HW 1, Math 320, Spring 2017

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Contents

1 HW 1

1.1 Section 1.3 problem 12

Determine whether existence of at least one solution of given initial value problem is guaranteed and is so, whether solution is unique.

$$\frac{dy}{dx} = x \ln y; y(1) = 1$$

Solution

$$f(x,y) = x \ln y$$

Sine $f(x,y)$ is continuous in x for all x and continuous in y for $y > 0$ and since initial condition is at point $(1,1)$, then a solution exist in some interval that contains $(1,1)$.

$$\frac{\partial f(x,y)}{\partial y} = \frac{x}{y}$$

Since $\frac{\partial f(x,y)}{\partial y}$ is continuous in x for all x and continuous in y for $y \neq 0$ and since initial condition is at point $(1,1)$, then the solution is unique in some interval that contains $(1,1)$. The following the the slope field for $f(x,y) = x \ln y$ showing small interval that contains $(1,1)$

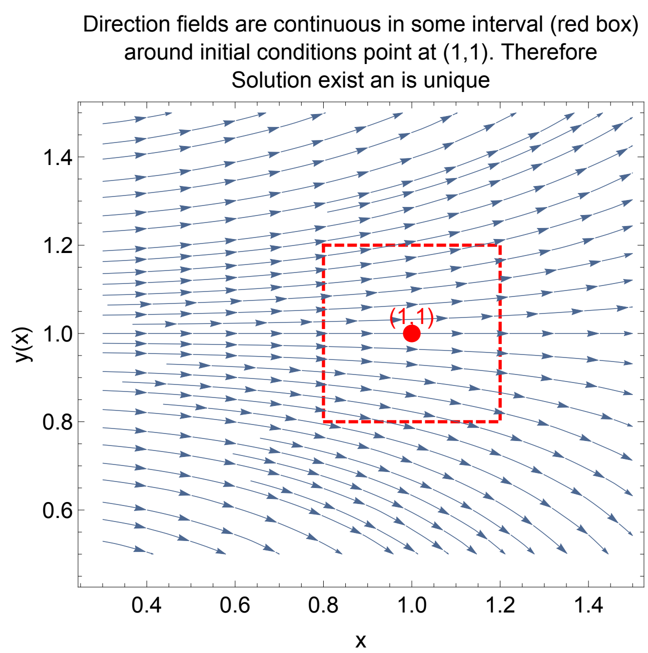

Figure 1: Problem 1.3, 11

1.2 Section 1.3 problem 17

Determine whether existence of at least one solution of given initial value problem is guaranteed and is so, whether solution is unique.

$$\frac{dy}{dx} = x - 1; y(0) = 1$$

Solution

$$f(x,y) = x - 1$$

$f(x,y)$ is continuous for all x (there is no y dependency to check), then a solution exist in some interval that contains $(0,1)$.

$$\frac{\partial f(x,y)}{\partial y} = 0$$

No dependency on x or y to check. Hence solution is unique in some interval that contains $(0,1)$. The following the the slope field for $f(x,y) = x-1$ showing small interval that contains $(0,1)$

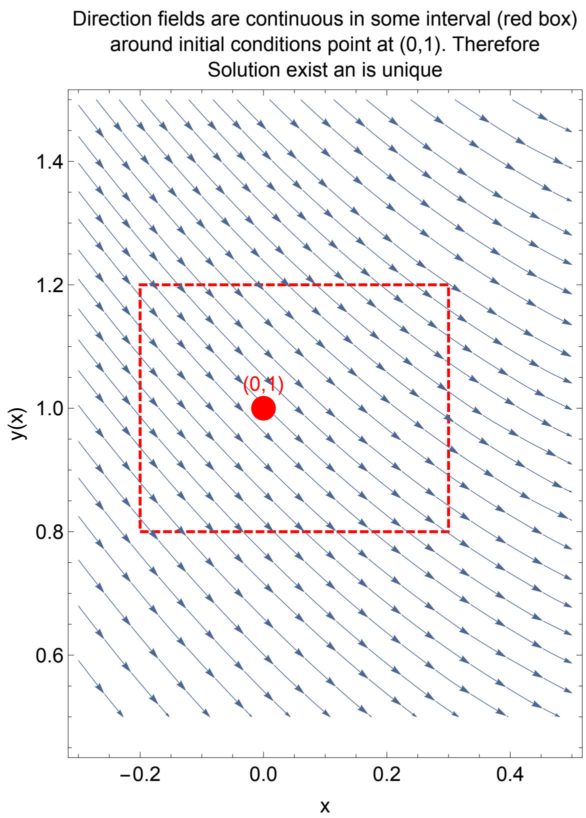

Figure 2: Problem 1.3, 17

1.3 Section 1.3 problem 18

Determine whether existence of at least one solution of given initial value problem is guaranteed and is so, whether solution is unique.

