HW 2, NE 548, Spring 2017

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Contents

0.1 problem 3.27 (page 138)

Problem derive 3.4.28. Below is a screen shot from the book giving 3.4.28 at page 88, and the context it is used in before solving the problem

Example 5 *Local behavior of solutions near an irregular singular point of a general nth-order Schrödinger equation.* In this example we derive an extremely simple and important formula for the leading behavior of solutions to the nth-order Schrödinger equation

$$\frac{d^n y}{dx^n} = Q(x)y \tag{3.4.27}$$

near an irregular singular point at x_0.

The exponential substitution $y = e^S$ and the asymptotic approximations $d^k S/dx^k \ll (S')^k$ as $x \to x_0$ for $k = 2, 3, \ldots, n$ give the asymptotic differential equation $(S')^n \sim Q(x)$ $(x \to x_0)$. Thus, $S(x) \sim \omega \int^x [Q(t)]^{1/n}\, dt$ $(x \to x_0)$, where ω is an nth root of unity. This result determines the n possible controlling factors of $y(x)$.

The leading behavior of $y(x)$ is found in the usual way (see Prob. 3.27) to be

$$y(x) \sim c[Q(x)]^{(1-n)/2n} \exp\left\{\omega \int^x [Q(t)]^{1/n}\, dt\right\}, \qquad x \to x_0. \tag{3.4.28}$$

If $x_0 \neq \infty$, (3.4.28) is valid if $|(x - x_0)^n Q(x)| \to \infty$ as $x \to x_0$. If $x_0 = \infty$, then (3.4.28) is valid if $|x^n Q(x)| \to \infty$ as $x \to \infty$. This important formula forms the basis of WKB theory and will be rederived perturbatively and in much greater detail in Sec. 10.2. If $Q(x) < 0$, solutions to (3.4.27) oscillate as $x \to \infty$; the nature of asymptotic relations between oscillatory functions is discussed in Sec. 3.7.

Here are some examples of the application of (3.4.28):

(*a*) For $y'' = y/x^5$, $y(x) \sim cx^{5/4} e^{\pm 2x^{-3/2}/3}$ $(x \to 0+)$.
(*b*) For $y''' = xy$, $y(x) \sim cx^{-1/3} e^{3\omega x^{4/3}/4}$ $(x \to +\infty)$, where $\omega^3 = 1$.
(*c*) For $d^4y/dy^4 = (x^4 + \sin x)y$, $y(x) \sim cx^{-3/2} e^{\omega x^2/2}$ $(x \to +\infty)$, where $\omega = \pm 1, \pm i$.

Solution

For n^{th} order ODE, $S_0(x)$ is given by

$$S_0(x) \sim \omega \int^x Q(t)^{\frac{1}{n}}\, dt$$

And (page 497, textbook)

$$S_1(x) \sim \frac{1-n}{2n} \ln(Q(x)) + c \tag{10.2.11}$$

Therefore

$$\begin{aligned} y(x) &\sim \exp(S_0 + S_1) \\ &\sim \exp\left(\omega \int^x Q(t)^{\frac{1}{n}} dt + \frac{1-n}{2n} \ln(Q(x)) + c\right) \\ &\sim c\,[Q(x)]^{\frac{1-n}{2n}} \exp\left(\omega \int^x Q(t)^{\frac{1}{n}} dt\right) \end{aligned}$$

Note: I have tried other methods to proof this, such as a proof by induction. But was not able to after many hours trying. The above method uses a given formula which the book did not indicate how it was obtained. (see key solution)

0.2 Problem 3.33(b) (page 140)

Problem Find leading behavior as $x \to 0^+$ for $x^4 y''' - 3x^2 y' + 2y = 0$

Solution Let

$$\begin{aligned} y(x) &= e^{S(x)} \\ y' &= S' e^S \\ y'' &= S'' e^S + (S')^2 e^S \\ y''' &= S''' e^S + S'' S' e^S + 2S' S'' e^S + (S')^3 e^S \end{aligned}$$

Hence the ODE becomes

$$x^4 \left[S''' + 3S'S'' + (S')^3\right] - 3x^2 S' = -2 \tag{1}$$

Now, we define $S(x)$ as sum of a number of leading terms, which we try to find

$$S(x) = S_0(x) + S_1(x) + S_2(x) + \cdots$$

Therefore (1) becomes (using only two terms for now $S = S_0 + S_1$)

$$\begin{aligned} \{S_0 + S_1\}''' + 3\left\{(S_0 + S_1)(S_0 + S_1)''\right\} + \left\{(S_0 + S_1)'\right\}^3 - \frac{3}{x^2}\{S_0 + S_1\}' &= -\frac{2}{x^4} \\ \left\{S_0''' + S_1'''\right\} + 3\left\{(S_0 + S_1)\left(S_0'' + S_1''\right)\right\} + \left\{S_0' + S_1'\right\}^3 - \frac{3}{x^2}\left(S_0' + S_1'\right) &= -\frac{2}{x^4} \\ \left\{S_0''' + S_1'''\right\} + 3\left\{S_0''S_0 + S_0''S_1 + S_1''S_0\right\} + \left\{\left(S_0'\right)^3 + 3\left(S_0'\right)^2 S_1' + 3S_0'\left(S_1'\right)^2\right\} - \frac{3}{x^2}\left\{S_0' + S_1'\right\} &= -\frac{2}{x^4} \end{aligned} \tag{2}$$

Assuming that $S_0' \ggg S_1', S_0''' \ggg S_1''', \left(S_0'\right)^3 \ggg 3\left(S_0'\right)^2 S_1'$ then equation (2) simplifies to

$$S_0''' + 3S_0''S_0' + \left(S_0'\right)^3 - \frac{3}{x^2}S_0' \sim -\frac{2}{x^4}$$

Assuming $\left(S_0'\right)^3 \ggg S_0''', \left(S_0'\right)^3 \ggg 3S_0''S_0', \left(S_0'\right)^3 \ggg \frac{3}{x^2}S_0'$ (which we need to verify later), then the above becomes

$$\left(S_0'\right)^3 \sim -\frac{2}{x^4}$$

Verification[1]

[1] When carrying out verification, all constant multipliers and signs are automticaly simplified and removed

Since $S_0' \sim \left(\frac{-2}{x^4}\right)^{\frac{1}{3}} = \frac{1}{x^{\frac{4}{3}}}$ then $S_0'' \sim \frac{1}{x^{\frac{7}{3}}}$ and $S_0''' \sim \frac{1}{x^{\frac{10}{3}}}$. Now we need to verify the three assumptions made above, which we used to obtain S_0'.

$$\begin{aligned} \left(S_0'\right)^3 &\ggg S_0''' \\ \frac{1}{x^4} &\ggg \frac{1}{x^{\frac{10}{3}}} \end{aligned}$$

Yes.

$$\begin{aligned} \left(S_0'\right)^3 &\ggg 3S_0''S_0' \\ \frac{1}{x^4} &\ggg \left(\frac{1}{x^{\frac{7}{3}}}\right)\left(\frac{1}{x^{\frac{4}{3}}}\right) \\ \frac{1}{x^4} &\ggg \left(\frac{1}{x^{\frac{11}{3}}}\right) \end{aligned}$$

Yes.

$$\begin{aligned} \left(S_0'\right)^3 &\ggg \frac{3}{x^2}S_0' \\ \frac{1}{x^4} &\ggg \left(\frac{1}{x^2}\right)\frac{1}{x^{\frac{4}{3}}} \\ \frac{1}{x^4} &\ggg \frac{1}{x^{\frac{10}{3}}} \end{aligned}$$

