$$

We expand this along the 3^{rd} row of B, since that is the one with most zeros.

$$M_{34} = \det(B) = b_{31}B_{31} + b_{32}B_{32} + b_{33}B_{33}$$

But $b_{31} = b_{33} = 0$. So the above simplifies to

$$\begin{aligned} M_{34} &= b_{32}B_{32} \\ &= b_{32}(-1)^{3+2} M_{32} \\ &= 4(-1)^{5} M_{32} \\ &= -4M_{32} \end{aligned} \tag{2}$$

But

$$\begin{aligned} M_{32} &= \begin{vmatrix} 5 & 8 \\ 3 & 6 \end{vmatrix} \\ &= 30 - 24 \\ &= 6 \end{aligned}$$

Therefore from (2), $M_{34} = -4(6) = -24$ and from (1)

$$\begin{aligned}\det(A) &= 3M_{34}\\ &= 3(-24)\end{aligned}$$

Hence

$$\boxed{\det(A) = -72}$$

0.2 Section 3.6 problem 8

Problem Evaluate determinant of

$$A = \begin{pmatrix} 2 & 3 & 4 \\ -2 & -3 & 1 \\ 3 & 2 & 7 \end{pmatrix}$$

after first simplifying the computation by adding multiple of some row of column to another.

solution The determinant of matrix do not change by adding multiple of one row or multiple of a column to another row or to another column. In the above, we see that adding the second row to the first row gives

$$B = \begin{pmatrix} 0 & 0 & 5 \\ -2 & -3 & 1 \\ 3 & 2 & 7 \end{pmatrix}$$

Now, expanding on the first row, since that is the one with most zeros, gives

$$\det(B) = b_{11}B_{11} + b_{12}B_{12} + b_{13}B_{13}$$

But $b_{11} = b_{12} = 0$, hence

$$\begin{aligned} \det(B) &= b_{13}B_{13} \\ &= 5\left(-1\right)^{1+3} M_{13} \\ &= 5M_{13} \end{aligned}$$

But

$$M_{13} = \begin{vmatrix} -2 & -3 \\ 3 & 2 \end{vmatrix} = -4 + 9 = 5$$

Hence $\det(B) = 5\,(5) = 25$. But since $\det\left(B\right) = \det\left(A\right)$, then

$$\boxed{\det\left(A\right) = 25}$$

0.3 Section 3.6 problem 19

Problem Use the method of elimination to evaluate the determinant of

$$A = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ -2 & 3 & -2 & 3 \\ 0 & -3 & 3 & 3 \end{pmatrix}$$

solution The idea is to use Forward elimination to produce an upper triangle matrix. The determinant of upper triangle matrix is then easily found as the product of elements on the diagonal. Since determinant do not change when adding multiple of a row to another, this method works. So we need first to produce the Echelon form (triangle matrix)

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ -2 & 3 & -2 & 3 \\ 0 & -3 & 3 & 3 \end{pmatrix} \xrightarrow{R_3=R_3+2R_1} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -2 & 9 \\ 0 & -3 & 3 & 3 \end{pmatrix} \xrightarrow[R4=R_4+3R_2]{R_3=R_3-3R_2}$$
$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & -3 & 3 \end{pmatrix} \xrightarrow{R_4=R_4+\frac{3}{4}R_3} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & \frac{39}{4} \end{pmatrix}$$

Hence

$$\begin{aligned} \det(A) &= 1 \times 1 \times 4 \times \frac{39}{4} \\ &= 39 \end{aligned}$$

0.4 Section 3.6 problem 46

Problem Verify the property

$$\begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

solution This property is saying that adding k times the second columns of A to the first column of A do not change the determinant. This is property 5. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and let $B = \begin{pmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{pmatrix}$. Then

$$\begin{aligned} \det(B) &= \begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} \end{aligned}$$

But $k \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} = 0$ since the first column is the same as the second column. Hence $\det(B) = \det(A)$. Now we will show this is true by actual expansion, since this is what the problem is asking. Expanding B along the first column, gives

$$\begin{aligned} \det(B) &= b_{11}B_{11} + b_{21}B_{21} + b_{31}B_{31} \\ &= (a_{11}+ka_{12})(-1)^{1+1}M_{11} + (a_{21}+ka_{22})(-1)^{2+1}M_{21} + (a_{31}+ka_{32})(-1)^{3+1}M_{31} \\ &= (a_{11}+ka_{12})M_{11} - (a_{21}+ka_{22})M_{21} + (a_{31}+ka_{32})M_{31} \\ &= (a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}) + k(a_{12}M_{11} - a_{22}M_{21} + a_{32}M_{31}) \end{aligned}$$

