Animation of figure 9.5

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Animation of figure 9.5 in text page 434, compare higher order boundary layer solutions

This is an animation of the percentage error between boundary layer solution and the exact solution for $\epsilon y'' + (1+x) y' + y = 0$ for different orders. Orders are given in text up y0,y1,y2,y3. We see the error is smaller when order increases as what one would expect.

The percentage error is largest around x=0.1 than other regions. Why? Location of matching between yin and yout. Due
to approximation made when doing matching? But all boundary layer solutions underestimate the exact solution since error is negative.

The red color is the most accurate. 3rd order.

First animation

In[45]:=

```
Clear["Global`*"];
yout[x_] := 2 / (1 + x);
yin[x_, eps_] :=
  Plot[2 - Exp[-x / eps], {x, 0, 1}, AxesOrigin → {0, 1}, PlotStyle → Black];
p1 = Plot[yout[x], {x, 0, 1}, PlotStyle → Blue];
combine[x_, eps_] := Plot[yout[x] - Exp[-x / eps], {x, 0, 1}, PlotStyle → {Thick, Red},
   AxesOrigin → {0, 1}, GridLines → Automatic, GridLinesStyle → LightGray];
Animate[
 Grid[{{TraditionalForm[
     NumberForm[eps, {Infinity, 4}] HoldForm[y''[x] + (1 + x) y'[x] + y[x] == 0]]},
   {Show[
     Legended[combine[x, eps], Style["combinbed solution", Red]],
     Legended[p1, Style["Outer solution", Blue]],
     Legended[yin[x, eps], Style["Inner solution", Black]]
     , PlotRange → {{0, 1}, {1, 2}}, ImageSize → 400]
   }}], {eps, 0.001, .1, .00001}]
```

second animation, y0 vs. exact

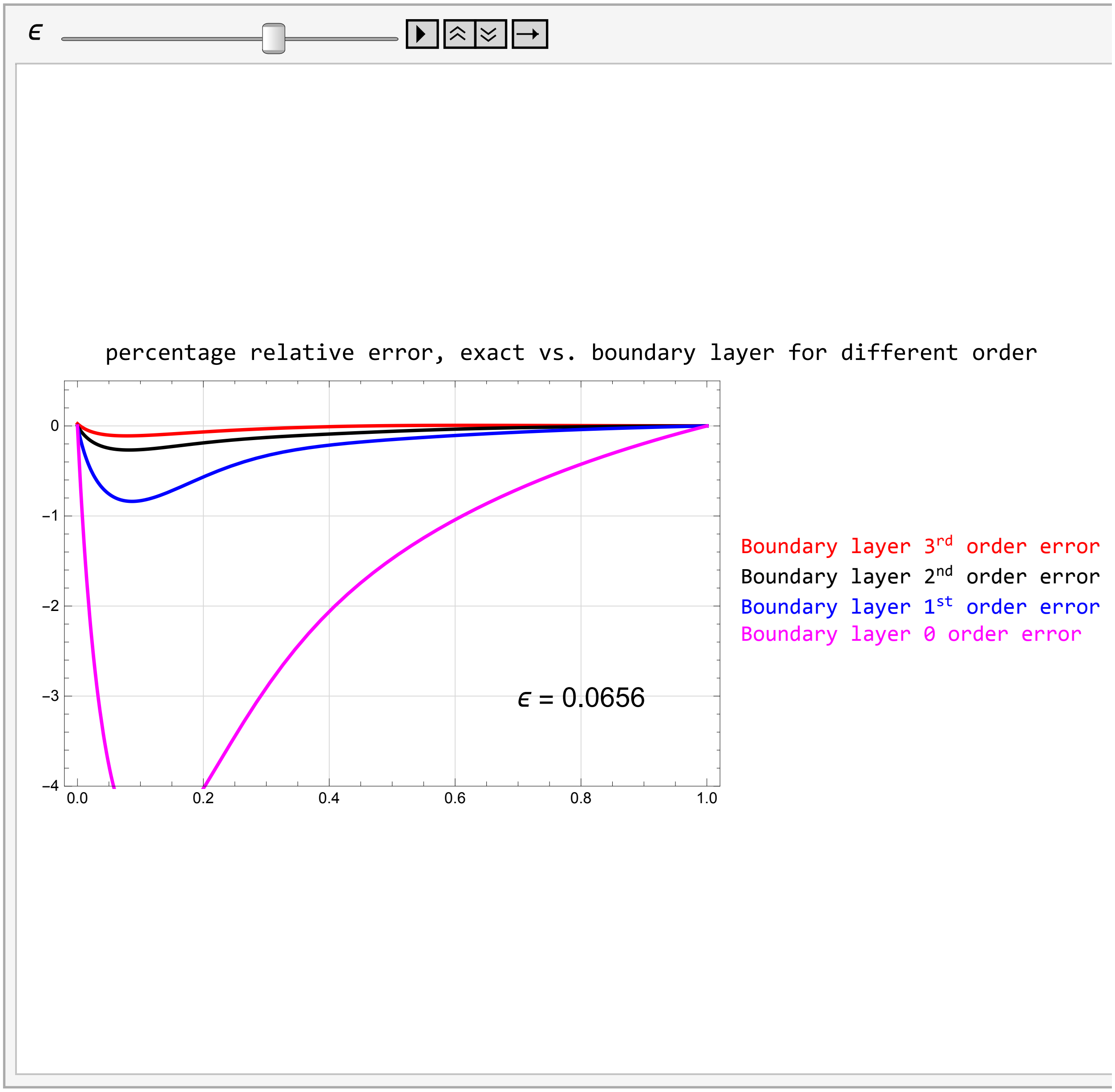
In[51]:=
```
Clear["Global`*"];
yout[x_] := 2/(1 + x);
sol =
  y[x] /. First@DSolve[{ϵ y''[x] + (1 + x) y'[x] + y[x] == 0, y[0] == 1, y[1] == 1}, y[x], x];
exact[x_, ϵ_] := Evaluate@sol;
yin[x_, eps_] :=
  Plot[2 - Exp[-x / eps], {x, 0, 1}, AxesOrigin → {0, 1}, PlotStyle → Black];
p1 = Plot[yout[x], {x, 0, 1}, PlotStyle → Blue];
combine[x_, eps_] := Plot[yout[x] - Exp[-x / eps], {x, 0, 1}, PlotStyle → {Thick, Red},
   AxesOrigin → {0, 1}, GridLines → Automatic, GridLinesStyle → LightGray];
Animate[
 Grid[{
   {TraditionalForm[
     NumberForm[ϵ, {Infinity, 4}] HoldForm[y''[x] + (1 + x) y'[x] + y[x] == 0]]},
   {Row[{"exact solution ", TraditionalForm[sol]}]},
   {Show[
     Legended[combine[x, ϵ], Style["Boundary layer solution, zero order", Red]],
     Legended[Plot[Evaluate@exact[x, ϵ], {x, 0, 1}, AxesOrigin → {0, 0}, PlotStyle → Blue],
      Style["Exact analytical solution", Blue]]
     , PlotRange → {{0, 1}, {1, 2}}, ImageSize → 400]
   }}], {ϵ, 0.0001, .1, .00001}]
```

