$\boxed{3.27}$ $\frac{d^{(n)}}{dx^{(n)}} y = Q(x) y \quad x \to x_0 \text{ ISP}$

Assume $Q(x) > 0$ for simplicity

Lets do 3^{rd}-order case 1^{st}:

$y''' = Q(x) y$, $y = \exp[S(x)]$, $S \sim S_0 + S_1 + S_2 + \cdots$

$y' = S' e^S$, $y'' = S'' e^S + (S')^2 e^S$

$y''' = \left[S''' + (S')^3 + 3S''S' \right] e^S$

Try the dominant balance $(S_0')^3 \sim Q(x) \quad x \to x_0$

$\Rightarrow S_0' \sim w[Q(x)]^{1/3} \quad w = (1)^{1/3}$

$\Rightarrow S_0 \sim \int^x w[Q(t)]^{1/3} dt + C_0$

Lets assume that all dropped terms are smaller than the terms we kept.

Continuing on ...

②

$$\left\{ S_0''' + S_1''' + \cdots \right\} + \left\{ \cancel{(S_0')^3} + 3(S_0')^2 S_1' + \cdots \right\}$$

$$+ \; 3\left\{ S_0'' S_0' + S_0'' S_1' + S_0' S_1'' + \cdots \right\} \sim \cancel{\phi(x)}$$

Next balance $\quad 3(S_0')^2 S_1' \sim -3 S_0'' S_0'$

$$\Rightarrow \quad S_1' \sim -\frac{S_0''}{S_0'} \quad \Rightarrow \quad S_1 \sim -\ln|S_0'| + C_1$$

$$\Rightarrow \quad S_1 \sim -\ln\left[\omega\, \phi(x)^{1/3} \right] + C_1$$

$$y(x) \sim C^* \phi(x)^{-1/3} \exp\left[\int^x \omega\, \phi(t)^{1/3}\, dt \right]$$

where constants have been absorbed into C^*.

Check BO formula for $n = 3$

$$\frac{(1-n)}{2n} = \frac{(1-3)}{6} = -\frac{1}{3} \quad \checkmark$$

$$\frac{1}{n} = \frac{1}{3} \quad \checkmark$$

Now try 4th-order equation

$$y^{(IV)} = \Big\{ S^{(IV)} + (S')^4 + 4S'''S' + 6(S')^2 S'' + 3S''S'' \Big\} e^{S}$$

The dominant balance will be $(S_0')^4 \sim \phi(x)$

$$\Rightarrow \quad S_0' \sim \omega\, \phi(x)^{1/4}, \quad \omega = (1)^{1/4}$$

$$\Rightarrow \quad S_0 \sim \int^x \omega\, \phi(t)^{1/4}\, dt + C_0$$

Assuming consistency, the next order will give

$$4(S_0')^3 S_1' \sim -6(S_0')^2 S_0''$$

$$S_1' \sim -\frac{6}{4} \frac{S_0''}{S_0'}$$

$$S_1 \sim -\frac{6}{4} \ln|S_0'| + C_1$$

$$\sim -\frac{6}{4} \ln\left[\omega\, \phi(x)^{1/4}\right] + C_1$$

Note that we supress higher-order derivatives in favor of lower-order derivatives raised to some power.

$$y \sim C^* \left[Q(x)^{1/4}\right]^{-6/4} \exp\left[\int^x \omega\, Q(t)^{1/4}\, dt\right]$$

$x \to x_0$

$$y \sim C^* \left[Q(x)\right]^{-3/8} \exp\left[\int^x \omega Q(t)^{1/4}\, dt\right]$$

Check BO formula for $n=4$

$$\frac{1-n}{2n} = \frac{1-4}{8} = \frac{-3}{8} \quad \checkmark \qquad \frac{1}{n} = 1/4 \quad \checkmark$$

Generalizing to larger $n \Longrightarrow$

$$y(x) \sim C^* \left[Q(x)\right]^{\frac{(1-n)}{2n}} \exp\left[\int^x \omega\, Q(t)^{1/n}\, dt\right]$$

$x \to x_0$