# HW6 EMA 471 Intermediate Problem Solving for Engineers

Spring 2016
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December 30, 2019

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## 0.1 Problem 1

(1) (10 pts) Consider the following elliptic partial differential equation:

$$2\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - 5u = -5,$$

Solve this problem over the unit square, $0 \le x \le 1,\ 0 \le y \le 1$, subject to $u = 0$ on all boundaries. Approximate the first derivative by a centrally-differenced expression. You may use either a direct solution (simultaneous system of equations) or an iterative solution. Find the largest value of $u$ within the domain.

Figure 1: problem 1 description

We need to solve $2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) - 5u = 5$ on the unit square. The discretized algebraic equation resulting from approximating this PDE using standard 5 point Laplacian and centered difference for the first derivatives is given by

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{1}{h^2}\left(U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j}\right) \\
\frac{\partial u}{\partial x} &= \frac{1}{2h}\left(U_{i+1,j} - U_{i-1,j}\right) \\
\frac{\partial u}{\partial y} &= \frac{1}{2h}\left(U_{i,j+1} - U_{i,j-1}\right)
\end{aligned}$$

which has local truncation error $O\left(h^2\right)$. Therefore the PDE becomes

$$\begin{aligned}
2\left(\frac{1}{h^2}\left(U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j}\right)\right) - \frac{1}{2h}\left(U_{i+1,j} - U_{i-1,j} + U_{i,j+1} - U_{i,j-1}\right) - 5U_{i,j} &= -5 \\
U_{i-1,j}\left(\frac{2}{h^2} + \frac{1}{2h}\right) + U_{i+1,j}\left(\frac{2}{h^2} - \frac{1}{2h}\right) + U_{i,j-1}\left(\frac{2}{h^2} + \frac{1}{2h}\right) + U_{i,j+1}\left(\frac{2}{h^2} - \frac{1}{2h}\right) + U_{i,j}\left(\frac{-8}{h^2} - 5\right) &= -5
\end{aligned}$$

### 0.1.1 Numbering system and grid updating

The numbering system used for the grid is the following. The indices for the unknown $U_{i,j}$ are numbered row wise, left to right, bottom to top. This follows the standard Cartesian coordinates system.

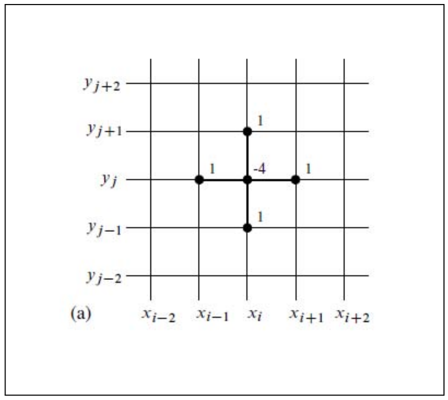



Figure 2: problem 1 grid

Lower case $n$ is used to indicate the number of unknowns along one dimension, and upper case $N$ is used to indicate the total number of unknowns. For example, if we use a grid with 5 points on each side, we obtain

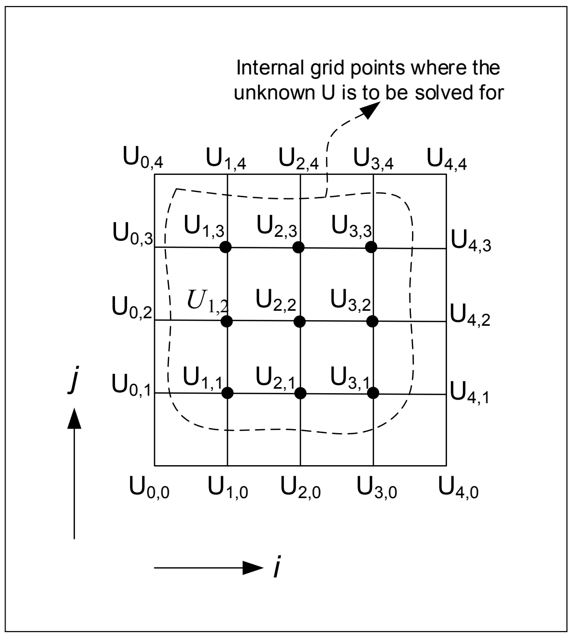



Figure 3: Example for $N = 3$

In the diagram above, $n = 3$ is the number of unknowns on each one row or each column, and since there are 3 internal rows, there will be 9 unknowns in total, all are located on internal grid points. There are a total of 25 grid points, 16 of which are on the boundaries (given as zero) and 9 are internal (which we need to solve for).