Yes. Assumed balance ise verified. Therefore

$$\begin{aligned} \left(S_0'\right)^3 &\sim -\frac{2}{x^4} \\ S_0' &\sim \omega x^{\frac{-4}{3}} \end{aligned}$$

Where $\omega^3 = -2$. Integrating

$$\begin{aligned} S_0 &\sim \omega \int x^{\frac{-4}{3}} dx \\ &\sim \omega \int x^{-\frac{4}{3}} dx \\ &\sim -3\omega x^{\frac{-1}{3}} \end{aligned}$$

Where we ignored the constant of integration since subdominant. To find leading behavior, we go back to equation (2) and now solve for S_1.

$$\left\{S_0''' + S_1'''\right\} + 3\left\{S_0''S_0' + S_0''S_1' + S_1''S_0'\right\} + \left\{\left(S_0'\right)^3 + 3\left(S_0'\right)^2 S_1' + 3S_0'\left(S_1'\right)^2\right\} - \frac{3}{x^2}\left\{S_0' + S_1'\right\} = -\frac{2}{x^4}$$

Moving all known quantities (those which are made of S_0 and its derivatives) to the RHS

going from one step to the next, as they do not affect the final result.

and simplifying, gives

$$\left\{S_1'''\right\} + 3\left\{S_0''S_1' + S_1''S_0'\right\} + \left\{3\left(S_0'\right)^2 S_1' + 3S_0'\left(S_1'\right)^2\right\} - \frac{3S_1'}{x^2} \sim -S_0''' - 3S_0''S_0' + \frac{3}{x^2}S_0'$$

Now we assume the following (then will verify later)

$$3\left(S_0'\right)^2 S_1' \ggg 3S_0'\left(S_1'\right)^2$$
$$3\left(S_0'\right)^2 S_1' \ggg S_1'''$$
$$3\left(S_0'\right)^2 S_1' \ggg S_1''S_0'$$
$$3\left(S_0'\right)^2 S_1' \ggg S_0''S_1'$$
$$3\left(S_0'\right)^2 S_1' \ggg S_1''S_0'$$

Hence

$$3\left(S_0'\right)^2 S_1' - \frac{3S_1'}{x^2} \sim -S_0''' - 3S_0''S_0' + \frac{3S_0'}{x^2} \tag{3}$$

But

$$S_0' \sim \omega x^{\frac{-4}{3}}$$
$$\left(S_0'\right)^2 \sim \omega^2 x^{\frac{-8}{3}}$$
$$S_0'' \sim -\frac{4}{3}\omega x^{\frac{-7}{3}}$$
$$S_0''' \sim \frac{28}{9}\omega x^{\frac{-10}{3}}$$

Hence (3) becomes

$$3\left(\omega^2 x^{\frac{-8}{3}}\right)S_1' - \frac{3S_1'}{x^2} \sim \frac{28}{9}\omega x^{\frac{-10}{3}} + 3\left(\frac{4}{3}\omega^2 x^{\frac{-7}{3}} x^{\frac{-4}{3}}\right) + \frac{3\omega x^{\frac{-4}{3}}}{x^2}$$
$$3\omega^2 x^{\frac{-8}{3}} S_1' - 3x^{-2}S_1' \sim \frac{28}{9}\omega x^{\frac{-10}{3}} + 4\omega^2 x^{\frac{-11}{3}} + 3\omega x^{\frac{-10}{3}}$$

For small x, $x^{\frac{-8}{3}}S_1' \ggg x^{-2}S_1'$ and $x^{\frac{-11}{3}} \ggg x^{\frac{-10}{3}}$, then the above simplifies to

$$3\omega^2 x^{\frac{-8}{3}} S_1' \sim 4\omega^2 x^{\frac{-11}{3}}$$
$$S_1' \sim \frac{4}{3}x^{-1}$$

$$S_1 \sim \frac{4}{3}\ln x$$

Where constant of integration was dropped, since subdominant.

<u>Verification</u> Using $S_0' \backsim x^{\frac{-4}{3}}, \left(S_0'\right)^2 \backsim x^{\frac{-8}{3}}, S_0'' \backsim x^{\frac{-7}{3}}, S_1' \sim \frac{1}{x}, S_1'' \sim \frac{1}{x^2}, S_1''' \sim \frac{1}{x^3}$

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg 3S_0'\left(S_1'\right)^2 \\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg x^{\frac{-4}{3}}\frac{1}{x^2} \\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_1''' \\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg \frac{1}{x^3} \\
\frac{1}{x^{\frac{8}{3}}} &\ggg \frac{1}{x^2}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_1'' S_0' \\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg \frac{1}{x^2} x^{\frac{-4}{3}} \\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_0'' S_1' \\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg x^{\frac{-7}{3}}\frac{1}{x} \\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes

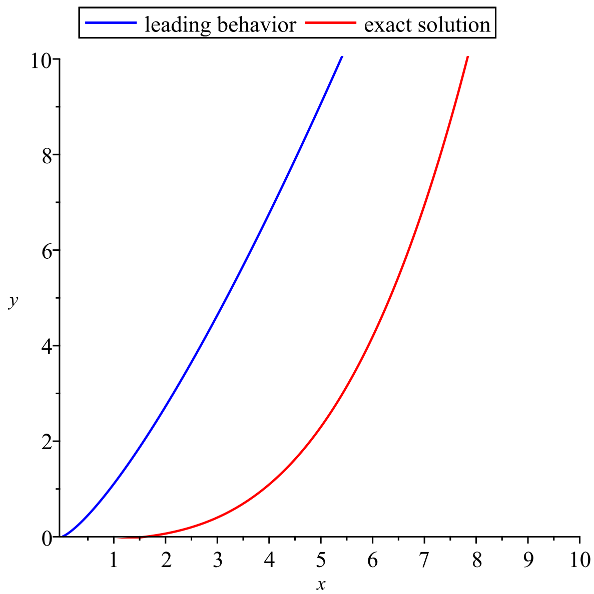
$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_1'' S_0' \\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg \frac{1}{x^2} x^{\frac{-4}{3}} \\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes. All verified. Leading behavior is

$$
\begin{aligned}
y(x) &\backsim e^{S_0(x)+S_1(x)} \\
&= \exp\left(c\omega x^{\frac{-1}{3}} + \frac{4}{3}\ln x\right) \\
&= x^{\frac{4}{3}} e^{c\omega x^{\frac{-1}{3}}}
\end{aligned}
$$

I now wanted to see how Maple solution to this problem compare with the leading behavior near $x = 0$. To obtain a solution from Maple, one have to give initial conditions a little bit removed from $x = 0$ else no solution could be generated. So using arbitrary initial conditions at $x = \frac{1}{100}$ a solution was obtained and compared to the above leading behavior. Another problem is how to select c in the above leading solution. By trial and error a constant was selected. Here is screen shot of the result. The exact solution generated by Maple is very complicated, in terms of hypergeom special functions.

```
ode:=x^4*diff(y(x),x$3)-3*x^2*diff(y(x),x)+2*y(x);
pt:=1/100:
ic:=y(pt)=500,D(y)(pt)=0,(D@@2)(y)(pt)=0:
sol:=dsolve({ode,ic},y(x)):
leading:=(x,c)->x^(4/3)*exp(c*x^(-1/3));
plot([leading(x,.1),rhs(sol)],x=pt..10,y=0..10,color=[blue,red],
legend=["leading behavior","exact solution"],legendstyle=[location=top]);
```