$$y\frac{dy}{dx} = x - 1; y(1) = 0$$

Solution

$$f(x,y) = \frac{x-1}{y}$$

$f(x,y)$ is continuous for all x, and continuous for all y except at $y = 0$. But since the initial point itself is at $(x = 1, y = 0)$, therefore, the theory can not decide on existence or uniqueness of solution in an intervals containing $(1,0)$.

1.4 Section 1.3 problem 22

Use the method of example 2 (page 20) to construct slope field then sketch solution curve corresponding to the given initial condition. Finally use this solution curve to estimate the desired value of the solution $y(x)$.

$$\frac{dy}{dx} = y - x$$
$$y(4) = 1$$
$$y(-4) = ?$$

Solution

$$f(x,y) = y - x$$

By making slope field for $f(x,y) = y - x$, then locating initial point $(4,1)$ and tracing the slope back to $x = -4$, we can then read the y value to be -3. Here is a plot showing trace of the slope field to the point $x = -4$, where $y = -3$. Hence $y(-4) \approx -3$.

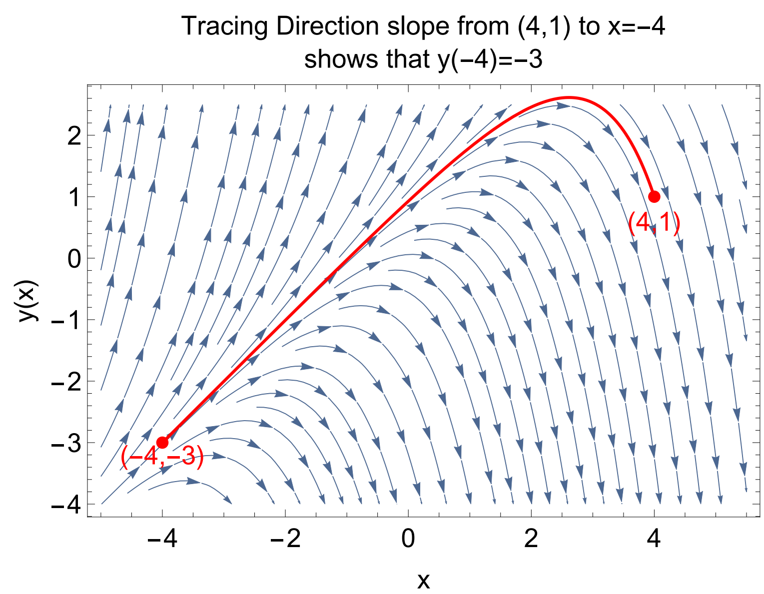

Figure 3: Problem 1.3, 22

1.5 Section 1.3 problem 26

Suppose the deer population $P(t)$ in small forest satisfies logistic equation $\frac{dp}{dt} = 0.0225p - 0.0003p^2$. Construct a slope field and appropriate solution curve to answer the following questions: If there are 25 deer at time $t = 0$ and t is measured in months, how long will it take for the number of deer to double? What will be the limiting deer population?

Solution

The slope field was first drawn. Then the point $(0,25)$ was located. Then the slope field was traced until $y = 50$, which is double the number of deer from the initial starting time. Now the t component was read from the slope field to answer the first part of the question.

$$f(t,p) = 0.0225p - 0.0003p^2$$

Here is a plot showing trace of the slope field. This shows at about $t = 60$ months, the deer population will be 50.

