HW 10, Math 320, Spring 2017

Nasser M. Abbasi (Discussion section 383, 8:50 AM - 9:40 AM Monday)

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0.1 Section 5.1 problem 10 (page 299)

problem

Verify that y_1, y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the initial conditions. $y'' - 10y' + 25y = 0$ with $y_1 = e^{5x}, y_2 = xe^{5x}$ and $y(0) = 3, y'(0) = 13$

solution

To verify that y_1 or y_2 is solution to the ODE, we plug it into the ODE and see if it gives zero, which is what the RHS is. Since $y_1' = 5e^{5x}, y_1'' = 25e^{5x}$, then substituting this into the ODE gives

$$\begin{aligned} y_1'' - 10y_1' + 25y_1 &= 0 \\ 25e^{5x} - 10\left(5e^{5x}\right) + 25\left(e^{5x}\right) &= 0 \\ 25e^{5x} - 50e^{5x} + 25e^{5x} &= 0 \\ 0 &= 0 \end{aligned}$$

Hence verified. Now we do the same for y_2. Since $y_2' = e^{5x} + 5xe^{5x}, y_2'' = 5e^{5x} + 5e^{5x} + 25xe^{5x}$, then substituting this into the ODE gives

$$\begin{aligned} y_2'' - 10y_2' + 25y_2 &= 0 \\ \left(5e^{5x} + 5e^{5x} + 25xe^{5x}\right) - 10\left(e^{5x} + 5xe^{5x}\right) + 25\left(xe^{5x}\right) &= 0 \\ 5e^{5x} + 5e^{5x} + 25xe^{5x} - 10e^{5x} - 50xe^{5x} + 25xe^{5x} &= 0 \\ 25xe^{5x} - 50xe^{5x} + 25xe^{5x} &= 0 \\ 0 &= 0 \end{aligned}$$

Hence verified. Therefore the general solution is

$$y(x) = c_1 y_1(x) + c_2 y(x)$$

Where the constants are found from initial conditions. Using the first initial condition gives

$$\begin{aligned} y(0) &= 3 \\ c_1 y_1(0) + c_2 y_2(0) &= 3 \\ c_1\left(e^{5x}\right)_{x=0} + c_2\left(xe^{5x}\right)_{x=0} &= 3 \\ c_1 &= 3 \end{aligned}$$

Hence the solution becomes

$$\begin{aligned} y(x) &= 3y_1(x) + c_2 y_2(x) \\ y' &= 3y_1' + c_2 y_2' \\ &= 3\left(5e^{5x}\right) + c_2\left(e^{5x} + 5xe^{5x}\right) \end{aligned}$$

Applying the second boundary conditions gives

$$\begin{aligned} y'(0) &= 13 \\ 3\left(5e^{5x}\right)_{x=0} + c_2\left(e^{5x} + 5xe^{5x}\right)_{x=0} &= 13 \\ 3(5) + c_2 &= 13 \\ c_2 &= 13 - 15 \\ &= -2 \end{aligned}$$

Therefore the particular solution is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y(x) \\ &= 3y_1(x) - 2y(x) \\ &= 3e^{5x} - 2xe^{5x} \\ &= e^{5x}(3 - 2x) \end{aligned}$$

0.2 Section 5.1 problem 19

problem Show that $y_1 = 1, y_2 = \sqrt{x}$ are solutions to $yy'' + \left(y'\right)^2 = 0$ but that their sum $y = y_1 + y_2$ is not a solution

solution To show that y_1 and y_2 are solution to the ODE, we plug them into the ODE and see if the result is the same as the RHS. Since $y_1 = 1$ then $y_1' = 0, y_1'' = 0$. Then ODE becomes

$$\begin{aligned} y_1 y_1'' + \left(y_1'\right)^2 &= 0 \\ 1(0) + 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Hence verified. For y_2, we have $y_2' = \frac{1}{2x^{\frac{1}{2}}}, y_2'' = \frac{-1}{4}\frac{1}{x^{\frac{3}{2}}}$. Hence the ODE becomes

$$
\begin{aligned}
y_2 y_2'' + \left(y_2'\right)^2 &= 0 \\
x^{\frac{1}{2}}\left(\frac{-1}{4}\frac{1}{x^{\frac{3}{2}}}\right) + \left(\frac{1}{2x^{\frac{1}{2}}}\right)^2 &= 0 \\
\left(\frac{-1}{4}\frac{1}{x}\right) + \left(\frac{1}{4x}\right) &= 0 \\
0 &= 0
\end{aligned}
$$

