$\boxed{\text{BO } 3.4 \text{ d}}$

$$x^2 y'' = \exp[1/x]\, y$$

consider $x_0 = 0$: $y'' - \frac{\exp[1/x]}{x^2} y = 0$

$x^2 q(x) = -\exp[1/x]$ not analytic at $x_0 = 0$

$\Longrightarrow$ $x_0 = 0$ is an irregular singular point

consider $x \to \infty$ with $t = \frac{1}{x}$:

$$t^4 \frac{d^2y}{dt^2} + 2t^3 \frac{dy}{dt} - \exp(t)\, t^2 y = 0$$

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} - \frac{\exp(t)}{t^2} y = 0$$

$t\,p(t) = 2$, $t^2 q(t) = -\exp(t)$

analytic at $t_0 = 0$ $\Longrightarrow$

$x = \infty$ is a regular singular point of the original equation

$\boxed{\text{BO } 3.4\text{ e}}$

$(\tan x)\, y' = y \;;\quad y' - \frac{\cos x}{\sin x}\, y = 0$

$x_0 = 0$: does $xp(x)$ have a Taylor series expansion about $x_0 = 0$?

$$x\,p(x) = \frac{-x\cos x}{\sin x}$$

$$= \frac{-x\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right\}}{\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right\}}$$

$$= \frac{-x\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right\}}{x\left\{1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right\}}$$

$$= -\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right\}\left\{1 + \frac{x^2}{3!} + \cdots\right\}$$

$$= -\left\{1 - \frac{x^2}{2!} + O(x^4)\right\}\left\{1 + \frac{x^2}{3!} + O(x^4)\right\}$$

converges for $|x| < 1$ $\Rightarrow$ $x_0 = 0$ is a regular singular point

Consider $x \to \infty$ with $t = \frac{1}{x}$:

$$-t^2 \frac{dy}{dt} - \frac{\cos(1/t)}{\sin(1/t)} y = 0$$

$$\frac{dy}{dt} + \frac{1}{t^2} \frac{\cos(1/t)}{\sin(1/t)} y = 0$$

$t=0$ is irregular singular $\Rightarrow$ $x \to \infty$ is an irregular singular point of the original equation.