HW 5, NE 548, Spring 2017

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Contents

NE548 Problem: Similarity solution for the 1D homogeneous heat equation

Due Thursday April 13, 2017

1. (a) Non-dimensionalize the 1D homogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \nu \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

with $-\infty < x < \infty$, and $u(x,t)$ bounded as $x \to \pm\infty$.
(b) Show that the non-dimensional equations motivate a similarity variable $\xi = x/t^{1/2}$.
(c) Find a similarity solution $u(x,t) = H(\xi)$ by solving the appropriate ODE for $H(\xi)$.
(d) Show that the similarity solution is related to the solution we found in class on 4/6/17 for initial condition

$$u(x,0) = 0, \quad x < 0 \qquad u(x,0) = C, \quad x > 0.$$

solution

0.1 Part a

Let $\bar{x}$ be the non-dimensional space coordinate and $\bar{t}$ the non-dimensional time coordinate. Therefore we need

$$\bar{x} = \frac{x}{l_0}$$
$$\bar{t} = \frac{t}{t_0}$$
$$\bar{u} = \frac{u}{u_0}$$

Where l_0 is the physical characteristic length scale (even if this infinitely long domain, l_0 is given) whose dimensions is $[L]$ and t_0 of dimensions $[T]$ is the characteristic time scale and $\bar{u}(\bar{x},\bar{t})$ is the new dependent variable, and u_0 characteristic value of u to scale against (typically this is the initial conditions) but this will cancel out. We now rewrite the PDE $\frac{\partial u}{\partial t} = \upsilon \frac{\partial^2 u}{\partial x^2}$ in terms of the new dimensionless variables.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \bar{u}} \frac{\partial \bar{u}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial t} \\ &= u_0 \frac{\partial \bar{u}}{\partial \bar{t}} \frac{1}{t_0}\end{aligned} \tag{1}$$

And

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \bar{u}} \frac{\partial \bar{u}}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x} \\ &= u_0 \frac{\partial \bar{u}}{\partial \bar{x}} \frac{1}{l_0}\end{aligned}$$

And

$$\frac{\partial^2 u}{\partial x^2} = u_0 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} \frac{1}{l_0^2} \tag{2}$$

Substituting (1) and (2) into $\frac{\partial u}{\partial t} = \upsilon \frac{\partial^2 u}{\partial x^2}$ gives

$$\begin{aligned}u_0 \frac{\partial \bar{u}}{\partial \bar{t}} \frac{1}{t_0} &= \upsilon u_0 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} \frac{1}{l_0^2} \\ \frac{\partial \bar{u}}{\partial \bar{t}} &= \left(\upsilon \frac{t_0}{l_0^2}\right) \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\end{aligned}$$

The above is now non-dimensional. Since υ has units $\left[\frac{L^2}{T}\right]$ and $\frac{t_0}{l_0^2}$ also has units $\left[\frac{T}{L^2}\right]$, therefore the product $\upsilon \frac{t_0}{l_0^2}$ is non-dimensional quantity.

If we choose t_0 to have same magnitude (not units) as l_0^2, i.e. $t_0 = l_0^2$, then $\frac{t_0}{l_0^2} = 1$ (with units $\left[\frac{T}{L^2}\right]$) and now we obtain the same PDE as the original, but it is non-dimensional. Where now $\bar{u} \equiv \bar{u}(\bar{t},\bar{x})$.

