Problem Statement: How many terms are needed in the Taylor series solution to $y''' = x^3 y$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$ are needed to evaluate $\int\limits_0^\infty y(x)dx$ correct to 3 decimal places?

Solution: Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n a_n (n-1) x^{n-1}$$

$$y''' = \sum_{n=0}^{\infty} n(n-1)(n-2) a_n x^{n-3} = x^3 y = x^3 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+3}.$$

Shift indices by letting $n \to (n-6)$ in the RHS:

$$\sum_{n=0}^{\infty} a_n x^{n+3} \to \sum_{n=6}^{\infty} a_{n-6} x^{n-3}$$

$$\sum_{n=0}^{\infty} n(n-1)(n-2) a_n x^{n-3} = 0 \cdot x^{-3} + 0 \cdot x^{-2} + 0 \cdot x^{-1} + 3 \cdot 2 \cdot 1 \cdot a_3 x^0 + 4 \cdot 3 \cdot 2 \cdot a_4 x^1$$

$$+ \; 5 \cdot 4 \cdot 3 \cdot a_5 x^2 + \sum_{n=6}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

From here,

$$a_3 = 0, \quad a_4 = 0, \quad a_5 = 0$$

$$x^{n-3}\{n(n-1)(n-2)a_n - a_{n-6}\} = 0 \;\text{ for }\; n = 6, 7, 8, ...$$

The recurrence relation becomes:

$$a_n = \frac{1}{n(n-1)(n-2)} a_{n-6} \quad n \geq 6$$

Solution to D.E. becomes:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \sum_{n=6}^{\infty} [\frac{1}{n(n-1)(n-2)} a_{n-6} x^n]$$

$$y(0) = a_0 = 1 ==> a_0 = 1$$

$$y'(0) = a_1 = 0 ==> a_1 = 0$$

$$y''(0) = 2 \cdot a_2 = 0 ==> a_2 = 0$$

$$y(x) = 1 + \sum_{n=6}^{\infty} \frac{1}{n(n-1)(n-2)} a_{n-6} x^n$$

$$\int_0^1 y(x)dx = [x + \sum_{n=6}^{\infty} \frac{1}{(n+1)n(n-1)(n-2)} a_{n-6} x^{n+1}]_0^1$$

$$= 1 + \sum_{n=6}^{\infty} \frac{1}{(n+1)(n)(n-1)(n-2)} a_{n-6}$$

$$= 1 + \frac{1}{7 \cdot 6 \cdot 5 \cdot 4}(1) + \frac{1}{13 \cdot 12 \cdot 11 \cdot 10}(\frac{1}{6 \cdot \cdot 4}(1)) + ...$$

$$= 1 + \frac{1}{840} + \frac{1}{2,059,200} + ...$$

$$\int_0^1 y(x)dx \simeq 1 + 0.00119 + 4.856 \times 10^{-7}$$

So the 2nd non-zero term is required. All other terms are below the specified tolerance of 0.001.