0.3 problem 3.33(c) (page 140)

Problem Find leading behavior as $x \to 0^+$ for $y'' = \sqrt{x}y$

Solution

Let $y(x) = e^{S(x)}$. Hence

$$\begin{aligned} y(x) &= e^{S_0(x)} \\ y'(x) &= S_0' e^{S_0} \\ y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\ &= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} \end{aligned}$$

Substituting in the ODE gives

$$S_0'' + \left(S_0'\right)^2 = \sqrt{x} \tag{1}$$

Assuming $S_0'' \sim \left(S_0'\right)^2$ then (1) becomes

$$S_0'' \sim -\left(S_0'\right)^2$$

Let $S_0' = z$ then the above becomes $z' = -z^2$. Hence $\frac{dz}{dx}\frac{1}{z^2} = -1$ or $\frac{dz}{z^2} = -dx$. Integrating $-\frac{1}{z} = -x + c$ or $z = \frac{1}{x+c_1}$. Hence $S_0' = \frac{1}{x+c_1}$. Integrating again gives

$$S_0(x) \sim \ln|x + c_1| + c_2$$

Verification

$S_0' = \frac{1}{x+c_1}, \left(S_0'\right)^2 = \frac{1}{(x+c_1)^2}, S_0'' = \frac{-1}{(x+c_1)^2}$

$$\begin{aligned} S_0'' &\ggg x^{\frac{1}{2}} \\ \frac{1}{(x+c_1)^2} &\ggg x^{\frac{1}{2}} \end{aligned}$$

Yes for $x \to 0^+$.

$$\begin{aligned} \left(S_0'\right)^2 &\ggg x^{\frac{1}{2}} \\ \frac{1}{(x+c_1)^2} &\ggg x^{\frac{1}{2}} \end{aligned}$$

Yes for $x \to 0^+$. Verified. Controlling factor is

$$\begin{aligned} y(x) &\sim e^{S_0(x)} \\ &\sim e^{\ln|x+c_1|+c_2} \\ &\sim Ax + B \end{aligned}$$

0.4 problem 3.35

Problem: Obtain the full asymptotic behavior for small x of solutions to the equation

$$x^2 y'' + (2x+1)\,y' - x^2\left(e^{\frac{2}{x}} + 1\right)y = 0$$

Solution

Let $y(x) = e^{S(x)}$. Hence

$$
\begin{aligned}
y(x) &= e^{S_0(x)} \\
y'(x) &= S_0' e^{S_0} \\
y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\
&= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0}
\end{aligned}
$$

Substituting in the ODE gives

$$
\begin{aligned}
x^2\left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} + (2x+1) S_0' e^{S_0} - x^2\left(e^{\frac{2}{x}} + 1\right) e^{S_0(x)} &= 0 \\
x^2\left(S_0'' + \left(S_0'\right)^2\right) + (2x+1) S_0' - x^2\left(e^{\frac{2}{x}} + 1\right) &= 0 \\
x^2\left(S_0'' + \left(S_0'\right)^2\right) + (2x+1) S_0' &= x^2\left(e^{\frac{2}{x}} + 1\right) \\
S_0'' + \left(S_0'\right)^2 + \frac{(2x+1)}{x^2} S_0' &= e^{\frac{2}{x}} + 1
\end{aligned}
$$

Assuming balance

$$
\begin{aligned}
\left(S_0'\right)^2 &\sim \left(e^{\frac{2}{x}} + 1\right) \\
\left(S_0'\right)^2 &\sim e^{\frac{2}{x}} \\
S_0' &\sim \pm e^{\frac{1}{x}}
\end{aligned}
$$

Where 1 was dropped since subdominant to $e^{\frac{1}{x}}$ for small x.

<u>Verification</u> Since $\left(S_0'\right)^2 \sim e^{\frac{2}{x}}$ then $S_0' \sim e^{\frac{1}{x}}$ and $S_0'' \sim -\frac{1}{x^2} e^{\frac{1}{x}}$, hence

$$
\begin{aligned}
\left(S_0'\right)^2 &\ggg S_0'' \\
e^{\frac{2}{x}} &\ggg \frac{1}{x^2} e^{\frac{1}{x}} \\
e^{\frac{1}{x}} &\ggg \frac{1}{x^2}
\end{aligned}
$$

<u>Yes</u>, As $x \to 0^+$

$$
\begin{aligned}
\left(S_0'\right)^2 &\ggg \frac{(2x+1) S_0'}{x^2} \\
e^{\frac{2}{x}} &\ggg \frac{(2x+1)}{x^2} e^{\frac{1}{x}} \\
e^{\frac{1}{x}} &\ggg \frac{(2x+1)}{x^2} \\
e^{\frac{1}{x}} &\ggg \frac{2}{x} + \frac{1}{x^2}
\end{aligned}
$$

Yes as $x \to 0^+$. Verified. Hence both assumptions used were verified OK. Hence

$$S_0' \sim \pm e^{\frac{2}{x}}$$
$$S_0 \sim \pm \int e^{\frac{1}{x}} dx$$

Since the integral do not have closed form, we will do asymptotic expansion on the integral. Rewriting $\int e^{\frac{1}{x}} dx$ as $\int \frac{e^{\frac{1}{x}}}{\left(-x^2\right)}\left(-x^2\right) dx$. Using $\int u dv = uv - \int v du$, where $u = -x^2, dv = \frac{-e^{\frac{1}{x}}}{x^2}$, gives $du = -2x$ and $v = e^{\frac{1}{x}}$, hence

$$\begin{aligned}
\int e^{\frac{1}{x}} dx &= uv - \int v du \\
&= -x^2 e^{\frac{1}{x}} - \int -2x e^{\frac{1}{x}} dx \\
&= -x^2 e^{\frac{1}{x}} + 2\int x e^{\frac{1}{x}} dx \qquad (1)
\end{aligned}$$

Now we apply integration by parts on $\int x e^{\frac{1}{x}} dx = \int x \frac{e^{\frac{1}{x}}}{-x^3}\left(-x^3\right) du = \int \frac{e^{\frac{1}{x}}}{-x^2}\left(-x^3\right) du$, where $u = -x^3, dv = \frac{e^{\frac{1}{x}}}{-x^2}$, hence $du = -3x^2, v = e^{\frac{1}{x}}$, hence we have

$$\begin{aligned}
\int x e^{\frac{1}{x}} dx &= uv - \int v du \\
&= -x^3 e^{\frac{1}{x}} + \int 3x^2 e^{\frac{1}{x}} dx
\end{aligned}$$

Substituting this into (1) gives

$$\begin{aligned}
\int e^{\frac{1}{x}} dx &= -x^2 e^{\frac{1}{x}} + 2\left(-x^3 e^{\frac{1}{x}} + \int 3x^2 e^{\frac{1}{x}} dx\right) \\
&= -x^2 e^{\frac{1}{x}} - 2x^3 e^{\frac{1}{x}} + 6\int 3x^2 e^{\frac{1}{x}} dx
\end{aligned}$$

And so on. The series will become

$$\begin{aligned}
\int e^{\frac{1}{x}} dx &= -x^2 e^{\frac{1}{x}} - 2x^3 e^{\frac{1}{x}} + 6x^4 e^{\frac{1}{x}} + 24x^5 e^{\frac{1}{x}} + \cdots + n! x^{n+1} e^{\frac{1}{x}} + \cdots \\
&= -e^{\frac{1}{x}}\left(x^2 + 2x^3 + 6x^4 + \cdots + n! x^{n+1} + \cdots\right)
\end{aligned}$$

