HW 5, Math 320, Spring 2017

Nasser M. Abbasi (Discussion section 383, 8:50 AM - 9:40 AM Monday)

December 30, 2019

Contents

0.1 Section 3.3 problem 8 (page 174)

Problem Find Reduced Echelon form for

$$\begin{pmatrix} 1 & -4 & -5 \\ 3 & -9 & 3 \\ 1 & -2 & 3 \end{pmatrix}$$

Solution

The first step is to obtain the Echelon form, then convert that to Reduced Echelon form

$$\begin{pmatrix} 1 & -4 & -5 \\ 3 & -9 & 3 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow[R_3=R_3-R_1]{R_2=R_2-3R_1} \begin{pmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 0 & 2 & 8 \end{pmatrix} \xrightarrow{R_3=R_3-\frac{2}{3}R_2} \begin{pmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 0 & 0 & -4 \end{pmatrix}$$

Now it is in Echelon form, we make it Reduced Echelon form. First we make each leading element 1

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 3 & 18 \\ 0 & 0 & -4 \end{pmatrix} \xrightarrow{R_2=\frac{1}{3}R_2} \begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & -4 \end{pmatrix} \xrightarrow{R_3=\frac{-1}{4}R_3} \begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Now we make all entries above each leading element zero

$$\begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1=R_1+4R_2} \begin{pmatrix} 1 & 0 & 19 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2=R_2-6R_3} \begin{pmatrix} 1 & 0 & 19 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1=R_1-19R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

0.2 Section 3.3 problem 9

<u>Problem</u> Find Reduced Echelon form for

$$\begin{pmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 4 & 1 & 12 \end{pmatrix}$$

<u>Solution</u>

The first step is to obtain the Echelon form, then we convert that to Reduced Echelon form

$$\begin{pmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 4 & 1 & 12 \end{pmatrix} \xrightarrow{R_3 = R_3 - \frac{4}{5}R_1} \begin{pmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 0 & -\frac{3}{5} & -\frac{12}{5} \end{pmatrix} \xrightarrow{R_3 = R_3 + \frac{3}{5}R_2} \begin{pmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Now it is in Echelon form, we make it Reduced Echelon form. First we make each leading element 1

$$\begin{pmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = \frac{1}{5}R_1} \begin{pmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we make all entries above each leading element zero

$$\begin{pmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 - \frac{2}{5}R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

0.3 Section 3.3 problem 31

Problem Show that the two matrices in (1) are both row equivalent to the 3×3 identity matrix (and hence by theorem 1, to each others)

Solution The two matrices in (1) are

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \tag{1}$$

We now reduce each matrix to Reduced Echelon form and see if we obtain the 3×3 identity matrix. Starting with the first matrix above, and since the matrices are already in Echelon form, we just need to do the reduction steps.

First we make each leading element 1

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \overset{R_2 = \frac{1}{4}R_2}{\longrightarrow} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 6 \end{pmatrix} \overset{R_3 = \frac{1}{6}R_3}{\longrightarrow} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

Now we make all entries above each leading element zero

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{pmatrix} \overset{R_1 = R_1 - 2R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{pmatrix} \overset{R_2 = R_2 - \frac{5}{4}R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{R_1 = R_1 - \frac{1}{2}R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Now we work on the second matrix. First we make each leading element 1

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \overset{R_2 = \frac{1}{2}R_2}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \overset{R_3 = \frac{1}{3}R_3}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now we make all entries above each leading element zero

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \overset{R_1 = R_1 - R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \overset{R_2 = R_2 - R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Comparing (2) and (3) we see that the Reduced Echelon form in both case came out to be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence both matrices in (1) are row equivalent.

0.4 Section 3.3 problem 32

Problem Show that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is row equivalent to the 2×2 identity matrix, provided $ad - bc \neq 0$

Solution let us convert the given matrix to Reduced Echelon form. Assuming $a \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_2 = R_2 - \frac{c}{a} R_1} \begin{pmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & \frac{ad-cb}{a} \end{pmatrix}$$

Now we need to make each leading element 1.

$$\begin{pmatrix} a & b \\ 0 & \frac{ad-cb}{a} \end{pmatrix} \xrightarrow{R_1 = \frac{1}{a} R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-cb}{a} \end{pmatrix} \xrightarrow{R_2 = \frac{a}{ad-cb} R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

Now, assuming that $ad - cb \neq 0$, only then we can do the next step, since we dividing by $ad - cb$

$$\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-cb}{a} \end{pmatrix} \xrightarrow{R_2 = \frac{a}{ad-cb} R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

Now we make all entries in column above each leading element zero.

$$\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 = R_1 - \frac{b}{a} R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So we see, that unless $ad - cb \neq 0$, we would not have been able to complete the Reduced Echelon form process, since in one the steps above, we would have divided by zero. We conclude that any 2×2 matrix is row equivalent to 2×2 identity matrix provided the determinant is not zero. Since $|A| = ad - cb$.

0.5 Section 3.3 problem 36

Problem Suppose that $ad-bc\neq0$ in the homogeneous system of problem 35. Use problem 32 to show that its only solution is the trivial solution.

Solution Problem 35 gives

$$
\begin{aligned}
ax+by&=0\\
cx+dy&=0
\end{aligned}
$$

In matrix form

$$
\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix} 0\\ 0\end{pmatrix}
$$

Since $ad-cb\neq0$, then using problem 32, we know A is row equivalent to 2×2 identity matrix. Which means the original system can now be written as

$$
\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix} 0\\ 0\end{pmatrix}
$$

Which means the solution is $x=0$ and $y=0$. The trivial solution.

$$
\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix} 0\\ 0\end{pmatrix}
$$

0.6 Section 3.3 problem 37

Problem Show that the system in problem 35 has a non-trivial solution iff $ad - bc = 0$

solution Problem 35 gives

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

In matrix form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The augment matrix is

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix}$$

We start by reducing it to Echelon form. We assume all along that $a \neq 0$.

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xrightarrow{R_2 = R_2 - \frac{c}{a} R_1} \begin{pmatrix} a & b & 0 \\ 0 & d - \frac{c}{a} b & 0 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ 0 & \frac{ad - cb}{a} & 0 \end{pmatrix}$$

Now we can solve by backward substitution. There are two cases to consider.

case 1 $ad - cb = 0$. In this case, the Echelon form becomes

$$\begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the second equation says $0 \left(y\right) = 0$. This implies infinite number of solutions, since any y will do the job.

case 2 $ad - cb \neq 0$. Lets say $ad - cb = N$, some non-zero value. In this case, the Echelon form becomes

$$\begin{pmatrix} a & b & 0 \\ 0 & N & 0 \end{pmatrix}$$

Hence the second equation says $N \left(y\right) = 0$. The solution to this is $y = 0$. Therefore, from the first equation we obtain

$$\begin{aligned} ax + by &= 0 \\ ax &= 0 \\ x &= 0 \end{aligned}$$

So we see that the solution vector is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is the trivial solution.

Conclusion The system has infinite number of solution iff $ad - cb = 0$ (this is the non-trivial solution case). And the system has unique solution, which is the trivial solution iff $ad - cb \neq 0$.