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This note solves in details the ODE

$$x^3 y''(x) = y(x)$$

Using asymptoes method using what is called the dominant balance submethod where it is assumed that $y(x) = e^{S(x)}$.

0.1 Solution

$x = 0$ is an irregular singular point. The solution is assumeed to be $y(x) = e^{S(x)}$. Therefore $y' = S' e^{S(x)}$ and $y'' = S'' e^{S(x)} + (S')^2 e^{S(x)}$ and the given ODE becomes

$$x^3 \left(S'' + (S')^2\right) = 1 \tag{1}$$

Assuming that

$$S'(x) \sim c x^{\alpha}$$

Hence $S'' \sim c\alpha x^{\alpha-1}$. and (1) becomes

$$\begin{aligned} x^3 \left(c\alpha x^{\alpha-1} + (c x^{\alpha})^2\right) &\sim 1 \\ c\alpha x^{\alpha+2} + c^2 x^{2\alpha+3} &\sim 1 \end{aligned}$$

Term $c\alpha x^{\alpha+2} \ggg c^2 x^{2\alpha+3}$, hence we set $\alpha = \frac{-3}{2}$ to remove the subdominant term. Therefore the above becomes, after substituting for the found α

$$\begin{aligned} \overbrace{\frac{-3}{2} c x^{\frac{1}{2}}}^{x\to 0} + c^2 &\sim 1 \\ c^2 &= 1 \end{aligned}$$

Therefore $c = \pm 1$. The result so far is $S'(x) \sim c x^{\frac{-3}{2}}$. Now another term is added. Let

$$S'(x) \sim c x^{\frac{-3}{2}} + A(x)$$

Now we will try to find $A(x)$. Hence $S''(x) \sim \frac{-3}{2} c x^{\frac{-5}{2}} + A'$ and $x^3 \left(S'' + (S')^2\right) = 1$ now becomes

$$\begin{aligned} x^3 \left(\frac{-3}{2} c x^{\frac{-5}{2}} + A' + \left(c x^{\frac{-3}{2}} + A\right)^2\right) &\sim 1 \\ x^3 \left(\frac{-3}{2} c x^{\frac{-5}{2}} + A' + c^2 x^{-3} + A^2 + 2Acx^{\frac{-3}{2}}\right) &\sim 1 \\ \left(\frac{-3}{2} c x^{\frac{1}{2}} + x^3 A' + c^2 + x^3 A^2 + 2Acx^{\frac{3}{2}}\right) &\sim 1 \end{aligned}$$

Since $c^2 = 1$ from the above, then

$$\frac{-3}{2} c x^{\frac{1}{2}} + x^3 A' + x^3 A^2 + 2Acx^{\frac{3}{2}} \sim 0$$

Dominant balance says to keep dominant term (but now looking at those terms in A only). From the above, since $A \ggg A^2$ and $A \ggg A'$ then from the above, we can cross out A^2 and A' resulting in

$$\frac{-3}{2} c x^{\frac{1}{2}} + 2Acx^{\frac{3}{2}} \sim 0$$

Hence we just need to find A to balance the above

$$\begin{aligned} \frac{-3}{2} c x^{\frac{1}{2}} + 2Acx^{\frac{3}{2}} &\sim 0 \\ 2Acx^{\frac{3}{2}} &\sim \frac{3}{2} c x^{\frac{1}{2}} \\ A &\sim \frac{3}{4x} \end{aligned}$$

We found $A(x)$ for the second term. Therefore, so far we have

$$S'(x) = c x^{\frac{-3}{2}} + \frac{3}{4x}$$

Or

$$S(x) = -2cx^{\frac{-1}{2}} + \frac{3}{4}\ln x + C_0$$

But C_0 can be dropped (subdominant to $\ln x$ when $x \to 0$) and so far then we can write the solution as

$$\begin{aligned}
y(x) &= e^{S(x)}W(x) \\
&= e^{S(x)}\sum_{n=0}^{\infty} a_n x^{nr} \\
&= \exp\left(-2cx^{\frac{-1}{2}} + \frac{3}{4}\ln x\right)\sum_{n=0}^{\infty} a_n x^{nr} \\
&= e^{-2cx^{\frac{-1}{2}}} x^{\frac{3}{4}} \sum_{n=0}^{\infty} a_n x^{nr} \\
&= e^{-\frac{2c}{\sqrt{x}}} \sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}} \\
&= e^{\pm\frac{2}{\sqrt{x}}} \sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}}
\end{aligned}$$