0.2 Part (b)

I Will use the Buckingham π theorem for finding expression for the solution in the form $u(x,t) = f(\xi)$ where ξ is the similarity variable. Starting with $\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial x^2}$, in this PDE, the diffusion substance is heat with units of Joule J. Hence the concentration of heat, which is what u represents, will have units of $[u] = \frac{J}{L^3}$. (heat per unit volume). From physics, we expect the solution $u(x,t)$ to depend on x, t, v and initial conditions u_0 as these are the only relevant quantities involved that can affect the diffusion. Therefore, by Buckingham theorem we say

$$u \equiv f(x,t,v,u_0) \tag{1}$$

We have one dependent quantity u and 4 independent quantities.The units of each of the above quantities is

$$\begin{aligned}
[u] &= \frac{J}{L^3} \\
[x] &= L \\
[t] &= T \\
[v] &= \frac{L^2}{T} \\
[u_0] &= \frac{J}{L^3}
\end{aligned}$$

Hence using Buckingham theorem, we write

$$[u] = \left[x^a t^b v^c u_0^d\right] \tag{2}$$

We now determine a, b, c, d, by dimensional analysis. The above is

$$\begin{aligned}
\frac{J}{L^3} &= L^a T^b \left(\frac{L^2}{T}\right)^c \left(\frac{J}{L^3}\right)^d \\
(J)\left(L^{-3}\right) &= \left(L^{a+2c-3d}\right)\left(T^{b-c}\right)\left(J^d\right)
\end{aligned}$$

Comparing powers of same units on both sides, we see that

$$\begin{aligned}
d &= 1 \\
b - c &= 0 \\
a + 2c - 3d &= -3
\end{aligned}$$

From second equation above, $b = c$, hence third equation becomes

$$\begin{aligned}
a + 2c - 3d &= -3 \\
a + 2c &= 0
\end{aligned}$$

Since $d = 1$. Hence

$$\begin{aligned}
c &= -\frac{a}{2} \\
b &= -\frac{a}{2}
\end{aligned}$$

Therefore, now that we found all the powers, (we have one free power a which we can set to any value), then from equation (1)

$$\begin{aligned}
[u] &= \left[x^a t^b v^c u_0^d\right] \\
\frac{u}{u_0} &= \bar{u} = x^a t^b v^c
\end{aligned}$$

Therefore $\bar{u}$ is function of all the variables in the RHS. Let this function be f (This is the same as H in problem statement). Hence the above becomes

$$\begin{aligned}
\bar{u} &= f\left(x^a t^{-\frac{a}{2}} v^{-\frac{a}{2}}\right) \\
&= f\left(\frac{x^a}{v^{\frac{a}{2}} t^{\frac{a}{2}}}\right)
\end{aligned}$$

Since a is free variable, we can choose

$$a = 1 \tag{2}$$

And obtain

$$\bar{u} \equiv f\left(\frac{x}{\sqrt{vt}}\right) \tag{3}$$

In the above $\frac{x}{\sqrt{vt}}$ is now non-dimensional quantity, which we call, the similarity variable

$$\xi = \frac{x}{\sqrt{vt}} \tag{4}$$

Notice that another choice of a in (2), for example $a = 2$ would lead to $\xi = \frac{x^2}{vt}$ instead of $\xi = \frac{x}{\sqrt{vt}}$ but we will use the latter for the rest of the problem.

0.3 Part (c)

Using $u \equiv f(\xi)$ where $\xi = \frac{x}{\sqrt{vt}}$ then

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{df}{d\xi}\frac{\partial \xi}{\partial t}\\ &= \frac{df}{d\xi}\frac{\partial}{\partial t}\left(\frac{x}{\sqrt{vt}}\right)\\ &= -\frac{1}{2}\frac{df}{d\xi}\left(\frac{x}{\sqrt{v}t^{\frac{3}{2}}}\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{df}{d\xi}\frac{\partial \xi}{\partial x}\\ &= \frac{df}{d\xi}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{vt}}\right)\\ &= \frac{df}{d\xi}\frac{1}{\sqrt{vt}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{df}{d\xi}\frac{1}{\sqrt{vt}}\right)\\ &= \frac{1}{\sqrt{vt}}\frac{\partial}{\partial x}\left(\frac{df}{d\xi}\right)\\ &= \frac{1}{\sqrt{vt}}\left(\frac{d^2f}{d\xi^2}\frac{\partial \xi}{\partial x}\right)\\ &= \frac{1}{\sqrt{vt}}\left(\frac{d^2f}{d\xi^2}\frac{1}{\sqrt{vt}}\right)\\ &= \frac{1}{vt}\frac{d^2f}{d\xi^2}\end{aligned}$$

