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This note solves in details the ODE

$$x^3 y''(x) = y(x)$$

Using asymptoes method using what is called the dominant balance submethod where it is assumed that $y(x) = e^{S(x)}$.

0.1 Solution

$x = 0$ is an irregular singular point. The solution is assumeed to be $y(x) = e^{S(x)}$. Therefore $y' = S' e^{S(x)}$ and $y'' = S'' e^{S(x)} + (S')^2 e^{S(x)}$ and the given ODE becomes

$$x^3 \left(S'' + (S')^2\right) = 1 \tag{1}$$

Assuming that

$$S'(x) \sim c x^{\alpha}$$

Hence $S'' \sim c\alpha x^{\alpha-1}$. and (1) becomes

$$\begin{aligned} x^3 \left(c\alpha x^{\alpha-1} + (cx^{\alpha})^2\right) &\sim 1 \\ c\alpha x^{\alpha+2} + c^2 x^{2\alpha+3} &\sim 1 \end{aligned}$$

Term $c\alpha x^{\alpha+2} \ggg c^2 x^{2\alpha+3}$, hence we set $\alpha = \frac{-3}{2}$ to remove the subdominant term. Therefore the above becomes, after substituting for the found α

$$\begin{aligned} \overbrace{\frac{-3}{2} c x^{\frac{1}{2}}}^{x\to 0} + c^2 &\sim 1 \\ c^2 &= 1 \end{aligned}$$

Therefore $c = \pm 1$. The result so far is $S'(x) \sim c x^{\frac{-3}{2}}$. Now another term is added. Let

$$S'(x) \sim c x^{\frac{-3}{2}} + A(x)$$

Now we will try to find $A(x)$. Hence $S''(x) \sim \frac{-3}{2} c x^{\frac{-5}{2}} + A'$ and $x^3 \left(S'' + (S')^2\right) = 1$ now becomes

$$\begin{aligned} x^3 \left(\frac{-3}{2} c x^{\frac{-5}{2}} + A' + \left(c x^{\frac{-3}{2}} + A\right)^2\right) &\sim 1 \\ x^3 \left(\frac{-3}{2} c x^{\frac{-5}{2}} + A' + c^2 x^{-3} + A^2 + 2 A c x^{\frac{-3}{2}}\right) &\sim 1 \\ \left(\frac{-3}{2} c x^{\frac{1}{2}} + x^3 A' + c^2 + x^3 A^2 + 2 A c x^{\frac{3}{2}}\right) &\sim 1 \end{aligned}$$

Since $c^2 = 1$ from the above, then

$$\frac{-3}{2}cx^{\frac{1}{2}} + x^3A' + x^3A^2 + 2Acx^{\frac{3}{2}} \sim 0$$

Dominant balance says to keep dominant term (but now looking at those terms in A only). From the above, since $A \ggg A^2$ and $A \ggg A'$ then from the above, we can cross out A^2 and A' resulting in

$$\frac{-3}{2}cx^{\frac{1}{2}} + 2Acx^{\frac{3}{2}} \sim 0$$

Hence we just need to find A to balance the above

$$\begin{aligned}\frac{-3}{2}cx^{\frac{1}{2}} + 2Acx^{\frac{3}{2}} &\sim 0\\ 2Acx^{\frac{3}{2}} &\sim \frac{3}{2}cx^{\frac{1}{2}}\\ A &\sim \frac{3}{4x}\end{aligned}$$

We found $A(x)$ for the second term. Therefore, so far we have

$$S'(x) = cx^{\frac{-3}{2}} + \frac{3}{4x}$$

Or

$$S(x) = -2cx^{\frac{-1}{2}} + \frac{3}{4}\ln x + C_0$$

But C_0 can be dropped (subdominant to $\ln x$ when $x \to 0$) and so far then we can write the solution as

$$\begin{aligned}y(x) &= e^{S(x)}W(x)\\ &= e^{S(x)}\sum_{n=0}^{\infty} a_n x^{nr}\\ &= \exp\left(-2cx^{\frac{-1}{2}} + \frac{3}{4}\ln x\right)\sum_{n=0}^{\infty} a_n x^{nr}\\ &= e^{-2cx^{\frac{-1}{2}}} x^{\frac{3}{4}} \sum_{n=0}^{\infty} a_n x^{nr}\\ &= e^{-\frac{2c}{\sqrt{x}}} \sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}}\\ &= e^{\pm\frac{2}{\sqrt{x}}} \sum_{n=0}^{\infty} a_n x^{nr+\frac{3}{4}}\end{aligned}$$