### 0.1.2 Structure of $Au = f$

We will derive the first few rows of the $A$ matrix to see the structure. For $i = 1, j = 1$

$$U_{0,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{2,1}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,0}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{1,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,1}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i = 2, j = 1$

$$U_{1,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,1}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,0}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{2,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,1}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=3,j=1$

$$U_{2,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{4,1}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{3,0}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{3,1}\left(\frac{-8}{h^2}-5\right)=-5$$

The above completes one row (it is the bottom row in the grid) but it goes as the first row in the $A$ matrix. Since boundary conditions are zero, then we can eliminate $U_{0,1},U_{4,1},U_{1,0},U_{2,0},U_{3,0}$ from the first row. Hence the above becomes

For $i=1,j=1$

$$U_{2,1}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,1}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=2,j=1$

$$U_{1,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,1}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,1}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=3,j=1$

$$U_{2,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{3,1}\left(\frac{-8}{h^2}-5\right)=-5$$

Looking at the second row in the grid (the second row from the bottom up), we find

For $i=1,J=2$

$$U_{0,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{2,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{1,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,2}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=2,j=2$

$$U_{1,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{2,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,2}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=3,j=2$

$$U_{2,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{4,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{3,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{3,2}\left(\frac{-8}{h^2}-5\right)=-5$$

Removing boundary conditions entries the above becomes

For $i=1,J=2$

$$U_{2,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{1,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,2}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=2, j=2$

$$U_{1,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,2}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{2,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,2}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=3, j=2$

$$U_{2,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,1}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{3,2}\left(\frac{-8}{h^2}-5\right)=-5$$

And finally for the third row from the bottom up, (this will be the last row in the $A$ matrix) we have

For $i=1, J=3$

$$U_{0,3}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{2,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{1,4}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,3}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=2, j=3$

$$U_{1,3}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{2,4}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,3}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=3, j=3$

$$U_{2,3}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{4,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{3,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,4}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{3,3}\left(\frac{-8}{h^2}-5\right)=-5$$

Removing boundary conditions entries the above becomes

For $i=1, J=3$

$$U_{2,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{1,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{1,3}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=2, j=3$

$$U_{1,3}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,3}\left(\frac{2}{h^2}-\frac{1}{2h}\right)+U_{2,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{2,3}\left(\frac{-8}{h^2}-5\right)=-5$$

For $i=3, j=3$

$$U_{2,3}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,2}\left(\frac{2}{h^2}+\frac{1}{2h}\right)+U_{3,3}\left(\frac{-8}{h^2}-5\right)=-5$$

