

Manual Verification of 3.4.28 in Bender and Orszag textbook

Nasser M. Abbasi

December 30, 2019

Current rules I am using in simplifications are

1. $S'_0 \ggg S'_1$
2. $S'_0 S'_1 \ggg (S'_1)^2$
3. $S_0 \ggg S_1$
4. $(S_0)^n \ggg S_0^{(n)}$ (the first is power, the second is derivative order).

Verify the above are valid for $x \rightarrow 0^+$ and well for $x \rightarrow \infty$. What can we say about (S') compared to $(S')^2$?

0.1 problem (a), page 88

$$y'' = \frac{1}{x^5} y$$

Irregular singular point at $x \rightarrow 0^+$. Let $y = e^{S_0(x)}$ and the above becomes

$$\begin{aligned} y(x) &= e^{S_0(x)} \\ y'(x) &= S'_0 e^{S_0} \\ y'' &= S''_0 e^{S_0} + (S'_0)^2 e^{S_0} \\ &= (S''_0 + (S'_0)^2) e^{S_0} \end{aligned}$$

Substituting back into $\frac{d^2}{dx^2} y = x^{-5} y$ gives

$$\begin{aligned} (S''_0 + (S'_0)^2) e^{S_0} &= x^{-5} e^{S_0(x)} \\ S''_0 + (S'_0)^2 &= x^{-5} \end{aligned}$$

Before solving for S_0 , we can do one more simplification. Using the approximation that $(S'_0)^2 \ggg S''_0$ for $x \rightarrow x_0$, the above becomes

$$(S'_0)^2 \sim x^{-5}$$

Now we are ready to solve for S_0

$$\begin{aligned} S'_0 &\sim \omega x^{-\frac{5}{2}} \\ S_0 &\sim \omega \int x^{-\frac{5}{2}} dx \\ &\sim \omega \frac{x^{-\frac{3}{2}}}{-\frac{3}{2}} \\ &\sim -\frac{2}{3} \omega x^{-\frac{3}{2}} \end{aligned}$$

To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$y'(x) = (S_0(x) + S_1(x))' e^{S_0+S_1}$$

$$y''(x) = ((S_0 + S_1)')^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1}$$

Using the above, the ODE $\frac{d^2}{dx^2}y = x^{-5}y$ now becomes

$$((S_0 + S_1)')^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1} \sim x^{-5}e^{S_0+S_1}$$

$$((S_0 + S_1)')^2 + (S_0 + S_1)'' \sim x^{-5}$$

$$(S_0' + S_1')^2 + S_0'' + S_1'' \sim x^{-5}$$

$$(S_0')^2 + (S_1')^2 + 2S_0'S_1' + S_0'' + S_1'' \sim x^{-5}$$

But $S_0' = \omega x^{-\frac{5}{2}}$, found before, hence $(S_0')^2 = x^{-5}$ and the above simplifies to

$$(S_1')^2 + 2S_0'S_1' + S_0'' + S_1'' = 0$$

Using approximation $S_0'S_1' \gg (S_1')^2$ the above simplifies to

$$2S_0'S_1' + S_0'' + S_1'' = 0$$

Finally, using approximation $S_0'' \gg S_1''$, the above becomes

$$2S_0'S_1' + S_0'' = 0$$

$$S_1' \sim -\frac{S_0''}{2S_0'}$$

$$S_1 \sim -\frac{1}{2} \ln S_0' + c$$

$$S_1' \sim -\frac{1}{2} \ln x^{-\frac{5}{2}} + c$$

$$S_1 \sim \frac{5}{4} \ln x + c$$

Hence, the leading behavior is

$$\begin{aligned} y(x) &= e^{S_0(x)+S_1(x)} \\ &= \exp\left(-\frac{2}{3}\omega x^{-\frac{3}{2}} + \frac{5}{4} \ln x + c\right) \\ &= cx^{\frac{5}{4}} \exp\left(-\omega \frac{2}{3}x^{-\frac{3}{2}}\right) \end{aligned} \quad (1)$$

To verify, using the formula 3.4.28, which is

$$y(x) \sim cQ^{\frac{1-n}{2n}} \exp\left(\omega \int^x Q(t)^{\frac{1}{n}} dt\right)$$

In this case, $n = 2$, since the ODE $y'' = x^{-5}y$ is second order. Here we have $Q(x) = x^{-5}$, therefore, plug-in into the above gives

$$\begin{aligned} y(x) &\sim c(x^{-5})^{\frac{1-2}{4}} \exp\left(\omega \int^x (t^{-5})^{\frac{1}{2}} dt\right) \\ &\sim c(x^{-5})^{\frac{-1}{4}} \exp\left(\omega \int^x t^{-\frac{5}{2}} dt\right) \\ &\sim cx^{\frac{5}{4}} \exp\left(\omega \left(\frac{x^{-\frac{3}{2}}}{-\frac{3}{2}}\right)\right) \\ &\sim cx^{\frac{5}{4}} \exp\left(-\omega \frac{2}{3}x^{-\frac{3}{2}}\right) \end{aligned} \quad (2)$$

Comparing (1) and (2), we see they are the same.