Now as $x \to 0^+$, we can decide how many terms to keep in the RHS, If we keep one term, then we can say

$$\begin{aligned}
S_0 &\sim \pm \int e^{\frac{1}{x}} dx \\
&\sim \pm x^2 e^{\frac{1}{x}}
\end{aligned}$$

For two terms

$$S_0 \sim \pm \int e^{\frac{1}{x}} dx \sim \pm e^{\frac{1}{x}}\left(x^2 + 2x^3\right)$$

And so on. Let us use one term for now for the rest of the solution.

$$S_0 \sim \pm x^2 e^{\frac{1}{x}}$$

To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$y'(x) = (S_0(x) + S_1(x))' e^{S_0+S_1}$$
$$y''(x) = \left((S_0 + S_1)'\right)^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1}$$

Substituting into the given ODE gives

$$x^2\left[\left((S_0 + S_1)'\right)^2 + (S_0 + S_1)''\right] + (2x+1)(S_0(x) + S_1(x))' - x^2\left(e^{\frac{2}{x}} + 1\right) = 0$$
$$x^2\left[\left(S_0' + S_1'\right)^2 + (S_0 + S_1)''\right] + (2x+1)\left(S_0'(x) + S_1'(x)\right) - x^2\left(e^{\frac{2}{x}} + 1\right) = 0$$
$$x^2\left[\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + \left(S_0'' + S_1''\right)\right] + (2x+1)S_0'(x) + (2x+1)S_1'(x) - x^2\left(e^{\frac{2}{x}} + 1\right) = 0$$
$$x^2\left(S_0'\right)^2 + x^2\left(S_1'\right)^2 + 2x^2S_0'S_1' + x^2S_0'' + x^2S_1'' + (2x+1)S_0'(x) + (2x+1)S_1'(x) = x^2\left(e^{\frac{2}{x}} + 1\right)$$

But $x^2\left(S_0'\right)^2 \sim x^2\left(e^{\frac{2}{x}} + 1\right)$ since we found that $S_0' \sim e^{\frac{1}{x}}$. Hence the above simplifies to

$$x^2\left(S_1'\right)^2 + 2x^2S_0'S_1' + x^2S_0'' + x^2S_1'' + (2x+1)S_0'(x) + (2x+1)S_1'(x) = 0$$
$$\left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' + \frac{(2x+1)}{x^2}S_0'(x) + \frac{(2x+1)}{x^2}S_1'(x) = 0 \tag{2}$$

Now looking at $S_0'' + \frac{(2x+1)}{x^2}S_0'(x)$ terms in the above. We can simplify this since we know $S_0' = e^{\frac{1}{x}}, S_0'' = -\frac{1}{x^2}e^{\frac{1}{x}}$ This terms becomes

$$-\frac{1}{x^2}e^{\frac{1}{x}} + \frac{(2x+1)}{x^2}e^{\frac{1}{x}} = \frac{-e^{\frac{1}{x}} + 2xe^{\frac{1}{x}} + e^{\frac{1}{x}}}{x^2} = \frac{2xe^{\frac{1}{x}}}{x^2} = \frac{2e^{\frac{1}{x}}}{x}$$

Therefore (2) becomes

$$\left(S_1'\right)^2 + 2S_0'S_1' + S_1'' + \frac{(2x+1)}{x^2}S_1'(x) \sim \frac{-2e^{\frac{1}{x}}}{x}$$
$$\frac{\left(S_1'\right)^2}{S_0'} + 2S_1' + \frac{S_1''}{S_0'} + \frac{(2x+1)}{x^2S_0'}S_1'(x) \sim \frac{-2e^{\frac{1}{x}}}{xS_0'}$$
$$\frac{\left(S_1'\right)^2}{e^{\frac{1}{x}}} + 2S_1' + \frac{S_1''}{e^{\frac{1}{x}}} + \frac{(2x+1)}{x^2e^{\frac{1}{x}}}S_1'(x) \sim \frac{-2}{x}$$

Assuming the balance is

$$S_1' \sim \frac{-1}{x}$$

Hence

$$S_1(x) \sim -\ln(x) + c$$

Since c subdominant as $x \to 0^+$ then

$$\boxed{S_1(x) \sim -\ln(x)}$$

Verification

$$\begin{aligned} S_1' &\ggg \frac{\left(S_1'\right)^2}{e^{\frac{1}{x}}} \\ \frac{1}{x} &\ggg \frac{1}{x^2}\frac{1}{e^{\frac{1}{x}}} \\ e^{\frac{1}{x}} &\ggg \frac{1}{x} \end{aligned}$$

Yes, for $x \to 0^+$

$$\begin{aligned} S_1' &\ggg \frac{S_1''}{e^{\frac{1}{x}}} \\ \frac{1}{x} &\ggg \frac{1}{x^2}\frac{1}{e^{\frac{1}{x}}} \end{aligned}$$

Yes.

$$\begin{aligned} S_1' &\ggg \frac{(2x+1)}{x^2 e^{\frac{1}{x}}} S_1'(x) \\ \frac{1}{x} &\ggg \frac{(2x+1)}{x^2 e^{\frac{1}{x}}}\frac{1}{x} \\ \frac{1}{x} &\ggg \frac{1}{x^3 e^{\frac{1}{x}}} \end{aligned}$$

Yes. All assumptions verified. Hence leading behavior is

$$\begin{aligned} y(x) &\sim \exp\left(S_0(x) + S_1(x)\right) \\ &\sim \exp\left(\pm x^2 e^{\frac{1}{x}} - \ln(x)\right) \\ &\sim \frac{1}{x}\left(\exp\left(x^2 e^{\frac{1}{x}}\right) + \exp\left(-x^2 e^{\frac{1}{x}}\right)\right) \end{aligned}$$

For small x, then we ignore $\exp\left(-x^2 e^{\frac{1}{x}}\right)$ since much smaller than $\exp\left(x^2 e^{\frac{1}{x}}\right)$. Therefore

$$y(x) \sim \frac{1}{x}\exp\left(x^2 e^{\frac{1}{x}}\right)$$

0.5 problem 3.39(h)

problem Find leading asymptotic behavior as $x \to \infty$ for $y'' = e^{-\frac{3}{x}} y$

<u>solution</u> Let $y(x)=e^{S(x)}$. Hence

$$\begin{aligned} y(x) &= e^{S_0(x)} \\ y'(x) &= S_0' e^{S_0} \\ y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\ &= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} \end{aligned}$$

Substituting in the ODE gives

$$\begin{aligned} \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} &= e^{-\frac{3}{x}} e^{S_0} \\ S_0'' + \left(S_0'\right)^2 &= e^{-\frac{3}{x}} \end{aligned}$$

<u>Assuming</u> $\left(S_0'\right)^2 \ggg S_0''$ the above becomes

$$\begin{aligned} \left(S_0'\right)^2 &\sim e^{-\frac{3}{x}} \\ S_0' &\sim \pm e^{-\frac{3}{2x}} \end{aligned}$$

Hence

$$S_0 \sim \pm \int e^{-\frac{3}{2x}} dx$$

Integration by parts. Since $\frac{d}{dx} e^{\frac{-3}{2x}} = \frac{3}{2x^2} e^{-\frac{3}{2x}}$, then we rewrite the integral above as

$$\int e^{-\frac{3}{2x}} dx = \int \frac{3}{2x^2} e^{-\frac{3}{2x}} \left(\frac{2x^2}{3}\right) dx$$