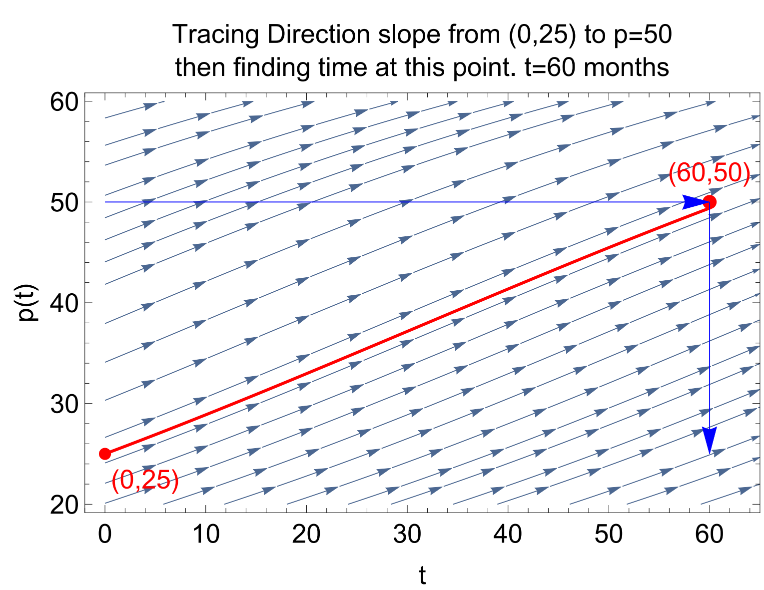

Figure 4: Problem 1.3, 26

1.6 Section 1.3 problem 28

Verify that if k is constant, then the function $y(x) = kx$ satisfies the differential equation $xy' = y$ for all x. Construct a slope field and several of the these straight line solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $xy' = y; y(a) = b$ has. One, none or infinitely many.

Solution

To verify that $y(x) = kx$ satisfies the differential equation, we plug-in this solution into the ODE and check that we get the same RHS as given. We see that $y'(x) = k$. Therefore $xy' = y$ becomes $x(k) = y = kx$. Hence satisfied.

$$f(x, y) = \frac{y}{x}$$

This is continuous for all x except at $x = 0$ and continuous for all y. Therefore solution exist in interval which do not contain $x = 0$. In addition $\frac{\partial f(x,y)}{\partial y} = \frac{1}{x}$ which is continuous for all x except at $x = 0$. Hence there is a solution and the solution is unique in an interval that do not contain $x = 0$. Here is a plot of the slope field in region around the origin.

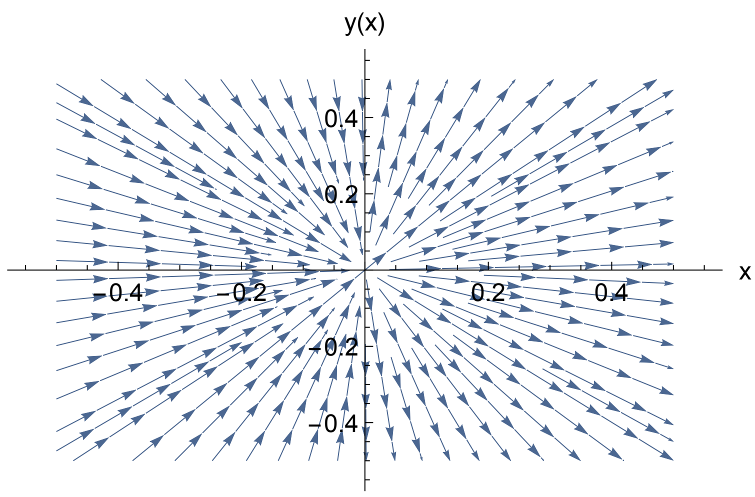

Figure 5: Problem 1.3, 28

We see from the above, that if we start from $x = 0, y = 0$, then there are ∞ number of solutions, since there are ∞ number of slope lines starting or ending at $(0, 0)$. For any point (a, b) where $a \neq 0$, there is unique solution, since we can find interval around (a, b) in this case with unique slope line. Finally, if $a = 0$ but $b \neq 0$, which means the initial condition is at the y axis, then there is no solution, since the slop is ∞ in this case. Hence

1. Infinite number of solution if $a = 0$ and $b = 0$
2. No solution if $a = 0, b \neq 0$
3. Unique solution if $a \neq 0$ and $b \neq 0$.

1.7 Section 1.3 problem 30

Verify that if c is constant, then the function defined piecewise by

$$y(x) = \begin{cases} 1 & x \leq c \\ \cos(x - c) & c < x < c + \pi \\ -1 & x \geq c + \pi \end{cases}$$

Satisfies $y' = -\sqrt{1 - y^2}$ for all x. (Perhaps an preliminary sketch with $c = 0$ will be helpful). Sketch a variety of such solution curves. Then determine (in terms of a and b how many different solutions the initial value problem $y' = -\sqrt{1 - y^2}; y(a) = b$ has.

Solution

The solution $y(x)$ is plotted for $c = 0, -1, +1$. The following show the result. The effect of c is that it causes a shift to the left or right depending on value of c.