But $(a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}) = \det(A)$, hence above becomes

$$\det(B) = \det(A) + k(a_{12}M_{11} - a_{22}M_{21} + a_{32}M_{31}) \tag{1}$$

But

$$\begin{aligned}
a_{12}M_{11} - a_{22}M_{21} + a_{32}M_{31} &= a_{12}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{32}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\
&= a_{12}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{22}\left(a_{12}a_{33} - a_{13}a_{32}\right) + a_{32}\left(a_{12}a_{23} - a_{13}a_{22}\right) \\
&= \overbrace{a_{12}a_{22}a_{33}} - \widehat{a_{12}a_{23}a_{32}} - \overbrace{a_{22}a_{12}a_{33}} + \overleftrightarrow{a_{22}a_{13}a_{32}} + \widehat{a_{32}a_{12}a_{23}} - \overleftrightarrow{a_{32}a_{13}a_{22}}
\end{aligned}$$

We see from the above, that all terms cancel out, and we obtain

$$a_{12}M_{11} - a_{22}M_{21} + a_{32}M_{31} = 0$$

Hence (1) becomes

$$\begin{aligned}
\det(B) &= \det(A) + k(0) \\
&= \det(A)
\end{aligned}$$

QED.

0.5 Section 3.6 problem 49

Problem Let $A = (a_{ij})$ be 3×3 matrix. Show that $\det(A^T) = \det(A)$ by expanding $\det(A)$ along its first row and $\det(A^T)$ along its first column.

solution Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Expanding $\det(A)$ along first row gives

$$\begin{aligned} \det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \qquad (1) \end{aligned}$$

But

$$B = A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

Finding $\det(A^T)$ by expanding along first column gives

$$\begin{aligned} \det(B) &= b_{11}B_{11} + b_{21}B_{21} + b_{31}B_{31} \\ &= a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{21} + a_{13}(-1)^{1+3}M_{31} \\ &= a_{11}M_{11} - a_{12}M_{21} + a_{13}M_{31} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \qquad (2) \end{aligned}$$

Examining (1) and (2) shows that they are the same expression. Hence

$$\det(A) = \det(A^T)$$

QED.

0.6 Section 3.6 problem 52

Problem The square matrix A is called orthogonal provided that $A^T = A^{-1}$. Show that the determinant of such matrix must be either +1 or −1.

solution

We are given $A^T = A^{-1}$. Premultiplying both sides by A gives

$$AA^T = AA^{-1}$$
$$AA^T = I$$

Taking the determinant of both sides gives

$$\det\left(AA^T\right) = \det(I)$$

But $\det(I) = 1$ hence

$$\det\left(AA^T\right) = 1$$

But $\det\left(AA^T\right) = \det\left(A^T\right)\det(A)$ by property of determinant of products, therefore the above becomes

$$\det\left(A^T\right)\det(A) = 1$$

But by property of determinant, we know that $\det(A) = \det\left(A^T\right)$, therefore the above becomes

$$\det(A)\det(A) = 1$$
$$(\det(A))^2 = 1$$

Therefore

$$\det(A) = \pm 1$$

QED

0.7 Section 3.6 problem 53

Problem The matrices A, B are said to be similar provided that $A = P^{-1}BP$ for some invertible matrix P. Show that if A and B are similar then $|A| = |B|$

solution

Since

$$A = P^{-1}BP \tag{1}$$

Pre multiplying both sides by P gives

$$\begin{aligned} PA &= PP^{-1}BP \\ &= \left(PP^{-1}\right) BP \\ &= IBP \\ &= BP \end{aligned}$$

Now, taking determinant of both sides gives

$$\begin{aligned} \det(PA) &= \det(BP) \\ \det(P)\det(A) &= \det(B)\det(P) \end{aligned}$$

Since P is invertible, then $\det(P) \neq 0$, therefore, we can divide both sides by $\det(P)$ and this gives

$$\det(A) = \det(B)$$

QED.