And now apply integration by parts. Let $dv = \frac{3}{2x^2} e^{-\frac{3}{2x}} \rightarrow v = e^{-\frac{3}{2x}}, u = \frac{2x^2}{3} \rightarrow du = \frac{4}{3} x$, hence

$$\begin{aligned} \int e^{-\frac{3}{2x}} dx &= [uv] - \int v du \\ &= \frac{2x^2}{3} e^{-\frac{3}{2x}} - \int \frac{4}{3} x e^{-\frac{3}{2x}} dx \end{aligned}$$

Ignoring higher terms, then we use

$$S_0 \sim \pm \frac{2x^2}{3} e^{-\frac{3}{2x}}$$

<u>Verification</u>

$$\begin{aligned} \left(S_0'\right)^2 &\ggg S_0'' \\ \left(e^{-\frac{3}{2x}}\right)^2 &\ggg \frac{3}{2x^2} e^{-\frac{3}{2x}} \\ e^{-\frac{3}{x}} &\ggg \frac{3}{2x^2} e^{-\frac{3}{2x}} \end{aligned}$$

<u>Yes</u>, as $x \rightarrow \infty$. To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$y'(x) = (S_0(x) + S_1(x))' e^{S_0+S_1}$$
$$y''(x) = \left((S_0 + S_1)'\right)^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1}$$

Using the above, the ODE $y'' = e^{-\frac{3}{x}} y$ now becomes

$$\left((S_0 + S_1)'\right)^2 + (S_0 + S_1)'' = e^{-\frac{3}{x}}$$
$$\left(S_0' + S_1'\right)^2 + S_0'' + S_1'' = e^{-\frac{3}{x}}$$
$$\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' = e^{-\frac{3}{x}}$$

But $\left(S_0'\right)^2 \sim e^{-\frac{3}{x}}$ hence the above simplifies to

$$\left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' = 0$$

<u>Assuming</u> $\left(2S_0'S_1'\right) \ggg S_1''$ the above becomes

$$\left(S_1'\right)^2 + 2S_0'S_1' + S_0'' = 0$$

<u>Assuming</u> $2S_0'S_1' \ggg \left(S_1'\right)^2$

$$2S_0'S_1' + S_0'' = 0$$
$$S_1' \sim -\frac{S_0''}{2S_0'}$$
$$S_1 \sim -\frac{1}{2} \ln\left(S_0'\right)$$

But $S_0' \sim e^{-\frac{3}{2x}}$, hence the above becomes

$$S_1 \sim -\frac{1}{2} \ln\left(e^{-\frac{3}{2x}}\right) + c$$
$$\sim \frac{3}{4x} + c$$

<u>Verification</u>

$$\left(2S_0'S_1'\right) \ggg S_1''$$
$$\left(2e^{-\frac{3}{2x}} \frac{-3}{4x^2}\right) \ggg \frac{3}{2x^3}$$
$$\frac{3}{2} \frac{e^{-\frac{3}{2x}}}{x^2} \ggg \frac{3}{2x^3}$$

For large x the above simplifies to

$$\frac{1}{x^2} \ggg \frac{1}{x^3}$$

Yes.

$$\left(2S_0'S_1'\right) \ggg \left(S_1'\right)^2$$
$$\left(2e^{-\frac{3}{2x}}\frac{-3}{4x^2}\right) \ggg \left(\frac{-3}{4x^2}\right)^2$$
$$\frac{3}{2}\frac{e^{-\frac{3}{2x}}}{x^2} \ggg \frac{9}{16x^4}$$

For large x the above simplifies to

$$\frac{1}{x^2} \ggg \frac{1}{x^4}$$

Yes. All verified. Therefore, the leading behavior is

$$\begin{aligned} y(x) &\sim \exp\left(S_0(x) + S_1(x)\right) \\ &\sim \exp\left(\pm\frac{2x^2}{3}e^{-\frac{3}{2x}} - \frac{1}{2}\ln\left(e^{-\frac{3}{2x}}\right) + c\right) \\ &\sim ce^{\frac{3}{4x}}\exp\left(\pm\frac{2x^2}{3}e^{-\frac{3}{2x}}\right) \end{aligned} \tag{1}$$

Check if we can use 3.4.28 to verify:

$$\lim_{x\to\infty}\left|x^nQ(x)\right| = \lim_{x\to\infty}\left|x^2e^{-\frac{3}{x}}\right| \to \infty$$

We can use it. Lets verify using 3.4.28

$$y(x) \sim c\left[Q(x)\right]^{\frac{1-n}{2n}}\exp\left(\omega\int^x Q[t]^{\frac{1}{n}}\,dt\right)$$

Where $\omega^2 = 1$. For $n = 2, Q(x) = e^{-\frac{3}{x}}$, the above gives

$$\begin{aligned} y(x) &\sim c\left[e^{-\frac{3}{x}}\right]^{\frac{1-2}{4}}\exp\left(\omega\int^x\left[e^{-\frac{3}{t}}\right]^{\frac{1}{2}}dt\right) \\ &\sim c\left[e^{-\frac{3}{x}}\right]^{\frac{-1}{4}}\exp\left(\omega\int^x e^{-\frac{3}{2t}}dt\right) \\ &\sim ce^{\frac{3}{4x}}\exp\left(\omega\int^x e^{-\frac{3}{2t}}dt\right) \end{aligned} \tag{2}$$

We see that (1,2) are the same. Verified OK. Notice that in (1), we use the approximation for the $e^{-\frac{3}{2x}}dx \approx \frac{2x^2}{3}e^{-\frac{3}{2x}}$ we found earlier. This was done, since there is no closed form solution for the integral.

QED.

0.6 problem 3.42(a)

Problem: Extend investigation of example 1 of section 3.5 (a) Obtain the next few corrections to the leading behavior (3.5.5) then see how including these terms improves the numerical approximation of $y(x)$ in 3.5.1.

Solution Example 1 at page 90 is $xy'' + y' = y$. The leading behavior is given by 3.5.5 as $(x \to \infty)$

$$y(x) \sim cx^{\frac{-1}{4}} e^{2x^{\frac{1}{2}}} \tag{3.5.5}$$

Where the book gives $c = \frac{1}{2}\pi^{\frac{-1}{2}}$ on page 91. And 3.5.1 is

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \tag{3.5.1}$$

To see the improvement, the book method is followed. This is described at end of page 91. This is done by plotting the leading behavior as ratio to $y(x)$ as given in 3.5.1. Hence for the above leading behavior, we need to plot

$$\frac{\frac{1}{2}\pi^{\frac{-1}{2}} x^{\frac{-1}{4}} e^{2x^{\frac{1}{2}}}}{y(x)}$$

We are given $S_0(x), S_1(x)$ in the problem. They are

$$\begin{aligned} S_0(x) &= 2x^{\frac{1}{2}} \\ S_1(x) &= -\frac{1}{4}\ln x + c \end{aligned}$$

Hence

$$\begin{aligned} S_0'(x) &= x^{\frac{-1}{2}} \\ S_0'' &= \frac{-1}{2} x^{-\frac{3}{2}} \\ S_1'(x) &= -\frac{1}{4}\frac{1}{x} \\ S_1''(x) &= \frac{1}{4x^2} \end{aligned} \tag{1}$$