Therefor the $Au=f$ structure is the following

$$\begin{pmatrix} \left(\frac{-8}{h^2}-5\right) & \left(\frac{2}{h^2}-\frac{1}{2h}\right) & 0 & \left(\frac{2}{h^2}-\frac{1}{2h}\right) & 0 & 0 & 0 & 0 & 0 \\ \left(\frac{2}{h^2}+\frac{1}{2h}\right) & \left(\frac{-8}{h^2}-5\right) & \left(\frac{2}{h^2}-\frac{1}{2h}\right) & 0 & \left(\frac{2}{h^2}-\frac{1}{2h}\right) & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{2}{h^2}+\frac{1}{2h}\right) & \left(\frac{-8}{h^2}-5\right) & 0 & 0 & \left(\frac{2}{h^2}-\frac{1}{2h}\right) & 0 & 0 & 0 \\ \left(\frac{2}{h^2}+\frac{1}{2h}\right) & 0 & 0 & \left(\frac{-8}{h^2}-5\right) & \left(\frac{2}{h^2}-\frac{1}{2h}\right) & 0 & \left(\frac{2}{h^2}-\frac{1}{2h}\right) & 0 & 0 \\ 0 & \left(\frac{2}{h^2}+\frac{1}{2h}\right) & 0 & \left(\frac{2}{h^2}+\frac{1}{2h}\right) & \left(\frac{-8}{h^2}-5\right) & \left(\frac{2}{h^2}-\frac{1}{2h}\right) & 0 & \left(\frac{2}{h^2}-\frac{1}{2h}\right) & 0 \\ 0 & 0 & \left(\frac{2}{h^2}+\frac{1}{2h}\right) & 0 & \left(\frac{2}{h^2}+\frac{1}{2h}\right) & \left(\frac{-8}{h^2}-5\right) & 0 & 0 & \left(\frac{2}{h^2}-\frac{1}{2h}\right) \\ 0 & 0 & 0 & \left(\frac{2}{h^2}+\frac{1}{2h}\right) & 0 & 0 & \left(\frac{-8}{h^2}-5\right) & \left(\frac{2}{h^2}-\frac{1}{2h}\right) & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{2}{h^2}+\frac{1}{2h}\right) & 0 & \left(\frac{2}{h^2}+\frac{1}{2h}\right) & \left(\frac{-8}{h^2}-5\right) & \left(\frac{2}{h^2}-\frac{1}{2h}\right) \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{2}{h^2}+\frac{1}{2h}\right) & 0 & \left(\frac{2}{h^2}+\frac{1}{2h}\right) & \left(\frac{-8}{h^2}-5\right) \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \end{pmatrix}$$

To simplify, let $c_1 = \left(\frac{-8}{h^2} - 5\right), c_2 = \left(\frac{2}{h^2} - \frac{1}{2h}\right), c_3 = \left(\frac{2}{h^2} + \frac{1}{2h}\right)$) since these repeat everywhere.

$$\begin{pmatrix} c_1 & c_2 & 0 & c_2 & 0 & 0 & 0 & 0 & 0 \\ c_3 & c_1 & c_2 & 0 & c_2 & 0 & 0 & 0 & 0 \\ 0 & c_3 & c_1 & 0 & 0 & c_2 & 0 & 0 & 0 \\ c_3 & 0 & 0 & c_1 & c_2 & 0 & c_2 & 0 & 0 \\ 0 & c_3 & 0 & c_3 & c_1 & c_2 & 0 & c_2 & 0 \\ 0 & 0 & c_3 & 0 & c_3 & c_1 & 0 & 0 & c_2 \\ 0 & 0 & 0 & c_3 & 0 & 0 & c_1 & c_2 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & c_3 & c_1 & c_2 \\ 0 & 0 & 0 & 0 & 0 & c_3 & 0 & c_3 & c_1 \end{pmatrix} \begin{pmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \\ -5 \end{pmatrix}$$

So now we see the structure of $Au = f$. For the number of unknowns being $n$ in one row, we have the following layout