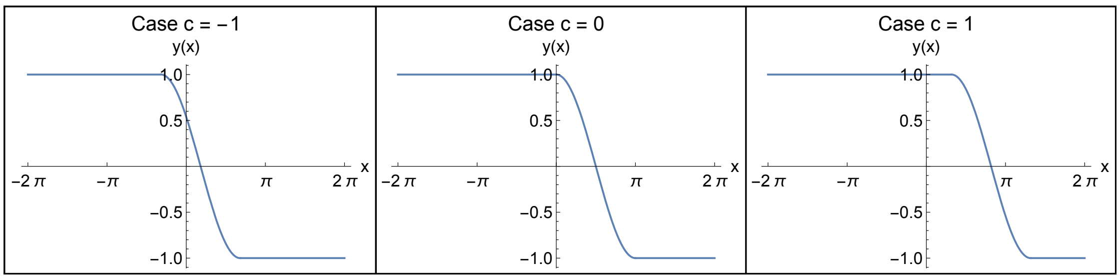

Figure 6: Problem 1.3, 30

Since

$$f(x,y) = -\sqrt{1-y^2}$$

Then the above is real, when $|y| < 1$ otherwise the value under the root will be become negative. To show that $y(x)$ satisfies the ODE, we plug-in each branch of the piecewise, one at a time, into the ODE and see if it satisfies it. When $x \leq c$, then $y(x) = 1$. Plugging this into the ODE gives $0 = 0$. Verified. When $c < x < c + \pi$, then $y(x) = \cos(x-c)$. Plugging this into the ODE gives

$$\begin{aligned} -\sin(x-c) &= -\sqrt{1-(\cos(x-c))^2} \\ &= -\sqrt{\sin(x-c)^2} \\ &= -\sin(x-c) \end{aligned}$$

Hence satisfied. When $x \geq c + \pi$ then $y(x) = -1$ and plugging this into the ODE gives

$$\begin{aligned} 0 &= -\sqrt{1-(-1)^2} \\ &= -\sqrt{1-1} \\ &= 0 \end{aligned}$$

Hence solution $y(x)$ satisfies the ODE. The slope field is now plotted

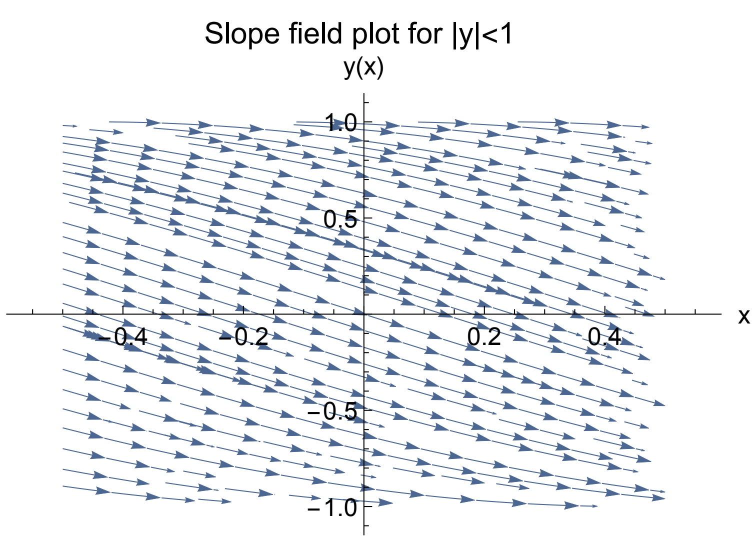

Figure 7: Problem 1.3, 30

We see from the slope plot, that starting at any point in a region, as long as $|y| < 0$, then the solution is unique. When $y = 1$ or $y = -1$, then $y' = 0$, and this gives infinite number of solutions since $y = c$ for any constant is a solution. For real solution, y can not be larger than 1. Hence in summary

1. Infinite number of solutions if $b = \pm 1$
2. Unique solution for any (a, b) where $|b| < 1$
3. No real solution for $|b| > 1$

1.8 Section 1.4 problem 6

Final general solution of $\frac{dy}{dx} = 3\sqrt{xy}$

Solution

This is separable.

$$\frac{dy}{\sqrt{y}} = 3\sqrt{x}dx$$
$$y^{\frac{-1}{2}}dy = 3x^{\frac{1}{2}}dx$$

Integrating

$$\frac{y^{\frac{1}{2}}}{\frac{1}{2}} = 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$2y^{\frac{1}{2}} = 2x^{\frac{3}{2}} + c$$
$$y^{\frac{1}{2}} = x^{\frac{3}{2}} + c_1$$
$$y = \left(x^{\frac{3}{2}} + c_1\right)^2$$