We need to find $S_2(x), S_3(x), \cdots$ to see that this will improve the solution $y(x) \sim \exp(S_0 + S_1 + S_2 + \cdots)$ as $x \to x_0$ compared to just using leading behavior $y(x) \sim \exp(S_0 + S_1)$. So now we need to find $S_2(x)$

Let $y(x) = e^S$, then the ODE becomes

$$x\left(S'' + (S')^2\right) + S' = 1$$

Replacing S by $S_0(x) + S_1(x) + S_2(x)$ in the above gives

$$\begin{aligned} (S_0 + S_1 + S_2)'' + \left[(S_0 + S_1 + S_2)'\right]^2 + \frac{1}{x}(S_0 + S_1 + S_2)' &\sim \frac{1}{x} \\ \left\{S_0'' + S_1'' + S_2''\right\} + \left[\left(S_0' + S_1' + S_2'\right)\right]^2 + \frac{1}{x}\left(S_0' + S_1' + S_2'\right) &\sim \frac{1}{x} \\ \left\{S_0'' + S_1'' + S_2''\right\} + \left\{\left[S_0'\right]^2 + 2S_0'S_1' + 2S_0'S_2' + \left[S_1'\right]^2 + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_0' + S_1' + S_2'\right\} &\sim \frac{1}{x} \end{aligned}$$

Moving all known quantities to the RHS, these are $S_0'', S_1'', \left[S_0'\right]^2, 2S_0'S_1', S_0', S_1', \left[S_1'\right]^2$ then the above reduces to

$$\left\{S_2''\right\} + \left\{+2S_0'S_2' + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_2'\right\} \sim \frac{1}{x} - S_0'' - S_1'' - \left[S_0'\right]^2 - 2S_0'S_1' - \frac{1}{x}S_0' - \frac{1}{x}S_1' - \left[S_1'\right]^2$$

Replacing known terms, by using (1) into the above gives

$$\left\{S_2''\right\} + \left\{2S_0'S_2' + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_2'\right\} \sim$$
$$\frac{1}{x} + \frac{1}{2}x^{-\frac{3}{2}} - \frac{1}{4x^2} - \left[x^{\frac{-1}{2}}\right]^2 - 2\left(x^{\frac{-1}{2}}\right)\left(-\frac{1}{4}\frac{1}{x}\right) - \frac{1}{x}\left(x^{\frac{-1}{2}}\right) - \frac{1}{x}\left(-\frac{1}{4}\frac{1}{x}\right) - \left(-\frac{1}{4}\frac{1}{x}\right)^2$$

Simplifying gives

$$\left\{S_2''\right\} + \left\{2S_0'S_2' + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_2'\right\} \sim \frac{1}{x} + \frac{1}{2}x^{-\frac{3}{2}} - \frac{1}{4x^2} - x^{-1} + \frac{1}{2}x^{\frac{-3}{2}} - x^{\frac{-3}{2}} + \frac{1}{4}\frac{1}{x^2} - \frac{1}{16}\frac{1}{x^2}$$

Hence

$$\left\{S_2''\right\} + \left\{2S_0'S_2' + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_2'\right\} \sim -\frac{1}{16}\frac{1}{x^2}$$

Lets assume now that

$$2S_0'S_2' \sim -\frac{1}{16}\frac{1}{x^2} \tag{2}$$

Therefore

$$\begin{aligned} S_2' &\sim -\frac{1}{32}\frac{1}{S_0'x^2} \\ &\sim -\frac{1}{32}\frac{1}{\left(x^{\frac{-1}{2}}\right)x^2} \\ &\sim -\frac{1}{32}x^{\frac{-3}{2}} \end{aligned}$$

We can now verify this before solving the ODE. We need to check that (as $x \to \infty$)

$$\begin{aligned} 2S_0'S_2' &\ggg S_2'' \\ 2S_0'S_2' &\ggg 2S_1'S_2' \\ 2S_0'S_2' &\ggg \left[S_2'\right]^2 \\ 2S_0'S_2' &\ggg \frac{1}{x}S_2' \end{aligned}$$

Where $S_2'' \sim x^{\frac{-5}{2}}$, Hence

$$\begin{aligned} 2S_0'S_2' &\ggg S_2'' \\ x^{\frac{-1}{2}}\left(x^{\frac{-3}{2}}\right) &\ggg x^{\frac{-5}{2}} \\ x^{-2} &\ggg x^{\frac{-5}{2}} \end{aligned}$$

Yes.

$$2S_0'S_2' \ggg 2S_1'S_2'$$
$$x^{-2} \ggg \left(\frac{1}{x}\right)\left(x^{\frac{-3}{2}}\right)$$
$$x^{-2} \ggg x^{\frac{-5}{2}}$$

Yes

$$2S_0'S_2' \ggg \left[S_2'\right]^2$$
$$x^{-2} \ggg \left(x^{\frac{-3}{2}}\right)^2$$
$$x^{-2} \ggg x^{-3}$$

Yes

$$2S_0'S_2' \ggg \frac{1}{x}S_2'$$
$$x^{-2} \ggg \frac{1}{x}x^{\frac{-3}{2}}$$
$$x^{-2} \ggg x^{\frac{-5}{2}}$$

Yes. All assumptions are verified. Therefore we can g ahead and solve for S_2 using (2)

$$2S_0'S_2' \sim -\frac{1}{16}\frac{1}{x^2}$$
$$S_2' \sim -\frac{1}{32}\frac{1}{x^2}\frac{1}{S_0'}$$
$$\sim -\frac{1}{32}\frac{1}{x^2}\frac{1}{x^{\frac{-1}{2}}}$$
$$\sim -\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}$$

Hence

$$\boxed{S_2 \sim \frac{1}{16}\frac{1}{\sqrt{x}}}$$

The leading behavior now is

$$y(x) \sim \exp\left(S_0 + S_1 + S_2\right)$$
$$\sim \exp\left(2x^{\frac{1}{2}} - \frac{1}{4}\ln x + c + \frac{1}{16}\frac{1}{\sqrt{x}}\right)$$

Now we will find S_3. From

$$x\left(S'' + (S')^2\right) + S' = 1$$

Replacing S by $S_0+S_1+S_2+S_3$ in the above gives

$$(S_0+S_1+S_2+S_3)''+\left[(S_0+S_1+S_2+S_3)'\right]^2+\frac{1}{x}(S_0+S_1+S_2+S_3)'\sim\frac{1}{x}$$

$$\left\{S_0''+S_1''+S_2''+S_3''\right\}+\left[\left(S_0'+S_1'+S_2'+S_3'\right)\right]^2+\frac{1}{x}\left(S_0'+S_1'+S_2'+S_3'\right)\sim\frac{1}{x}$$

Hence

$$\left\{S_0''+S_1''+S_2''+S_3''\right\}+\left\{\left[S_0'\right]^2+2S_0'S_1'+2S_0'S_2'+2S_1'S_2'+\left[S_1'\right]^2+\left[S_2'\right]^2+2S_0'S_3'+2S_1'S_3'+2S_2'S_3'+\left[S_3'\right]^2\right\}+\frac{1}{x}\left\{S_0'+S_1'+S_2'+S_3'\right\}\sim\frac{1}{x}$$

Moving all known quantities to the RHS gives

$$\left\{S_3''\right\}+\left\{2S_0'S_3'+2S_1'S_3'+2S_2'S_3'+\left[S_3'\right]^2\right\}+\frac{1}{x}\left\{S_3'\right\}$$
$$\sim\frac{1}{x}-S_0''-S_1''-S_2''-\left[S_0'\right]^2-2S_0'S_1'-2S_0'S_2'-2S_1'S_2'-\left[S_1'\right]^2-\left[S_2'\right]^2-\frac{1}{x}S_0'-\frac{1}{x}S_1'-\frac{1}{x}S_2' \tag{3}$$

