1 Closed form of Problem 3.24e

In problem 3.24e, we find Frobenius series solutions

$$y_\alpha(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha} \tag{1}$$

for $\alpha = 0, \frac{3}{2}$, and recurrence relation on the coefficients:

$$a_{n+3} = \frac{-a_n}{(n+\alpha+3)(2n+2\alpha+3)}.$$

For each α, $a_1 = a_2 = 0$, so only the a_{3k} survive. Rewriting the recurrence relation:

$$a_{3k+3} = \frac{-a_{3k}}{(3k+3+\alpha)(6k+3+2\alpha)} = \frac{-a_{3k}}{9(k+1+\alpha/3)(2k+1+2\alpha/3)}.$$

To identify a closed-form sum, we need to find the general form of the coefficients a_{3k}. Towards this end, consider the case $\alpha = 0$. Now

$$a_{3k} = \frac{-1/9}{k(2k-1)} a_{3k-3} = \frac{(-1/9)^2}{k(k-1)\cdot(2k-1)(2k-3)} a_{3k-6} = \cdots = \frac{(-1/9)^k}{k!\cdot(2k-1)!!} a_0.$$

Now note that

$$(2k-1)!! = \frac{(2k)!}{(2k)!!} = \frac{(2k)!}{2^k \cdot k!} \implies k!\cdot(2k-1)!! = 2^{-k}(2k)!,$$

so

$$a_{3k} = \frac{(-1/9)^k}{2^{-k}(2k)!} a_0 = \frac{(-2/9)^k}{(2k)!} a_0.$$

Therefore,

$$y_0(x) = a_0 \sum_{k=0}^{\infty} \frac{(-2/9)^k}{(2k)!} x^{3k} = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x^{3/2}\sqrt{2}/3)^{2k} =: a_0 \sum_{k=0}^{\infty} \frac{-z^{2k}}{(2k)!},$$

where $z := x^{3/2}\sqrt{2}/3$. We recognize the remaining series as the Taylor series for $\cos z$, so

$$y_0(x) = a_0 \cos z(x) = a_0 \cos\left(\frac{\sqrt{2}}{3} x^{3/2}\right).$$

For $\alpha = \frac{3}{2}$,

$$a_{3k+3} = \frac{-a_{3k}}{9(k+3/2)(2k+2)} = \frac{-a_{3k}}{9(k+1)(2k+3)}.$$

Following the same argument as before,

$$a_{3k} = \frac{(-1/9)^k}{k!\cdot(2k+1)!!} a_0.$$

This is the same as before, but divided by $2k+1$, so

$$y_{3/2}(x) = a_0 \sum_{k=0}^{\infty} \frac{(-2/9)^k}{(2k+1)!} x^{3k} = a_0 \sin\left(\frac{\sqrt{2}}{3} x^{3/2}\right).$$

Therefore, the general solution is

$$y(x) = y_0(x) + y_{3/2}(x) = c_1 \cos\left(\frac{\sqrt{2}}{3} x^{3/2}\right) + c_2 \sin\left(\frac{\sqrt{2}}{3} x^{3/2}\right).$$

2 Closed form of Problem 3.24f

Problem 3.24f asks us to solve $(\sin x)y'' - 2(\cos x)y' - (\sin x)y = 0$. Dividing by $\sin x$,

$$y'' - 2(\cot x)y' - y = 0,$$

so $x = 0$ is a regular singular point ($\cot x$ is not analytic at $x = 0$, but $x \cot x$ is). Thus, we can look for a Frobenius series solution.

Expanding $\sin x$ and $\cos x$:

$$0 = \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right) \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2} - 2\left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}\right) \sum_{n=0}^{\infty} (n+\alpha)a_n x^{n+\alpha-1}$$
$$- \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right) \sum_{n=0}^{\infty} a_n x^{n+\alpha}.$$

The lowest order is $x^{\alpha-1}$:

$$0 = \alpha(\alpha-1)a_0 - 2\alpha a_0 = \alpha(\alpha-3)a_0,$$

so $\alpha = 0, 3$. We are in case II(b) on page 72.