1.9 Section 1.4 problem 10

Final general solution of $(1+x)^2 \frac{dy}{dx} = \left(1+y\right)^2$

Solution

Before solving, it is good idea to check if the solution exist and if it is unique.

$$f\left(x,y\right) = \frac{\left(1+y\right)^2}{(1+x)^2}$$

$f\left(x,y\right)$ is continuous for all y but not continuous for $x = -1$. Therefore solution exist as long as solution interval or initial conditions do not include $x = -1$.

$$\frac{\partial f\left(x,y\right)}{\partial y} = \frac{2\left(1+y\right)}{(1+x)^2}$$

$f\left(x,y\right)$ is continuous for all y but not continuous for $x = -1$. Therefore solution exist and is unique as long as solution interval or initial conditions do not include $x = -1$. The slope field is given below

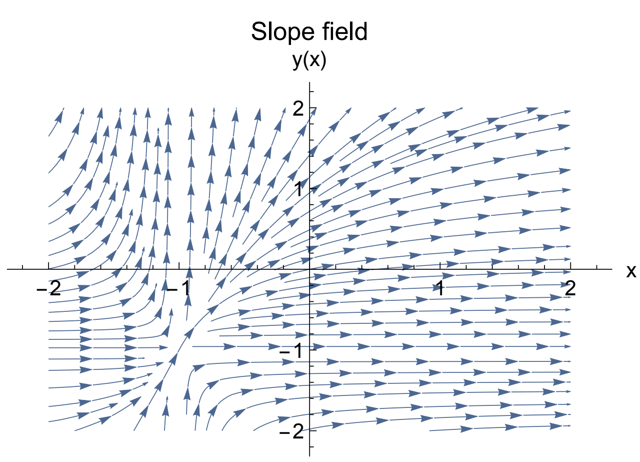

Figure 8: Problem 1.4, 10

Now the ODE is solved. This is separable.

$$\frac{dy}{\left(1+y\right)^2} = \frac{dx}{(1+x)^2}$$

Integrating

$$\int \frac{dy}{\left(1+y\right)^2} = \int \frac{dx}{(1+x)^2}$$

Let $u = 1 + y$ then $\frac{du}{dy} = 1$. Hence $\int \frac{dy}{\left(1+y\right)^2} \rightarrow \int \frac{du}{u^2} = -\frac{1}{u} \rightarrow \frac{-1}{1+y}$. Similarly, $\int \frac{dx}{(1+x)^2} = \frac{-1}{1+x}$.

Therefore the above becomes

$$\begin{aligned}
\frac{-1}{1+y} &= \frac{-1}{1+x} + c \\
\frac{1}{1+y} &= \frac{1}{1+x} + c_1 \\
\frac{1}{1+y} &= \frac{1 + c_1 (1+x)}{1+x} \\
1 + y &= \frac{1+x}{1 + c_1 (1+x)}
\end{aligned}$$

Hence

$$\boxed{y = \frac{1+x}{1+c_1(1+x)} - 1}$$

For $x \neq -1$.

1.10 Section 1.4 problem 22

Find explicit particular solution of $\frac{dy}{dx} = 4x^3 y - y; y(1) = -3$

Solution

Before solving, it is good idea to check if the solution exist and if it is unique.

$$f\left(x,y\right) = 4x^3 y - y$$

$f\left(x,y\right)$ is continuous for all y and continuous for all x.

$$\frac{\partial f\left(x,y\right)}{\partial y} = 4x^3 - 1$$

$\frac{\partial f(x,y)}{\partial y}$ is continuous for all x. It does not depend on y. Hence solution is exist and is unique in some interval that contain initial point $(1,-3)$. Now the ODE is solved.

$$\frac{dy}{dx} = y\left(4x^3 - 1\right)$$

This is now separable

$$\frac{dy}{y} = \left(4x^3 - 1\right) dx$$

Integrating

$$\begin{aligned}
\ln\left|y\right| &= 4\frac{x^4}{4} - x + c \\
\ln\left|y\right| &= x^4 - x + c \\
y &= e^{x^4 - x + c}
\end{aligned}$$