Now we will simplify the RHS, since it is all known. Using

$$\begin{aligned}
S_0'(x)&=x^{\frac{-1}{2}}\\
\left[S_0'\right]^2&=x^{-1}\\
S_0''&=\frac{-1}{2}x^{-\frac{3}{2}}\\
S_1'(x)&=-\frac{1}{4}\frac{1}{x}\\
\left[S_1'(x)\right]^2&=\frac{1}{16}\frac{1}{x^2}\\
S_1''(x)&=\frac{1}{4x^2}\\
S_2'(x)&=-\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}\\
\left[S_2'(x)\right]^2&=\frac{1}{1024x^3}\\
S_2''(x)&=\frac{3}{64}\frac{1}{x^{\frac{5}{2}}}\\
2S_0'S_1'&=2\left(x^{\frac{-1}{2}}\right)\left(-\frac{1}{4}\frac{1}{x}\right)=-\frac{1}{2}\frac{1}{x^{\frac{3}{2}}}\\
2S_0'S_2'&=2\left(x^{\frac{-1}{2}}\right)\left(-\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}\right)=-\frac{1}{16x^2}\\
2S_1'S_2'&=2\left(-\frac{1}{4}\frac{1}{x}\right)\left(-\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}\right)=\frac{1}{64x^{\frac{5}{2}}}
\end{aligned}$$

Hence (3) becomes

$$\left\{S_3''\right\} + \left\{2S_0'S_3' + 2S_1'S_3' + 2S_2'S_3' + \left[S_3'\right]^2\right\} + \frac{1}{x}\left\{S_3'\right\} \sim$$
$$\frac{1}{x} + \frac{1}{2x^{\frac{3}{2}}} - \frac{1}{4x^2} - \frac{3}{64}\frac{1}{x^{\frac{5}{2}}} - \frac{1}{x} + \frac{1}{2}\frac{1}{x^{\frac{3}{2}}} + \frac{1}{16x^2} - \frac{1}{64x^{\frac{5}{2}}} - \frac{1}{16}\frac{1}{x^2} - \frac{1}{1024x^3} - \frac{1}{x^{\frac{3}{2}}} + \frac{1}{4}\frac{1}{x^2} + \frac{1}{32}\frac{1}{x^{\frac{5}{2}}}$$

Simplifying gives

$$\left\{S_3''\right\} + \left\{2S_0'S_3' + 2S_1'S_3' + 2S_2'S_3' + \left[S_3'\right]^2\right\} + \frac{1}{x}\left\{S_3'\right\} \sim -\left(\frac{32}{1024x^{\frac{5}{2}}} + \frac{1}{1024x^3}\right)$$

Let us now assume that

$$S_0'S_3' \ggg S_1'S_3'$$
$$S_0'S_3' \ggg S_2'S_3'$$
$$S_0'S_3' \ggg \left[S_3'\right]^2$$
$$S_0'S_3' \ggg \frac{1}{x}\left\{S_3'\right\}$$
$$S_0'S_3' \ggg S_3''$$

Therefore, we end up with the balance

$$2S_0'S_3' \sim -\left(\frac{32}{1024x^{\frac{5}{2}}} + \frac{1}{1024x^3}\right)$$
$$S_3' \sim -\left(\frac{32}{1024x^{\frac{5}{2}}S_0'} + \frac{1}{1024x^3S_0'}\right)$$
$$\sim -\left(\frac{32}{1024x^{\frac{5}{2}}\left(x^{\frac{-1}{2}}\right)} + \frac{1}{1024x^3\left(x^{\frac{-1}{2}}\right)}\right)$$
$$\sim -\left(\frac{1}{32x^2} + \frac{1}{1024x^{\frac{5}{2}}}\right)$$

Hence

$$\boxed{S_3 \sim \frac{1}{1024}\left(\frac{2}{32x^2} + \frac{32}{x}\right)}$$

Where constant of integration was ignored. Let us now verify the assumptions made

$$S_0'S_3' \ggg S_1'S_3'$$
$$S_0' \ggg S_1'$$

Yes.

$$S_0'S_3' \ggg S_2'S_3'$$
$$S_0' \ggg S_2'$$

Yes

$$S_0' S_3' \ggg \left[S_3'\right]^2$$
$$S_0' \ggg S_3'$$

Yes.

$$S_0' S_3' \ggg \frac{1}{x}\left\{S_3'\right\}$$
$$S_0' \ggg \frac{1}{x}$$
$$x^{\frac{-1}{2}} \ggg \frac{1}{x}$$

Yes, as $x \to \infty$, and finally

$$S_0' S_3' \ggg S_3''$$
$$x^{\frac{-1}{2}}\left(\frac{1}{32x^2} + \frac{1}{1024x^{\frac{5}{2}}}\right) \ggg \left(\frac{5}{2048x^{\frac{7}{2}}} + \frac{1}{16x^3}\right)$$
$$\frac{\left(32\sqrt{x}+1\right)}{1024x^3} \ggg \frac{128\sqrt{x}+5}{2048x^{\frac{7}{2}}}$$

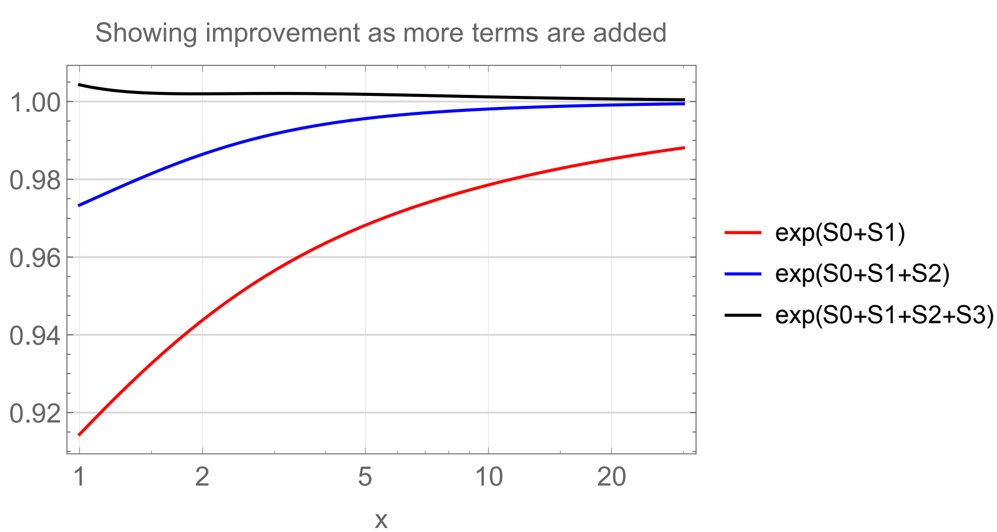
Yes, as $x \to \infty$. All assumptions verified. The leading behavior now is

$$\begin{aligned} y(x) &\sim \exp\left(S_0 + S_1 + S_2 + S_3\right) \\ &\sim \exp\left(2x^{\frac{1}{2}} - \frac{1}{4}\ln x + c + \frac{1}{16}\frac{1}{\sqrt{x}} + \frac{1}{1024}\left(\frac{2}{32x^2} + \frac{32}{x}\right)\right) \\ &\sim cx^{\frac{-1}{4}}\exp\left(2x^{\frac{1}{2}} + \frac{1}{16}\frac{1}{\sqrt{x}} + \frac{1}{1024}\left(\frac{2}{32x^2} + \frac{32}{x}\right)\right) \end{aligned}$$