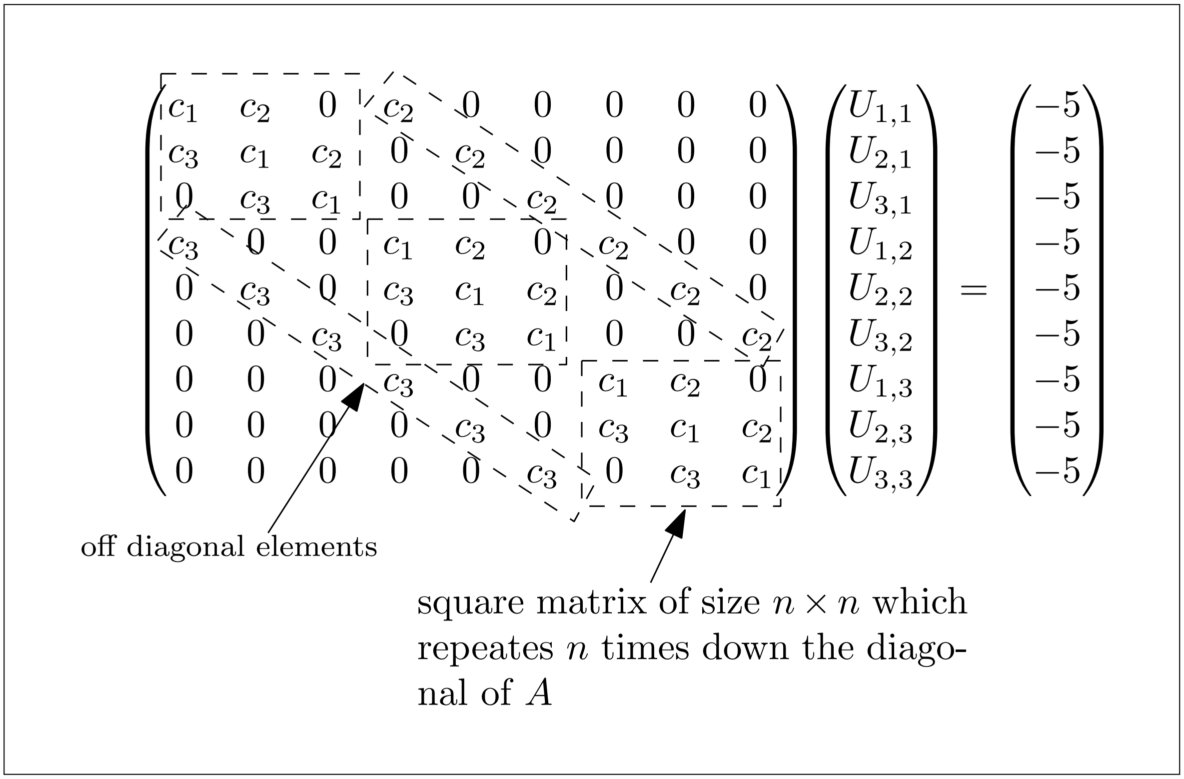



Figure 4: $A$ matrix structure

The program `nma_EMA_471_HW6_problem_1.m` solves this $Au = f$ using the direct method and plots the solution. Maximum value found was