Hence the PDE $\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial x^2}$ becomes

$$\begin{aligned}-\frac{1}{2}\frac{df}{d\xi}\left(\frac{x}{\sqrt{v}t^{\frac{3}{2}}}\right) &= v\frac{1}{vt}\frac{d^2f}{d\xi^2}\\ \frac{1}{t}\frac{d^2f}{d\xi^2} + \frac{1}{2}\frac{x}{\sqrt{v}t^{\frac{3}{2}}}\frac{df}{d\xi} &= 0\\ \frac{d^2f}{d\xi^2} + \frac{1}{2}\frac{x}{\sqrt{vt}}\frac{df}{d\xi} &= 0\end{aligned}$$

But $\frac{x}{\sqrt{vt}} = \xi$, hence we obtain the required ODE as

$$\begin{aligned}\frac{d^2f(\xi)}{d\xi^2} + \frac{1}{2}\xi\frac{df(\xi)}{d\xi} &= 0\\ f'' + \frac{\xi}{2}f' &= 0\end{aligned}$$

We now solve the above ODE for $f(\xi)$. Let $f' = z$, then the ODE becomes

$$z' + \frac{\xi}{2}z = 0$$

Integrating factor is $\mu = e^{\int \frac{\xi}{2} d\xi} = e^{\frac{\xi^2}{4}}$, hence

$$\begin{aligned}\frac{d}{d\xi}\left(z\mu\right) &= 0\\ z\mu &= c_1\\ z &= c_1 e^{\frac{-\xi^2}{4}}\end{aligned}$$

Therefore, since $f' = z$, then

$$f' = c_1 e^{\frac{-\xi^2}{4}}$$

Integrating gives

$$f(\xi) = c_2 + c_1 \int_0^{\xi} e^{\frac{-s^2}{4}} ds$$

0.4 Part (d)

For initial conditions of step function

$$u(x,0) = \begin{cases} 0 & x < 0 \\ C & x > 0 \end{cases}$$

The solution found in class was

$$u(x,t) = \frac{C}{2} + \frac{C}{2}\operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right) \tag{1}$$

Where $\operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4\nu t}}} e^{-z^2} dz$. The solution found in part (c) earlier is

$$f(\xi) = c_1 \int_0^{\xi} e^{\frac{-s^2}{4}} ds + c_2$$

Let $s = \sqrt{4}z$, then $\frac{ds}{dz} = \sqrt{4}$, when $s = 0, z = 0$ and when $s = \xi, z = \frac{\xi}{\sqrt{4}}$, therefore the integral becomes

$$f(\xi) = c_1\sqrt{4}\int_0^{\frac{\xi}{\sqrt{4}}} e^{-z^2} dz + c_2$$

But $\frac{2}{\sqrt{\pi}}\int_0^{\frac{\xi}{\sqrt{4}}} e^{-z^2} dz = \operatorname{erf}\left(\frac{\xi}{\sqrt{4}}\right)$, hence $\int_0^{\frac{\xi}{\sqrt{4}}} e^{-z^2} dz = \frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{\xi}{\sqrt{4}}\right)$ and the above becomes

$$\begin{aligned}f(\xi) &= c_1\sqrt{\pi}\operatorname{erf}\left(\frac{\xi}{\sqrt{4}}\right) + c_2\\ &= c_3\operatorname{erf}\left(\frac{\xi}{\sqrt{4}}\right) + c_2\end{aligned}$$

Since $\xi = \frac{x}{\sqrt{\nu t}}$, then above becomes, when converting back to $u(x,t)$

$$u(x,t) = c_3\operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right) + c_2 \tag{2}$$

Comparing (1) and (2), we see they are the same. Constants of integration are arbitrary and can be found from initial conditions.