Hence verified. Now we plugin the sum into the ODE.

$$
\begin{aligned}
\left(y_1 + y_2\right)\left(y_1 + y_2\right)'' + \left(\left(y_1 + y_2\right)'\right)^2 &= 0 \\
\left(y_1 + y_2\right)\left(y_1'' + y_2''\right) + \left(y_1' + y_2'\right)^2 &= 0 \\
\left(y_1 y_1'' + y_1 y_2''\right) + \left(y_2 y_1'' + y_2 y_2''\right) + \left(y_1'\right)^2 + \left(y_2'\right)^2 + 2y_1' y_2' &= 0 \\
y_1 y_1'' + y_1 y_2'' + y_2 y_1'' + y_2 y_2'' + \left(y_1'\right)^2 + \left(y_2'\right)^2 + 2y_1' y_2' &= 0
\end{aligned}
$$

But we found that $y_1 y_1'' + \left(y_1'\right)^2 = 0$ and $y_2 y_2'' + \left(y_2'\right)^2 = 0$ from earlier. Using these into the LHS of the above simplifies it to

$$
y_1 y_2'' + y_2 y_1'' + 2y_1' y_2' = 0
$$

But $y_2'' = \frac{-1}{4}\frac{1}{x^{\frac{3}{2}}}, y_1'' = 0, y_1' = 0, y_1 = 1$, then the above becomes

$$
\frac{-1}{4}\frac{1}{x^{\frac{3}{2}}} = 0
$$

We see that the LHS is not zero. Hence $y_1 + y_2$ is not a solution to the ODE.

0.3 Section 5.1 problem 24

problem Determine whether the pairs of functions are linearly independent or not on the real line.
$f(x) = \sin^2 x, g(x) = 1 - \cos 2x$

solution The two functions are L.I. if $c_1 f(x) + c_1 g(x) = 0$ for each x, only when $c_1 = c_2 = 0$. Or stated differently, two functions are L.D. if there exist c_1, c_2 not all zero, such that $c_1 f(x) + c_1 g(x) = 0$ for each x. To show this, we set up the Wronskian W and see if it is zero or not. If $W = 0$ then this mean that the functions are L.D.

$$
\begin{aligned}
W &= \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \\
&\begin{vmatrix} \sin^2 x & 1-\cos 2x \\ 2\sin x\cos x & 2\sin 2x \end{vmatrix} \\
&= 2\sin^2 x\sin 2x - (1-\cos 2x)(2\sin x\cos x) \\
&= 2\sin^2 x\sin 2x - 2\sin x\cos x + 2\cos 2x\sin x\cos x
\end{aligned}
$$

The RHS of the above simplifies to 0.

$$W = 0$$

Therefore, the functions are linearly dependent.

0.4 Section 5.1 problem 26

problem Determine whether the pairs of functions are linearly independent or not on the real line. $f(x) = 2\cos x + 3\sin x, g(x) = 3\cos x - 2\sin x$

solution To show this, we set up the Wronskian W and see if it is zero or not. If $W = 0$ then this mean that the functions are L.D.

$$
\begin{aligned}
W &= \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \\
&\begin{vmatrix} 2\cos x + 3\sin x & 3\cos x - 2\sin x \\ -2\sin x + 3\cos x & -3\sin x - 2\cos x \end{vmatrix} \\
&= (2\cos x + 3\sin x)(-3\sin x - 2\cos x) - (3\cos x - 2\sin x)(-2\sin x + 3\cos x) \\
&= -13\cos^2 x - 13\sin^2 x \\
&= -13\left(\cos^2 x + \sin^2 x\right) \\
&= -13
\end{aligned}
$$

Since $W \neq 0$ then the functions are Linearly independent.