Let $\alpha = 0$. Then

$$0 = \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right) \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2\left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}\right) \sum_{n=0}^{\infty} n a_n x^{n-1}$$
$$- \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right) \sum_{n=0}^{\infty} a_n x^{n}.$$

We solve order-by-order:

$$x^0 : 0 = -2a_1 \implies a_1 = 0,$$
$$x^1 : 0 = 2a_2 - 2\cdot 2a_2 - a_0 \implies a_2 = -\frac{a_0}{2},$$
$$x^{2k} : 0 = \sum_{\ell=0}^{k} (-1)^{k-\ell} \frac{2\ell(2\ell+1)}{(2k-2\ell+1)!} a_{2\ell+1} - 2\sum_{\ell=0}^{k} (-1)^{k-\ell} \frac{2\ell+1}{(2k-2\ell)!} a_{2\ell+1} - \sum_{\ell=0}^{k-1} \frac{(-1)^{k-\ell-1}}{(2k-2\ell-1)!} a_{2\ell+1},$$
$$x^{2k+1} : 0 = \sum_{\ell=0}^{k+1} (-1)^{k-\ell+1} \frac{2\ell(2\ell-1)}{(2k-2\ell+3)!} a_{2\ell} - 2\sum_{\ell=0}^{k+1} (-1)^{k-\ell+1} \frac{2\ell}{(2k-2\ell+2)!} a_{2\ell} - \sum_{\ell=0}^{k} \frac{(-1)^{k-\ell}}{(2k-2\ell+1)!} a_{2\ell},$$

where $k \geq 1$. From the x^{2k} terms, we see that a_{2n+1} depends only on $a_1, a_3, ..., a_{2n-1}$. As $a_1 = 0$, it is a simple (strong) induction to show that each $a_{2n+1} = 0$.

We use the x^{2k+1} terms for $k = 1$ to see that

$$0 = \left(0a_0 - \frac{2}{6}a_2 + \frac{12}{1}a_4\right) - 2\left(0a_0 - \frac{2}{2}a_2 + \frac{4}{1}a_4\right) - \left(\frac{-1}{6}a_0 + \frac{1}{1}a_2\right) \implies a_4 = -\frac{1}{4}\left(\frac{1}{6}a_0 - \frac{2}{3}a_2\right).$$

As $a_2 = -\frac{1}{2}a_0$, we see that $a_4 = \frac{1}{24}a_0$. We notice a pattern that these coefficients are starting to look like those in the Taylor series expansion of $y_0(x) = a_0 \cos x$, which we now verify by plugging into the ODE:

$$(\sin x)y_0''(x) - 2(\cos x)y_0'(x) - (\sin x)y_0(x) = -a_0 \sin x \cos x + 2a_0 \cos x \sin x - a_0 \sin x \cos x = 0.$$

To find the other solution, we use reduction of order: look for a solution of the form $y_3(x) = v(x)\cos x$ (we keep the $\alpha = 3$ subscript to remind us that we expect the Taylor series of our solution to start at the x^3-term). Plugging into the ODE:

$$(\sin x)(v''(x)\cos x - 2v'(x)\sin x - v(x)\cos x) - (2\cos x)(v'(x)\cos x - v(x)\sin x) - (\sin x)(v(x)\cos x) = 0.$$

Simplifying,

$$(\sin x\cos x)v''(x) - 2v'(x) = 0 \implies v'(x) = c_1\tan^2 x \implies v(x) = c_1(\tan x - x) + c_0.$$

We set $c_0 = 0$, since this corresponds to $y_0(x)$. Then $y_3(x) = c_1(\tan x - x)\cos x = c_1(\sin x - x\cos x)$.

The full solution is therefore

$$y(x) = c_0\cos x + c_1(\sin x - x\cos x).$$