add more terms

In[60]:=
```
Clear["Global`*"];
sol =
  y[x] /. First@DSolve[{ϵ y''[x] + (1 + x) y'[x] + y[x] == 0, y[0] == 1, y[1] == 1}, y[x], x];
exact[x_, ϵ_] := Evaluate@sol;
boundary0[x_, eps_] := (2/(1 + x) - Exp[-x / eps]);
boundary1[x_, eps_] :=
  (2/(1 + x) - Exp[-x / eps]) + eps (2/(1 + x)^3 - 1/(2 (1 + x)) + (1/2 (x / eps)^2 - 3/2) Exp[-x / eps]);
boundary2[x_, eps_] := (2/(1 + x) - Exp[-x / eps]) +
   eps (2/(1 + x)^3 - 1/(2 (1 + x)) + (1/2 (x / eps)^2 - 3/2) Exp[-x / eps]) +
   eps^2 (6/(1 + x)^5 - 1/(2 (1 + x)^3) - 1/(4 (1 + x)) - (1/8 (x / eps)^4 - 3/4 (x / eps)^2 + 21/4) Exp[-x / eps]);
boundary3[x_, eps_] := (2/(1 + x) - Exp[-x / eps]) +
   eps (2/(1 + x)^3 - 1/(2 (1 + x)) + (1/2 (x / eps)^2 - 3/2) Exp[-x / eps]) +
   eps^2 (6/(1 + x)^5 - 1/(2 (1 + x)^3) - 1/(4 (1 + x)) - (1/8 (x / eps)^4 - 3/4 (x / eps)^2 + 21/4) Exp[-x / eps]) +
   eps^3 (30/(1 + x)^7 - 3/(2 (1 + x)^5) - 1/(4 (1 + x)^3) - 5/(16 (1 + x)) +
      (1/48 (x / eps)^6 - 3/16 (x / eps)^4 + 21/8 (x / eps)^2 - 1949/72) Exp[-x / eps]);
```

In[67]:=
```
Animate[
 ex = exact[x, ϵ];
 Grid[{
   {Style[
     "percentage relative error, exact vs. boundary layer for different order", 14]},
   {Show[
     Legended[Plot[100 * ((boundary3[x, ϵ] - ex)/ex), {x, 0, 1},
       PlotStyle → {Thick, Red}, AxesOrigin → {0, 1}, GridLines → Automatic,
       GridLinesStyle → LightGray, PlotRange → {Automatic, {-4, .5}}, Frame → True,
       Epilog → Text[Style[Row[{"ϵ = ", NumberForm[ϵ, {Infinity, 4}]}], 16], {.8, -3}]],
      Style["Boundary layer 3rd order error", Red]],
     Legended[Plot[100 * ((boundary2[x, ϵ] - ex)/ex), {x, 0, 1}, PlotStyle → {Thick, Black},
       AxesOrigin → {0, 1}, GridLines → Automatic, GridLinesStyle → LightGray],
      Style["Boundary layer 2nd order error", Black]],
     Legended[Plot[100 * ((boundary1[x, ϵ] - ex)/ex), {x, 0, 1}, PlotStyle → {Thick, Blue},
       AxesOrigin → {0, 1}, GridLines → Automatic, GridLinesStyle → LightGray],
      Style["Boundary layer 1st order error", Blue]],
     Legended[Plot[100 * ((boundary0[x, ϵ] - ex)/ex), {x, 0, 1}, PlotStyle → {Thick, Magenta},
       AxesOrigin → {0, 1}, GridLines → Automatic, GridLinesStyle → LightGray],
      Style["Boundary layer 0 order error", Magenta]]
     , PlotRange → {{0, 1}, {-4, .5}}, ImageSize → 400]
   }}], {ϵ, 0.001, .1, .0001}]
```

Out[67]=


ϵ
percentage relative error, exact vs. boundary layer for different order
0
−1
−2
−3
−4
0.0
0.2
0.4
0.6
0.8
1.0
ϵ = 0.0656
Boundary layer 3rd order error
Boundary layer 2nd order error
Boundary layer 1st order error
Boundary layer 0 order error