0.5 Section 5.1 problem 27

problem Ley y_p be a particular solution of the nonhomogeneous equation $y'' + py' + qy = f(x)$ and let y_h be the homogenous solution. Show that $y = y_h + y_p$ is a solution of the given ODE.

solution since y_h satisfies the homogenous ODE then we can write

$$y_h'' + py_h' + qy_h = 0 \tag{1}$$

And since y_p satisfies the nonhomogeneous ODE then we can write

$$y_p'' + py_p' + qy_p = f(x) \tag{2}$$

Adding (1)+(2) gives

$$\left(y_p'' + y_h''\right) + p\left(y_p' + y_p'\right) + q\left(y_p + y_h\right) = f(x)$$

But due to linearity of differentiation, then the above can be written as

$$\left(y_p + y_h\right)'' + p\left(y_p + y_p\right)' + q\left(y_p + y_h\right) = f(x)$$

Let $Y = y_p + y_h$ then

$$Y'' + pY' + qY = f(x)$$

Therefore we showed that $Y = y_p + y_h$ satisfies the original ODE, hence it is a solution. QED

0.6 Section 5.1 problem 31

problem Show that $y_1 = \sin x^2$ and $y = \cos x^2$ are L.I. functions, but their Wronskian vanishes are $x = 0$. Why does this implies that there is no differential equation of the form $y'' + p(x)y' + q(x)y = 0$ with both p, q continuous everywhere, having both y_1, y_2 are solutions?

solution

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_2' & y_2' \end{vmatrix} = \\ &\begin{vmatrix} \sin x^2 & \cos x^2 \\ (2x)\cos x^2 & -(2x)\sin x^2 \end{vmatrix} \\ &= -2x\sin x^2 \sin x^2 - 2x\cos^2 \cos x^2 \\ &= -2x\left(\left(\sin x^2\right)^2 + \left(\cos x^2\right)^2\right) \\ &= -2x \end{aligned}$$

The Wronskian is zero at $x = 0$ but not zero at other points. It is only when $W = 0$ everywhere, we say that y_1, y_2 are L.D. We can have L.I. functions, but also have $W(x_0) = 0$ at some x_0 as in this problem. What this mean, is that $x = 0$ can not be in the domain of the solution for y_1, y_2 to be solutions to the ODE. Hence, since the domain of the solution is everywhere, this means $x = 0$ is part of the domain, then we conclude that y_1, y_2 can not be both solutions, since they are L.I. at $x = 0$.

0.7 Section 5.1 problem 32

problem Let y_1, y_2 be two solutions of $A(x)y'' + B(x)y' + C(x)y = 0$ on open interval I where A, B, C are continuous and $A(x)$ is never zero. (a) Let $W = W(y_1, y_2)$. Show that $A(x)\frac{dW}{dx} = y_1(Ay_2'') - y_2(Ay_1'')$ then substitute for Ay_2'' and Ay_1'' from the original ODE to show that $A(x)\frac{dW}{dx} = -B(x)W(x)$ (b) Solve this first order ODE equation to deduce Abel's formula $W(x) = k\exp\left(-\int \frac{B(x)}{A(x)}dx\right)$ where k is constant. (c) Why does Abel's formula imply that the Wronskian $W(y_1, y_2)$ is either zero everywhere or non-zero everywhere (as stated in theorem 3)?

solution

0.7.1 Part (a)

By definition

$$W(x) = y_1 y_2' - y_2 y_1'$$

Hence

$$\begin{aligned}\frac{dW}{dx} &= y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' \\ &= y_1 y_2'' - y_2 y_1''\end{aligned}$$

Therefore

$$\begin{aligned}A(x)\frac{dW}{dx} &= A(x)\left(y_1 y_2'' - y_2 y_1''\right) \\ &= y_1\left(A(x)y_2''\right) - y_2\left(A(x)y_1''\right) \qquad (1)\end{aligned}$$