Let $e^c = c_1$, then the above can be written as

$$y = c_1 e^{x^4 - x}$$

Now the constant of integration is found from initial conditions. $y(1) = -3$, therefore

$$-3 = c_1 e^{1-1} = c_1$$

Hence the solution becomes

$$y(x) = -3e^{x^4 - x}$$

Here is a plot of the solution in small interval around $x = 1$

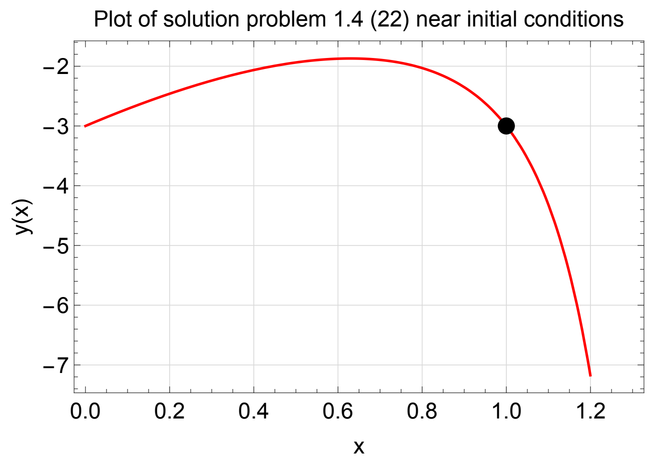

Figure 9: Problem 1.4, 22

1.11 Section 1.4 problem 26

Find explicit particular solution of $\frac{dy}{dx} = 2xy^2 + 3x^2y^2; y(1) = -1$

Solution

Before solving, it is good idea to check if the solution exist and if it is unique.

$$f(x, y) = 2xy^2 + 3x^2y^2$$

$f(x, y)$ is continuous for all y and continuous for all x.

$$\frac{\partial f(x, y)}{\partial y} = 4xy + 6x^2y$$

$\frac{\partial f(x,y)}{\partial y}$ is continuous for all x and for all y. Hence a solution is exist and is unique in some interval that contain initial point $(1, -1)$. Now the ODE is solved.

$$\frac{dy}{dx} = y^2 \left(2x + 3x^2\right)$$

This is separable.

$$\frac{dy}{y^2} = 2x + 3x^2 dx$$

Integrating

$$\begin{aligned} -\frac{1}{y} &= x^2 + x^3 + c \\ \frac{1}{y} &= -\left(x^2 + x^3 + c\right) \\ y &= \frac{-1}{x^2 + x^3 + c} \end{aligned}$$

Applying initial conditions to find c gives

$$\begin{aligned} -1 &= \frac{-1}{1 + 1 + c} \\ -2 - c &= -1 \\ c &= -1 \end{aligned}$$

Hence solution is

$$\begin{aligned} y &= \frac{-1}{x^2 + x^3 - 1} \\ &= \frac{1}{1 - x^2 - x^3} \end{aligned}$$

Here is a plot of the solution in small interval around $x = 1$

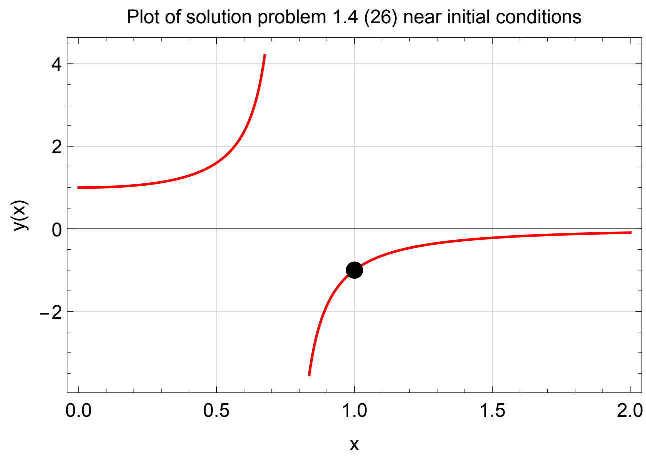

Figure 10: Problem 1.4, 26

We notice that at the real root of $1 - x^2 - x^3$, the solution $y(x)$ goes to $\pm\infty$. This happens at $x \approx 0.75487$.

1.12 Section 1.4 problem 30

Solve $\left(\frac{dy}{dx}\right)^2 = 4y$ to verify the general solution curves and singular solution curve that are illustrated in fig 1.4.5. Then determine the points (a, b) in the plane for which the initial value problem $\left(y'\right)^2 = 4y; y(a) = b$ has (a) No solution, (b) infinitely many solutions that are defined for all x, (c) on some neighborhood of the point $x = a$, only finitely many solutions.