Since $c = \pm 1$. We can now try adding one more term to $S(x)$. Let

$$S'(x) = cx^{\frac{-3}{2}} + \frac{3}{4x} + B(x)$$

Hence

$$S'' = \frac{-3}{2}cx^{\frac{-5}{2}} - \frac{3}{4x^2} + B'(x)$$

And $x^3\left(S'' + (S')^2\right) \sim 1$ now becomes

$$\begin{aligned}
x^3\left(\left(\frac{-3}{2}cx^{\frac{-5}{2}} - \frac{3}{4x^2} + B'(x)\right) + \left(cx^{\frac{-3}{2}} + \frac{3}{4x} + B(x)\right)^2\right) &\sim 1 \\
x^3\left(\frac{c^2}{x^3} - \frac{3}{16}x^{-2} + 2cBx^{-\frac{3}{2}} + \frac{3}{2}Bx^{-1} + B^2 + B'\right) &\sim 1 \\
\left(\overset{=1}{c^2} - \frac{3}{16}x + 2cBx^{\frac{3}{2}} + \frac{3}{2}Bx^2 + x^3B^2 + x^3B'\right) &\sim 1 \\
-\frac{3}{16}x + 2cBx^{\frac{3}{2}} + \frac{3}{2}Bx^2 + x^3B^2 + x^3B' &\sim 0
\end{aligned}$$

From the above, since $B(x) \ggg B^2(x)$ and $B(x) \ggg B'(x)$ and for small x, then we can cross out terms with B^2 and B' from above, and we are left with

$$-\frac{3}{16}x + 2cBx^{\frac{3}{2}} + \frac{3}{2}Bx^2 \sim 0$$

Between $2cBx^{\frac{3}{2}}$ and $\frac{3}{2}Bx^2$, for small x, then $2cBx^{\frac{3}{2}} \ggg \frac{3}{2}Bx^2$, so we can cross out $\frac{3}{2}Bx^2$ from above

$$\begin{aligned}
-\frac{3}{16}x + 2cBx^{\frac{3}{2}} &\sim 0 \\
2cBx^{\frac{3}{2}} &\sim \frac{3}{16}x \\
B &\sim \frac{3}{32c}x^{-\frac{1}{2}}
\end{aligned}$$

We found $B(x)$, Hence now we have

$$S'(x) = cx^{\frac{-3}{2}} + \frac{3}{4x} + \frac{3}{32c}x^{-\frac{1}{2}}$$

Or

$$S(x) = -2cx^{\frac{-1}{2}} + \frac{3}{4}\ln x + \frac{3}{16c}x^{\frac{1}{2}} + C_1$$

But C_1 can be dropped (subdominant to $\ln x$ when $x \to 0$) and so far then we can write the solution as

$$\begin{aligned}
y(x) &= e^{S(x)}W(x) \\
&= e^{S(x)}\sum_{n=0}^{\infty} a_n x^{nr} \\
&= \exp\left(-2cx^{\frac{-1}{2}} + \frac{3}{4}\ln x + \frac{3}{16c}x^{\frac{1}{2}}\right)\sum_{n=0}^{\infty} a_n x^{nr} \\
&= e^{-2cx^{\frac{-1}{2}} + \frac{3}{16c}x^{\frac{1}{2}}} x^{\frac{3}{4}} \sum_{n=0}^{\infty} a_n x^{nr} \\
&= e^{-2cx^{\frac{-1}{2}} + \frac{3}{16c}x^{\frac{1}{2}}} \sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}}
\end{aligned}$$

For $c = 1$

$$y_1(x) = e^{-2x^{\frac{-1}{2}} + \frac{3}{16}x^{\frac{1}{2}}} \sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}}$$

For $c = -1$

$$y_2(x) = e^{2x^{\frac{-1}{2}} - \frac{3}{16}x^{\frac{1}{2}}} \sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}}$$

Hence

$$y(x) \sim Ay_1(x) + By_2(x)$$

Reference

1. Page 80-82 Bender and Orszag textbook.
2. Lecture notes, Lecture 5, Tuesday Janurary 31, 2017. EP 548, University of Wisconsin, Madison by Professor Smith.
3. Lecture notes from http://www.damtp.cam.ac.uk/