But from original ODE, $A(x)y_1'' + B(x)y_1' + C(x)y_1 = 0$, therefore

$$A(x)y_1'' = -B(x)y_1' - C(x)y_1 \qquad (2)$$

And also from original ODE, $A(x)y_2'' + B(x)y_2' + C(x)y_2 = 0$, therefore

$$A(x)y_2'' = -B(x)y_2' - C(x)y_2 \qquad (3)$$

Substituting (2,3) into (1) gives

$$\begin{aligned}A(x)\frac{dW}{dx} &= y_1\left(-B(x)y_2' - C(x)y_2\right) - y_2\left(-B(x)y_1' - C(x)y_1\right) \\ &= -B(x)y_1y_2' - C(x)y_1y_2 + B(x)y_2y_1' + C(x)y_2y_1 \\ &= -B(x)y_1y_2' + B(x)y_2y_1' \\ &= -B(x)\left(y_1y_2' - y_2y_1'\right) \\ &= -B(x)W(x) \qquad (4)\end{aligned}$$

QED.

0.7.2 Part (b)

Solving (4).

$$\begin{aligned} A(x)\frac{dW}{dx} + B(x)W(x) &= 0 \\ \frac{dW}{dx} + \frac{B(x)}{A(x)}W(x) &= 0 \end{aligned}$$

Integrating factor is $\mu = e^{\int \frac{B(x)}{A(x)}dx}$, hence the above becomes

$$\frac{d}{dx}\left(\mu W(x)\right) = 0$$

Integrating gives

$$\begin{aligned} \mu W(x) &= k \\ W(x) &= ke^{-\int \frac{B(x)}{A(x)}dx} \end{aligned}$$

0.7.3 Part (c)

Since an exponential function is never zero (for bounded $\frac{B(x)}{A(x)}$), then $W(x) = ke^{(\cdot)}$ can only be zero if $k = 0$. This makes $W = 0$ everywhere when $k = 0$. But if $k \neq 0$, then $W \neq 0$ everywhere. So W can only be zero everywhere, or not zero everywhere.

0.8 Section 5.1 problem 34

problem Apply theorem 5 and 6 to find general solutions of the differential equation $y'' + 2y' - 15y = 0$

solution The characteristic equation is $r^2 + 2r - 15 = 0$, and the roots are

$$\begin{aligned} r_1 &= 3 \\ r_2 &= -5 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ &= c_1 e^{3x} + c_2 e^{-5x} \end{aligned}$$

0.9 Section 5.1 problem 42

problem Apply theorem 5 and 6 to find general solutions of the differential equation $35y'' - y' - 12y = 0$

solution The characteristic equation is $35r^2 - r - 12 = 0$, and the roots are

$$r_1 = \frac{3}{5}$$
$$r_2 = -\frac{4}{7}$$

Therefore the solution is

$$\begin{aligned} y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ &= c_1 e^{\frac{3}{5}x} + c_2 e^{-\frac{4}{7}x} \end{aligned}$$

0.10 Section 5.1 problem 48

problem Problem gives a general solution $y(x)$ of a homogeneous second order ODE $ay'' + by' + cy = 0$ with constant coefficients. Find such an equation $y(x) = e^x \left(c_1 e^{x\sqrt{2}} + c_2 e^{-x\sqrt{2}}\right)$

solution We compare the above solution to the general form of the solution given by

$$\begin{aligned} y &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ &= c_1 e^{x\left(1+\sqrt{2}\right)} + c_2 e^{x\left(1-\sqrt{2}\right)} \end{aligned}$$

We see that

$$r_1 = 1 + \sqrt{2}$$
$$r_2 = 1 - \sqrt{2}$$

This implies that the characteristic equation is

$$\begin{aligned} (r - r_1)(r - r_2) &= 0 \\ \left(r - \left(1 + \sqrt{2}\right)\right)\left(r - \left(1 - \sqrt{2}\right)\right) &= 0 \\ r^2 - 2r - 1 &= 0 \end{aligned}$$

Therefore the ODE is

$$y'' - 2y' - y = 0$$

Where $a = 1, b = -2, c = -1$.