Solution

Figure 1.4.5 is below

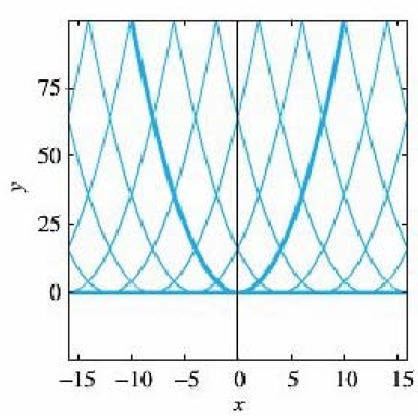

FIGURE 1.4.5. The general solution curves $y = (x - C)^2$ and the singular solution curve $y = 0$ of the differential equation $(y')^2 = 4y$.

$$f(x, y) = \pm 2\sqrt{y}$$

Hence $f(x, y)$ is continuous in y for $y > 0$. Hence solutions exist for $y > 0$. $\frac{\partial f(x,y)}{\partial y} = \pm 2\frac{1}{\sqrt{y}}$ and this is also continuous in y for $y > 0$. Therefore, unique solution exist for $y > 0$. (Interval can be found around initial conditions (a, b) as long as $b > 0$). Here is slope field plot

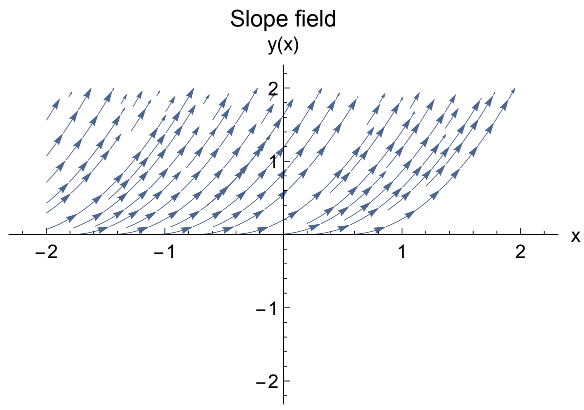

Figure 11: Problem 1.4, 30

$$\frac{dy}{dx} = \pm 2\sqrt{y}$$

For the negative case, we obtain

$$\begin{aligned} y^{\frac{-1}{2}}dy &= -2dx \\ 2y^{\frac{1}{2}} &= -2x + c \\ y^{\frac{1}{2}} &= -x + c_1 \\ y &= (c_1 - x)^2 \end{aligned}$$

For the positive case

$$\begin{aligned} y^{\frac{-1}{2}}dy &= 2dx \\ 2y^{\frac{1}{2}} &= 2x + c \\ y^{\frac{1}{2}} &= x + c_1 \\ y &= (c_1 + x)^2 \end{aligned}$$

Hence the solutions are

$$y(x) = \begin{cases} (c_1 - x)^2 & \\ (c_1 + x)^2 & \\ 0 & \text{singular solution} \end{cases}$$

The solution $y(x) = 0$ is singular, since it can not be obtained from the general solution $(c_1 - x)^2$ for arbitrary c. Summary:

1. No solution for $y < 0$
2. singular solution for $y = 0$
3. Two general solutions $(c_1 - x)^2$ and $(c_1 + x)^2$ for all x and $y > 0$.

The following is plot of $y(x) = (c_1 - x)^2$ for few values of c_1 to show the shape of the solution curves. This agrees with the figure given in the book.

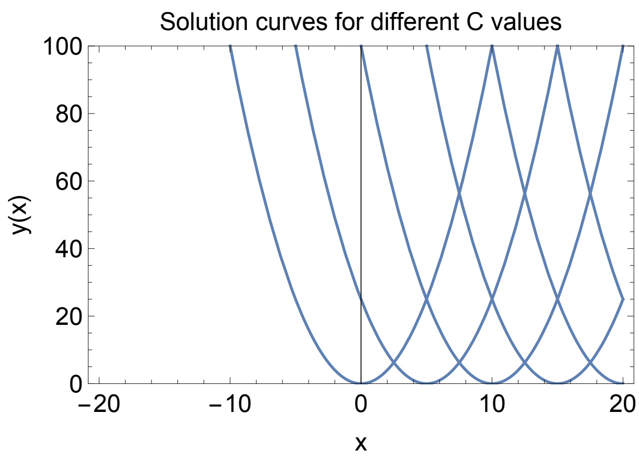

Figure 12: Problem 1.4, 30

1.13 Section 1.4 problem 42

A certain moon rock was found to contain equal number of potassium and argon atoms. Assume that all the argon is the result of radioactive decay of potassium (its half like is about 1.28×10^9 years) and that one of every nine potassium atom disintegrations yields an argon atom. What is the age of the rock, measured from the time it contained only potassium?