0.2 problem (b), page 88

$$y''' = xy$$

Irregular singular point at $x \rightarrow +\infty$. Let $y = e^{S_0(x)}$ and the above becomes

$$y(x) = e^{S_0(x)}$$

$$y'(x) = S_0' e^{S_0}$$

$$y'' = S_0'' e^{S_0} + (S_0')^2 e^{S_0}$$

$$= (S_0'' + (S_0')^2) e^{S_0}$$

$$y''' = (S_0'' + (S_0')^2)' e^{S_0} + (S_0'' + (S_0')^2) S_0' e^{S_0}$$

$$= (S_0''' + 2S_0' S_0'') e^{S_0} + (S_0' S_0'' + (S_0')^3) e^{S_0}$$

$$= (S_0''' + 3S_0' S_0'' + (S_0')^3) e^{S_0}$$

Substituting back into $y''' = xy$ gives

$$(S_0''' + 3S_0' S_0'' + (S_0')^3) e^{S_0} = x e^{S_0(x)}$$

$$S_0''' + 3S_0' S_0'' + (S_0')^3 = x$$

Before solving for S_0 , we can do one more simplification. Using the approximation that $(S_0')^3 \gg S_0'''$ for $x \rightarrow x_0$, the above becomes

$$3S_0' S_0'' + (S_0')^3 \sim x$$

In addition, since $S_0' \gg S_0''$ then we can use the approximation $(S_0')^3 \gg S_0' S_0''$ and the above becomes

$$(S_0')^3 \sim x$$

$$S_0' \sim \omega x^{\frac{1}{3}}$$

$$S_0 \sim \omega \int x^{\frac{1}{3}} dx$$

$$S_0 \sim \omega \frac{3}{4} x^{\frac{4}{3}}$$

To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$y'(x) = (S_0(x) + S_1(x))' e^{S_0+S_1}$$

$$y''(x) = ((S_0 + S_1)')^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1}$$

$$= (S_0' + S_1')^2 e^{S_0+S_1} + (S_0'' + S_1'') e^{S_0+S_1}$$

$$= ((S_0')^2 + (S_1')^2 + 2S_0' S_1') e^{S_0+S_1} + (S_0'' + S_1'') e^{S_0+S_1}$$

$$= ((S_0')^2 + (S_1')^2 + 2S_0' S_1' + S_0'' + S_1'') e^{S_0+S_1}$$

We can take the third derivative

$$\begin{aligned}
y'''(x) &\sim \left((S'_0)^2 + (S'_1)^2 + 2S'_0S'_1 + S''_0 + S''_1 \right)' e^{S_0+S_1} \\
&\quad + \left((S'_0)^2 + (S'_1)^2 + 2S'_0S'_1 + S''_0 + S''_1 \right) (S_0 + S_1)' e^{S_0+S_1} \\
&\sim (2S'_0S''_0 + 2S'_1S''_1 + 2S''_0S'_1 + 2S'_0S''_1 + S'''_0 + S'''_1) e^{S_0+S_1} \\
&\quad + \left((S'_0)^2 + (S'_1)^2 + 2S'_0S'_1 + S''_0 + S''_1 \right) (S'_0 + S'_1) e^{S_0+S_1} \\
&\sim (2S'_0S''_0 + 2S'_1S''_1 + 2S''_0S'_1 + 2S'_0S''_1 + S'''_0 + S'''_1) e^{S_0+S_1} \\
&\quad + \left((S'_0)^3 + S'_0(S'_1)^2 + 2(S'_0)^2S'_1 + S'_0S''_0 + S'_0S''_1 \right) e^{S_0+S_1} \\
&\quad + \left(S'_1(S'_0)^2 + (S'_1)^3 + 2S'_0(S'_1)^2 + S'_1S''_0 + S'_1S''_1 \right) e^{S_0+S_1} \\
&\sim (2S'_0S''_0 + 2S'_1S''_1 + 2S''_0S'_1 + 2S'_0S''_1 + S'''_0 + S'''_1) e^{S_0+S_1} \\
&\quad + \left[(S'_0)^3 + 3S'_0(S'_1)^2 + 3(S'_0)^2S'_1 + S'_0S''_0 + S'_0S''_1 + (S'_1)^3 + S'_1S''_0 + S'_1S''_1 \right] e^{S_0+S_1} \\
&\sim \left(3S'_0S''_0 + 3S'_1S''_1 + 3S''_0S'_1 + 3S'_0S''_1 + S'''_0 + S'''_1 + (S'_0)^3 + 3S'_0(S'_1)^2 + 3(S'_0)^2S'_1 + (S'_1)^3 \right) e^{S_0+S_1} \\
&\sim \left((S'_0)^3 + (S'_1)^3 + 3S'_0(S'_1)^2 + 3(S'_0)^2S'_1 + 3S'_0S''_0 + 3S'_1S''_1 + 3S''_0S'_1 + 3S'_0S''_1 + S'''_0 + S'''_1 \right) e^{S_0+S_1}
\end{aligned}$$

