NEEP 548: Engineering Analysis II 2/6/2011

Problem 3.33: Bender & Orszag

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1 Problem Statement

Find the leading behavoir of the following equation as $x \to 0^+$:

$$x^4 y''' - 3x^2 y' + 2y = 0 \tag{1}$$

2 Solution

Need to make the substitution: $y = e^{S(x)}$

$$\begin{aligned} y' &= S' e^{S(x)} \\ y'' &= S'' e^{S(x)} + (S')^2 e^{S(x)} \\ y''' &= S''' e^{S(x)} + 3S''S' e^{S(x)} + (S')^3 e^{S(x)} \end{aligned}$$

After substituting and diving through by y, one obtains,

$$S''' + 3S''S' + (S')^3 - \frac{3}{x^2}S' + \frac{2}{x^4} \sim 0 \tag{2}$$

Typically, $S'' << (S')^2 \implies S''' << (S')^3$. Similarly, assume $S''' << (S')^3$
With these assumptions, 2 becomes:

$$(S')^3 - \frac{3}{x^2}S' \sim -\frac{2}{x^4}$$

which is still a difficult problem to solve. Therefore, also want to assume $(S')^3 >> \frac{3}{x^2}S'$ as $x \to 0^+$. Then,

$$\begin{aligned} (S')^3 &\sim \frac{-2}{x^4} && x \to 0^+ \\ S' &\sim wx^{-4/3} && x \to 0^+ \\ w &= (-2)^{1/3} \\ S_o(x) &\sim -3wx^{-1/3} && x \to 0^+ \\ S'' &\sim -\frac{4w}{3}x^{-7/3} \\ S''' &\sim \frac{28w}{9}x^{-10/3} \end{aligned}$$

Now, check our assumptions,

$$\begin{aligned}
&(S')^3 \sim -2x^{-12/3} >> -3S'x^{-2} \sim -3wx^{-10/3} && x \to 0^+ \quad \checkmark \\
&(S')^3 \sim -2x^{-12/3} >> S''' \sim \frac{28w}{9}x^{-10/3} && x \to 0^+ \quad \checkmark \\
&(S')^3 \sim -2x^{-12/3} >> S''S' \sim \frac{-4w^2}{3}x^{-11/3} && x \to 0^+ \quad \checkmark
\end{aligned}$$

Now estimate the inegrating function $C(x)$ by letting,

$$S(x) = S_o(x) + C(x) \tag{3}$$

and substitute this into 2.

$$S_o''' + C''' + 3\underbrace{(S_o'' + C'')(S_o' + C')}_{\text{term 1}} + \underbrace{(S_o' + C')^3}_{\text{term 2}} - \frac{3}{x^2}(S_o' + C') + \frac{2}{x^4} = 0 \tag{4}$$

term 1: $(S_o''S_o' + C''S_o' + S_o''C' + C''C')$

term 2: $(S_o')^3 + (C')^3 + 3(S')^2C' + 3(S')(C')^2$

From here, notice that we have already balanced the terms $(S_o')^3$ and $\frac{-2}{x^4}$, so they are removed at this point. Now we make the following assumptions:

$$S_o' >> C', \qquad S_o'' >> C'', \qquad S_o''' >> C''' \qquad \text{as} \qquad x \to 0^+$$

These assumptions result in,

$$S_o''' + 3S_o''S_o' + 3(S_o')^2C' \sim \frac{3S_o'}{x^2} + \frac{3C'}{x^2} \qquad x \to 0^+$$

Insert the value of S_o found in the previous step,

$$\frac{28w}{9}x^{-10/3} - 4w^2x^{-11/3} + 3w^2x^{-8/3}C' \sim 3wx^{-10/3} + 3C'x^{-6/3}$$