$$\boxed{u_{\text{max}} = 0.1609}$$

Here is plot of the solution

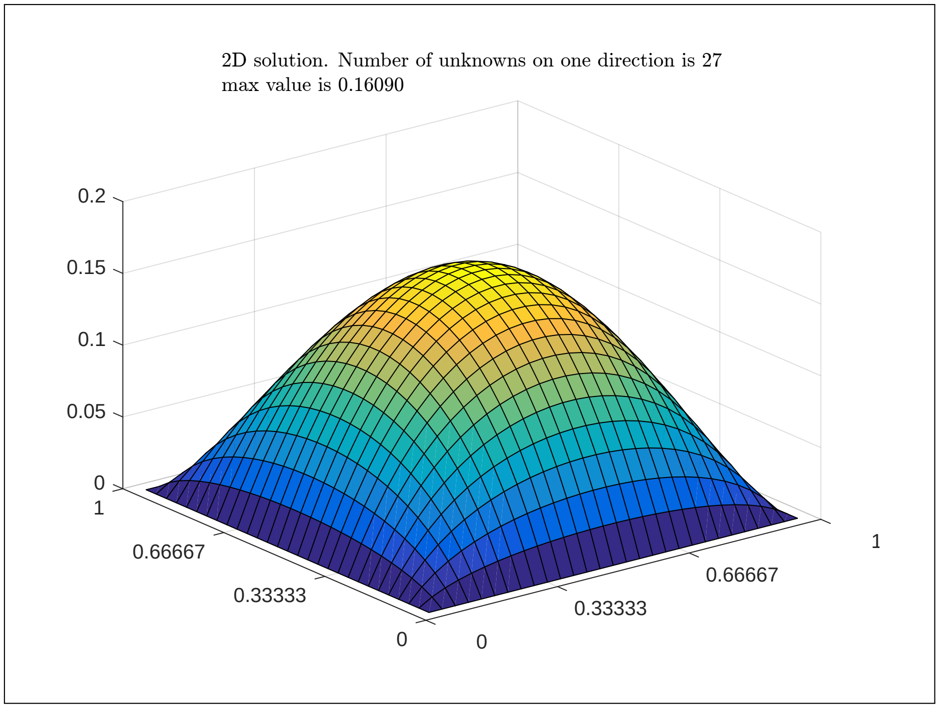



Figure 5: Solution plot

```
function nma_EMA_471_HW6_problem_1
%solution to probnlem 1, HW6, EMA 471

N  = 27;  %number of unknowns in one direction. change
%grid size as needed
h  = 1/(N+1); %grid spacing

c1 = (-8/h^2-5);
c2 = (2/h^2-1/(2*h));
c3 = (2/h^2+1/(2*h));
A  = lap2d(N,c1,c3,c2);   %make the A matrix

f  = -5*ones(N^2,1);  %RHS
u  = A\f; %direct solver
u  = reshape(u,N,N);
U  = zeros(N+2,N+2); %put the zero BC back in to plot
U(2:end-1,2:end-1) = u;  %put the solution into the larger grid
figure;
surf(U);

title({sprintf('2D solution. Number of unknowns on one direction is %d',N),  ...
       sprintf('max value is $%6.5f$',max(U(:)))},...
              'Interpreter', 'latex','fontsize',10);

%relabel ticks for 0..1 in both directions
r = get(gca,'XTickLabel');
set(gca,'XTickLabel',num2str((0:1/(length(r)-1):1)'));
r = get(gca,'YTickLabel');
set(gca,'YTickLabel',num2str((0:1/(length(r)-1):1)'));

```

```
%found a bug in Matlab!
%set(gca,'TickLabelInterpreter', 'Latex','fontsize',8);
end
%===================================
function L2 = lap2d(n,middle,left,right)
%function to construct the 2D A matrix for this problem
e  = ones(n,1);
B  = [e*left e*middle/2 e*right];
L  = spdiags(B,[-1 0 1],n,n);
I  = speye(n);
Lm = kron(I,L); %does the central diagonal
Lo = kron(L,I); %does the off diagonal
L2 = Lm+Lo;
end
```

## 0.2 Problem 2

(2) (15 pts) If heat conduction takes place within a body that is simultaneously immersed in a fluid, there are two heat removal mechanisms. One particular energy balance gives:

$$\nabla^2 T - 20T = -200, \qquad \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

Solve this problem over the unit square, $0 \le x \le 1$, $0 \le y \le 1$ subject to insulated boundary conditions at $x = 0$ and $y = 0$ (temperature gradients normal to the boundary are zero) and $T = 0$ at $x = 1$ and $y = 1$. Find the peak temperature in the medium.