Lets go ahead and plug-in this into the ODE

$$(S'_0)^3 + (S'_1)^3 + 3S'_0(S'_1)^2 + 3(S'_0)^2S'_1 + 3S'_0S''_0 + 3S'_1S''_1 + 3S''_0S'_1 + 3S'_0S''_1 + S'''_0 + S'''_1 \sim x$$

Now we do some simplification. $(S'_0)^3 \gg S'''_0$ and $(S'_1)^3 \gg S'''_1$, hence above becomes

$$(S'_0)^3 + (S'_1)^3 + 3S'_0(S'_1)^2 + 3(S'_0)^2S'_1 + 3S'_0S''_0 + 3S'_1S''_1 + 3S''_0S'_1 + 3S'_0S''_1 \sim x$$

Also, since $S''_0 \gg S'_1$ then $3S'_0S''_0 \gg 3S'_0S''_1$

$$(S'_0)^3 + (S'_1)^3 + 3S'_0(S'_1)^2 + 3(S'_0)^2S'_1 + 3S'_0S''_0 + 3S'_1S''_1 + 3S''_0S'_1 \sim x$$

Also, since $(S'_0)^2 \gg S''_0$ then $3(S'_0)^2S'_1 \gg 3S''_0S'_1$

$$(S'_0)^3 + (S'_1)^3 + 3S'_0(S'_1)^2 + 3(S'_0)^2S'_1 + 3S'_0S''_0 + 3S'_1S''_1 \sim x$$

Also since $S'_1 \gg S''_1$ then $3(S'_0)^2S'_1 \gg 3S'_1S''_1$

$$(S'_0)^3 + (S'_1)^3 + 3S'_0(S'_1)^2 + 3(S'_0)^2S'_1 + 3S'_0S''_0 \sim x$$

But $S'_0 \sim x^{\frac{1}{3}}$ hence $(S'_0)^3 \sim x$ and the above simplifies to

$$(S'_1)^3 + 3S'_0(S'_1)^2 + 3(S'_0)^2S'_1 + 3S'_0S''_0 = 0$$

Using $3S'_0(S'_1)^2 \gg (S'_1)^3$ since $S'_0 \gg S'_1$ then

$$3S'_0(S'_1)^2 + 3(S'_0)^2S'_1 + 3S'_0S''_0 = 0$$

Using $3(S'_0)^2S'_1 \gg S'_0(S'_1)^2$ since $(S'_0)^2 \gg S'_0$ then

$$3(S'_0)^2S'_1 + 3S'_0S''_0 = 0$$

No more simplification. We are ready to solve for S_1 .

$$\begin{aligned}
 S_1' &\sim \frac{-S_0' S_0''}{(S_0')^2} \\
 &\sim \frac{-S_0''}{S_0'}
 \end{aligned}$$

Hence

$$\begin{aligned}
 S_1 &\sim -\int \frac{S_0''}{S_0'} dx \\
 &\sim -\ln S_0' + c
 \end{aligned}$$

Since $S_0' \sim x^{\frac{1}{3}}$ then the above becomes

$$\begin{aligned}
 S_1 &\sim -\ln x^{\frac{1}{3}} + c \\
 S_1 &\sim -\frac{1}{3} \ln x + c
 \end{aligned}$$

Hence, the leading behavior is

$$\begin{aligned}
 y(x) &= e^{S_0(x)+S_1(x)} \\
 &= \exp\left(\omega \frac{3}{4} x^{\frac{4}{3}} - \frac{1}{3} \ln x + c\right) \\
 &= cx^{\frac{-1}{3}} \exp\left(\omega \frac{3}{4} x^{\frac{4}{3}}\right) \tag{1}
 \end{aligned}$$

To verify, using the formula 3.4.28, which is

$$y(x) \sim cQ^{\frac{1-n}{2n}} \exp\left(\omega \int^x Q(t)^{\frac{1}{n}} dt\right)$$

In this case, $n = 3$, since the ODE $y''' = xy$ is third order. Here we have $Q(x) = x$, therefore, plug-in into the above gives

$$\begin{aligned}
 y(x) &\sim c(x)^{\frac{1-3}{6}} \exp\left(\omega \int^x (t)^{\frac{1}{3}} dt\right) \\
 &\sim cx^{\frac{-1}{3}} \exp\left(\omega \frac{4}{3} x^{\frac{4}{3}}\right)
 \end{aligned}$$

Comparing (1) and (2), we see they are the same.