Since $c = \pm 1$. We can now try adding one more term to $S(x)$. Let

$$S'(x) = cx^{\frac{-3}{2}} + \frac{3}{4x} + B(x)$$

Hence

$$S'' = \frac{-3}{2}cx^{\frac{-5}{2}} - \frac{3}{4x^2} + B'(x)$$

And $x^3\left(S''+(S')^2\right)\sim 1$ now becomes

$$x^3\left(\left(\frac{-3}{2}cx^{\frac{-5}{2}}-\frac{3}{4x^2}+B'(x)\right)+\left(cx^{\frac{-3}{2}}+\frac{3}{4x}+B(x)\right)^2\right)\sim 1$$

$$x^3\left(\frac{c^2}{x^3}-\frac{3}{16}x^{-2}+2cBx^{-\frac{3}{2}}+\frac{3}{2}Bx^{-1}+B^2+B'\right)\sim 1$$

$$\left(\overbrace{c^2}^{=1}-\frac{3}{16}x+2cBx^{\frac{3}{2}}+\frac{3}{2}Bx^2+x^3B^2+x^3B'\right)\sim 1$$

$$-\frac{3}{16}x+2cBx^{\frac{3}{2}}+\frac{3}{2}Bx^2+x^3B^2+x^3B'\sim 0$$

From the above, since $B(x)\ggg B^2(x)$ and $B(x)\ggg B'(x)$ and for small x, then we can cross out terms with B^2 and B' from above, and we are left with

$$-\frac{3}{16}x+2cBx^{\frac{3}{2}}+\frac{3}{2}Bx^2\backsim 0$$

Between $2cBx^{\frac{3}{2}}$ and $\frac{3}{2}Bx^2$, for small x, then $2cBx^{\frac{3}{2}}\ggg\frac{3}{2}Bx^2$, so we can cross out $\frac{3}{2}Bx^2$ from above

$$-\frac{3}{16}x+2cBx^{\frac{3}{2}}\backsim 0$$

$$2cBx^{\frac{3}{2}}\backsim\frac{3}{16}x$$

$$B\backsim\frac{3}{32c}x^{-\frac{1}{2}}$$

We found $B(x)$, Hence now we have

$$S'(x)=cx^{\frac{-3}{2}}+\frac{3}{4x}+\frac{3}{32c}x^{-\frac{1}{2}}$$

Or

$$S(x)=-2cx^{\frac{-1}{2}}+\frac{3}{4}\ln x+\frac{3}{16c}x^{\frac{1}{2}}+C_1$$

But C_1 can be dropped (subdominant to $\ln x$ when $x\to 0$) and so far then we can write the solution as

$$\begin{aligned}
y(x)&=e^{S(x)}W(x)\\
&=e^{S(x)}\sum_{n=0}^{\infty}a_nx^{nr}\\
&=\exp\left(-2cx^{\frac{-1}{2}}+\frac{3}{4}\ln x+\frac{3}{16c}x^{\frac{1}{2}}\right)\sum_{n=0}^{\infty}a_nx^{nr}\\
&=e^{-2cx^{\frac{-1}{2}}+\frac{3}{16c}x^{\frac{1}{2}}}x^{\frac{3}{4}}\sum_{n=0}^{\infty}a_nx^{nr}\\
&=e^{-2cx^{\frac{-1}{2}}+\frac{3}{16c}x^{\frac{1}{2}}}\sum_{n=0}^{\infty}a_nx^{nr+\frac{3}{4}}
\end{aligned}$$

For $c=1$

$$y_1(x)=e^{-2x^{\frac{-1}{2}}+\frac{3}{16}x^{\frac{1}{2}}}\sum_{n=0}^{\infty}a_nx^{nr+\frac{3}{4}}$$

For $c=-1$

$$y_2(x)=e^{2x^{\frac{-1}{2}}-\frac{3}{16}x^{\frac{1}{2}}}\sum_{n=0}^{\infty}a_nx^{nr+\frac{3}{4}}$$

Hence

$$y(x)\sim Ay_1(x)+By_2(x)$$

Reference

1. Page 80-82 Bender and Orszag textbook.
2. Lecture notes, Lecture 5, Tuesday Janurary 31, 2017. EP 548, University of Wisconsin, Madison by Professor Smith.
3. Lecture notes from http://www.damtp.cam.ac.uk/