In keeping with the dominant balance idea, it is clear that the middle two terms dominate, thus leading to the following simplified relation,

$$\begin{aligned}
3w^2x^{-8/3}C' &\sim 4w^2x^{-11/3} && x \to 0^+ \\
C' &\sim \frac{4}{3}x^{-1} && x \to 0^+ \\
C &\sim \frac{4}{3}ln(x) && x \to 0^+
\end{aligned}$$

Then $C'' \sim \frac{-4}{3}x^{-2}$, $C''' \sim \frac{8}{3}x^{-3}$ and once again, check assumptions made,

$$S_o' \sim wx^{-4/3} >> C' \sim \frac{4}{3}x^{-1} \qquad x \to 0^+ \qquad \checkmark$$
$$S_o'' \sim \frac{-4}{3}x^{-7/3} >> C'' \sim \frac{-4}{3}x^{-2} \qquad x \to 0^+ \qquad \checkmark$$
$$S_o''' \sim \frac{28w}{9}x^{-10/3} >> C''' \sim \frac{8}{3}x^{-3} \qquad x \to 0^+ \qquad \checkmark$$

With the appearance of the ln(x) term, we have likely found the leading behavior already. However, lets find D, the third term, just to check.

Substitute $y = e^{(S_o+C_o+D)}$, then divide by y,

$$y' = [S_o' + C_o' + D']e^{(S_o+C_o+D)}$$
$$y'' = [S_o'' + C_o'' + D'']e^{(S_o+C_o+D)} + [S_o' + C_o' + D']^2 e^{(S_o+C_o+D)}$$
$$y''' = [S_o''' + C_o''' + D''']e^{(S_o+C_o+D)} + 3[S_o'' + C_o'' + D''][S_o' + C_o' + D']e^{(S_o+C_o+D)} + [S_o' + C_o' + D']^3 e^{(S_o+C_o+D)}$$

And...here we go...

$$S_o''' + \cancel{C_o'''} + \cancel{D'''} + 3[\cancel{S_o''S_o'} + S_o''C_o' + \cancel{S_o''D'} + C_o''S_o' + \cancel{C_o''C_o'} + \cancel{C_o''D'} + \cancel{D''S_o'} + \cancel{D''C_o'} + \cancel{D''D'}]$$
$$+ \cancel{6S_o'C_o'D'} + 3[\cancel{(S_o')^2C_o'} + \underbrace{(S_o')^2D' + (C_o')^2S_o'}_{\text{Unknown balance}} + \cancel{(C_o')^2D'} + \cancel{(D')^2S_o'} + \cancel{(D')2C_o'}] + \cancel{(S_o')^3}$$
$$+ \cancel{(C_o')^3} + \cancel{(D')^3} - \frac{3}{x^2}[S_o' + C_o' + D'] + \frac{\cancel{2}}{\cancel{x^4}} = 0$$

Assumptions: $S_o' >> C_o' >> D'$, $S_o'' >> C_o'' >> D''$, $S_o''' >> C_o''' >> D'''$ all as $x \to 0^+$ and remove previously balanced terms,

$$S_o''' + 3S_o''C_o' + 3C_o''S_o' + 3(S_o')^2D' \sim -3(C_o')S_o' \qquad x \to 0^+$$

Subbing in the previously found values for S_o, C_o,

$$-3wx^{-10/3} + \frac{28w}{9}x^{-10/3} - \cancel{\frac{16w}{3}x^{-10/3}} - 4wx^{-10/3} + 3w^2x^{-8/3}D' \sim \cancel{\frac{-16w}{3}x^{-10/3}}$$

$$-3w^2x^{-8/3}D' \sim \frac{35w}{9}x^{-10/3}$$
$$D' \sim \frac{35}{27w}x^{-2/3}$$
$$D \sim \frac{35}{27w}(\frac{1}{3})x^{1/3} + d$$

So since $x^{1/3} \to 0$ as $x \to 0^+$, the leading order behavior is

$$\boxed{y \sim \exp[-\frac{w}{3}x^{-1/3} + \frac{4}{3}\ln(x) + d] \qquad x \to 0^+}$$