Solution

Half life is the time for a quantity to reduce to half its original number. Let $T = 1.28 \times 10^9$ years in this example. Let $P(0)$ be the number of potassium atoms at time $t = 0$. Hence the formula for half life decay is

$$P(t) = P(0)\left(\frac{1}{2}\right)^{\frac{t}{T}}$$

Where in the above $P(t)$ is number of potassium atoms that remain after time t. Let $g(t)$ be the number of argon atoms at time t. Since $\frac{1}{9}$ of the decayed potassium atoms changed to argon, then

$$\begin{aligned} g(t) &= \frac{1}{9}(P(0) - P(t)) \\ &= \frac{1}{9}\left(P(0) - P(0)\left(\frac{1}{2}\right)^{\frac{t}{T}}\right) \\ &= \frac{1}{9}P(0)\left(1 - \left(\frac{1}{2}\right)^{\frac{t}{T}}\right) \end{aligned}$$

Since we want to find t when $g(t) = P(t)$, then we solve from

$$\begin{aligned}
g(t) &= P(t) \\
\frac{1}{9}P(0)\left(1-\left(\frac{1}{2}\right)^{\frac{t}{T}}\right) &= P(0)\left(\frac{1}{2}\right)^{\frac{t}{T}} \\
\frac{1}{9}\left(1-\left(\frac{1}{2}\right)^{\frac{t}{T}}\right) &= \left(\frac{1}{2}\right)^{\frac{t}{T}} \\
1-\left(\frac{1}{2}\right)^{\frac{t}{T}} &= 9\left(\frac{1}{2}\right)^{\frac{t}{T}} \\
1 &= 9\left(\frac{1}{2}\right)^{\frac{t}{T}} + \left(\frac{1}{2}\right)^{\frac{t}{T}} \\
1 &= 10\left(\frac{1}{2}\right)^{\frac{t}{T}} \\
\frac{1}{10} &= \left(\frac{1}{2}\right)^{\frac{t}{T}}
\end{aligned}$$

Taking log

$$\begin{aligned}
\log\left(\frac{1}{10}\right) &= \frac{t}{T}\log\left(\frac{1}{2}\right) \\
t &= T\frac{\log\left(\frac{1}{10}\right)}{\log\left(\frac{1}{2}\right)} \\
&= 1.28\times 10^{9}\left(\frac{-2.3}{-0.693}\right) \\
&= 4.2482\times 10^{9}
\end{aligned}$$

Hence <u>it will take 4.2482 billion years.</u>

1.14 Section 1.4 problem 46

The barometric pressure p (in inches of mercury) at an altitude x miles above sea level satisfies the initial value problem $\frac{dp}{dx} = (-0.2)p; p(0) = 29.92$. (a) Calculate the barometric pressure at $10,000$ ft. and again at $30,000$ ft. (b) Without prior conditioning, few people can survive when the pressure drops to less than 15 in. Of mercury. How high is that?

<u>Solution</u>

1.14.1 Part (a)

$$\frac{dp}{dx} = (-0.2)p$$

This is separable.

$$\begin{aligned}
\frac{dp}{p} &= -0.2dx \\
\ln|p| &= -0.2x + c \\
p &= ce^{-0.2x}
\end{aligned}$$

To find c, we apply initial conditions. At $x = 0, p = 29.92$ in, hence

$$29.92 = c$$

Therefore the general solution is

$$p = 29.92e^{-0.2x}$$

Now, when $x = 10000$ ft or $10000/5280 = 1.894$ miles, then

$$
\begin{aligned}
p &= 29.92e^{-0.2(1.894)} \\
&= 20.486 \text{ in}
\end{aligned}
$$

when $x = 30000$ ft or $30000/5280 = 5.6818$ miles, then

$$
\begin{aligned}
p &= 29.92e^{-0.2(5.6818)} \\
&= 9.6039 \text{ in}
\end{aligned}
$$

1.14.2 Part (b)

We solve for x from

$$
\begin{aligned}
15 &= 29.92e^{-0.2x} \\
\frac{15}{29.92} &= e^{-0.2x}
\end{aligned}
$$

Taking natural log

$$
\begin{aligned}
\ln \frac{15}{29.92} &= -0.2x \\
-0.69047 &= -0.2x
\end{aligned}
$$

Hence

$$
\begin{aligned}
x &= \frac{0.69047}{0.2} = 3.4524 \text{ miles} \\
&= (3.4524)(5280) = 18229 \text{ ft}
\end{aligned}
$$