Now we will show how adding more terms to leading behavior improved the $y(x)$ solution for large x. When plotting the solutions, we see that $\frac{\exp(S_0+S_1+S_2+S_3)}{y(x)}$ approached the ratio 1 sooner than $\frac{\exp(S_0+S_1+S_2)}{y(x)}$ and this in turn approached the ratio 1 sooner than just using $\frac{\exp(S_0+S_1)}{y(x)}$. So the effect of adding more terms, is that the solution becomes more accurate for larger range of x values. Below is the code used and the plot generated.

```
ClearAll[y, x];
s0[x_] := (1/(2 Pi^(1/2)) Exp[2 x^(1/2)]) / y[x, 300];
s0s1[x_] := (1/(2 Pi^(1/2)) Exp[2 x^(1/2) - 1/4 Log[x]]) / y[x, 300];
s0s1s2[x_] := (1/(2 Pi^(1/2)) Exp[2 x^(1/2) - 1/4 Log[x] + 1/16 1/Sqrt[x]]) / y[x, 300];
s0s1s2s3[x_] := (1/(2 Pi^(1/2)) Exp[2 x^(1/2) - 1/4 Log[x] + 1/16 1/Sqrt[x] + 1/1024 (2/(32 x^2) + 32/x)]) / y[x, 300];
y[x_, max_] := Sum[ x^n / (Factorial[n] ^2), {n, 0, max}];
LogLinearPlot[Evaluate[{s0s1[x], s0s1s2[x], s0s1s2s3[x]}], {x, 1, 30},
 PlotRange → All, Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
 PlotLegends → {"exp(S0+S1)", "exp(S0+S1+S2)", "exp(S0+S1+S2+S3)"},
 FrameLabel → {{None, None}, {"x", "Showing improvement as more terms are added"}},
 PlotStyle → {Red, Blue, Black}, BaseStyle → 14]
```

Out[59]=



0.7 problem 3.49(c)

Problem Find the leading behavior as $x \to \infty$ of the general solution of $y'' + xy = x^5$

Solution This is non-homogenous ODE. We solve this by first finding the homogenous solution (asymptotic solution) and then finding particular solution. Hence we start with

$$y_h'' + xy_h = 0$$

$x = \infty$ is ISP point. Therefore, we assume $y_h(x) = e^{S(x)}$ and obtain

$$S'' + (S')^2 + x = 0 \tag{1}$$

Let

$$S(x) = S_0 + S_1 + \cdots$$

Therefore (1) becomes

$$\left(S_0'' + S_1'' + \cdots\right) + \left(S_0' + S_1' + \cdots\right)^2 = -x$$
$$\left(S_0'' + S_1'' + \cdots\right) + \left(\left[S_0'\right]^2 + 2S_0'S_1' + \left[S_1'\right]^2 + \cdots\right) = -x \tag{2}$$

Assuming $\left[S_0'\right]^2 \ggg S_0''$ we obtain

$$\left[S_0'\right]^2 \sim -x$$
$$S_0' \sim \omega\sqrt{x}$$

Where $\omega = \pm i$

Verification

$$\left[S_0'\right]^2 \ggg S_0''$$
$$x \ggg \frac{1}{2}\frac{1}{\sqrt{x}}$$

Yes, as $x \to \infty$. Hence

$$S_0 \sim \frac{3}{2}\omega x^{\frac{3}{2}}$$

Now we will find S_1. From (2), and moving all known terms to RHS

$$\left(S_1'' + \cdots\right) + \left(2S_0'S_1' + \left[S_1'\right]^2 + \cdots\right) \sim -x - S_0'' - \left[S_0'\right]^2 \tag{3}$$

Assuming

$$2S_0'S_1' \ggg S_1''$$
$$2S_0'S_1' \ggg \left[S_1'\right]^2$$

Then (3) becomes (where $S_0' \sim \omega\sqrt{x}, \left[S_0'\right]^2 \sim -x, S_0'' \sim \frac{1}{2}\omega\frac{1}{\sqrt{x}}$)

$$\begin{aligned} 2S_0'S_1' &\sim -x - S_0'' - \left[S_0'\right]^2 \\ S_1' &\sim \frac{-x - S_0'' - \left[S_0'\right]^2}{2S_0'} \\ &\sim \frac{-x - \frac{1}{2}\omega\frac{1}{\sqrt{x}} - (-x)}{2\omega\sqrt{x}} \\ &\sim -\frac{1}{4x} \end{aligned}$$

<u>Verification</u> (where $S_1'' \sim \frac{1}{4}\frac{1}{x^2}$)

$$\begin{aligned} 2S_0'S_1' &\ggg S_1'' \\ \sqrt{x}\left(\frac{1}{4x}\right) &\ggg \frac{1}{4}\frac{1}{x^2} \\ \frac{1}{x^{\frac{1}{2}}} &\ggg \frac{1}{x^2} \end{aligned}$$

Yes, as $x \to \infty$

$$\begin{aligned} 2S_0'S_1' &\ggg \left[S_1'\right]^2 \\ \frac{1}{x^{\frac{1}{2}}} &\ggg \left(\frac{1}{4x}\right)^2 \\ \frac{1}{x^{\frac{1}{2}}} &\ggg \frac{1}{16x^2} \end{aligned}$$

Yes, as $x \to \infty$. All validated. We solve for S_1

$$\begin{aligned} S_1' &\sim -\frac{1}{4x} \\ S_1 &\sim -\frac{1}{4}\ln x + c \end{aligned}$$

y_h is found. It is given by

$$\begin{aligned} y_h(x) &\sim \exp\left(S_0(x) + S_1(x)\right) \\ &\sim \exp\left(\frac{3}{2}\omega x^{\frac{3}{2}} - \frac{1}{4}\ln x + c\right) \\ &\sim c x^{\frac{-1}{4}} \exp\left(\frac{3}{2}\omega x^{\frac{3}{2}}\right) \end{aligned}$$

Now that we have found y_h, we go back and look at

$$y'' + xy = x^5$$

And consider two cases (a) $y'' \sim x^5$ (b) $xy \sim x^5$. The case of $y'' \sim xy$ was covered above. This is what we did to find $y_h(x)$.

case (a)

$$y_p'' \sim x^5$$
$$y_p' \sim \frac{1}{5}x^4$$
$$y_p \sim \frac{1}{20}x^3$$

Where constants of integration are ignored since subdominant for $x \to \infty$. Now we check if this case is valid.

$$xy_p \lll x^5$$
$$x\frac{1}{20}x^3 \lll x^5$$
$$x^4 \lll x^5$$

No. Therefore case (a) did not work out. We try case (b) now

$$xy_p \sim x^5$$
$$y_p \sim x^4$$

Now we check is this case is valid.

$$y_p'' \lll x^5$$
$$12x^2 \lll x^5$$

Yes. Therefore, we found

$$y_p \sim x^4$$

Hence the complete asymptotic solution is

$$\begin{aligned} y(x) &\sim y_h(x) + y_p(x) \\ &\sim cx^{\frac{-1}{4}} \exp\left(\frac{3}{2}\omega x^{\frac{3}{2}}\right) + x^4 \end{aligned}$$