Figure 6: problem 2 description

On the left edge we have (where $i = 0$) we have

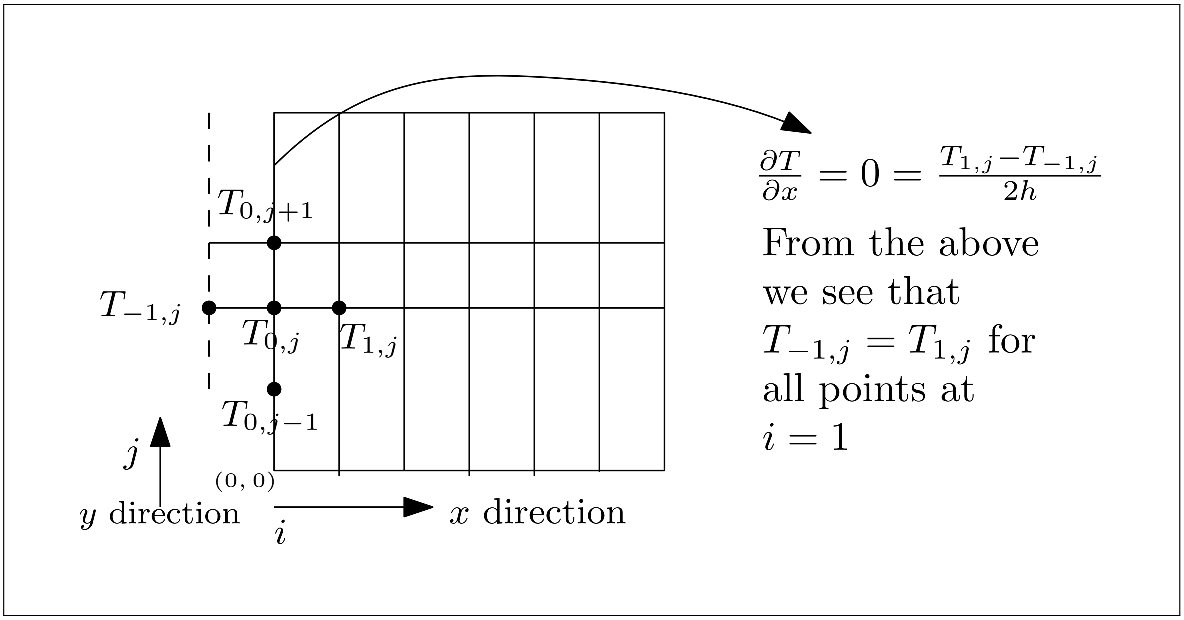



Figure 7: Left edge conditions

And on the bottom edge where $j = 0$ we have

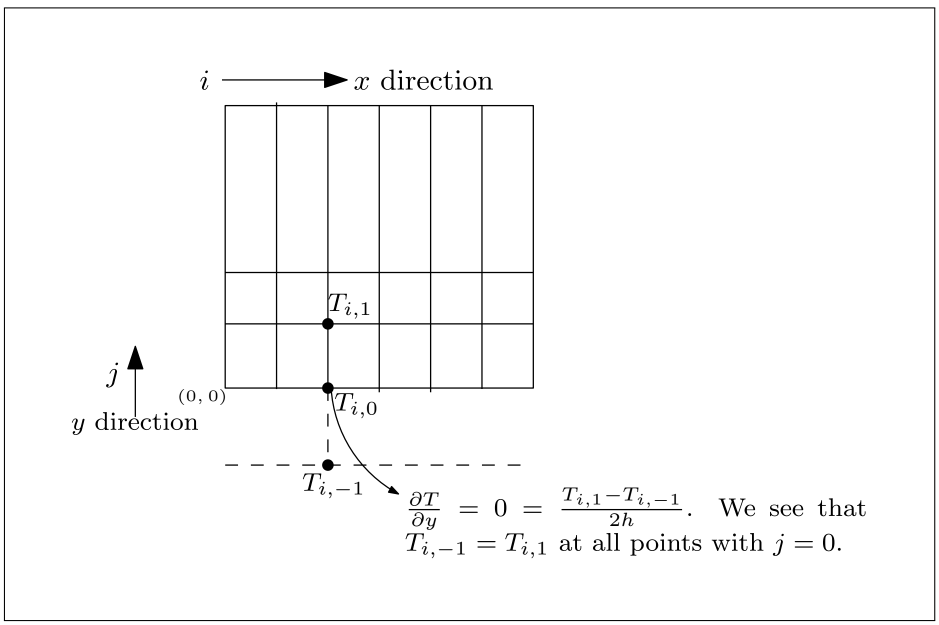



Figure 8: bottom edge conditions

Using the above relations, then at node $i = 0$,

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{1,j} - 2T_{0,j} + T_{-1,j}}{h^2}$$

Using the relation we found earlier, which said that $T_{-1,j} = T_{1,j}$, the above becomes

$$\frac{\partial^2 T}{\partial x^2} = \frac{2T_{1,j} - 2T_{0,j}}{h^2}$$

Therefore, the differential equation at $i = 0$ and for all $j$ becomes

$$\begin{aligned}
\nabla^2 T - 20T &= -200 \\
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - 20T &= -200 \\
\frac{2T_{1,j} - 2T_{0,j}}{h^2} + \frac{T_{0,j+1} - 2T_{0,j} + T_{0,j-1}}{h^2} - 20T_{0,j} &= -200 \qquad (1)
\end{aligned}$$

The above is what we will use on the left edge, for $j = 1 \cdots N$ where $N$ is the number of internal nodes. We now find the PDE on the lower edge in similar way. On the bottom edge, where $j = 0$, we have

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,1} - 2T_{i,0} + T_{i,-1}}{h^2}$$

Using the relation we found earlier, which said that $T_{i,-1} = T_{i,1}$, then the above becomes

$$\frac{\partial^2 T}{\partial y^2} = \frac{2T_{i,1} - 2T_{i,0}}{h^2}$$

Therefore, the differential equation at $j = 0$ and for all $i$ becomes

$$\begin{aligned}
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - 20T &= -200 \\
\frac{T_{i+1,0} - 2T_{i,0} + T_{i-1,0}}{h^2} + \frac{2T_{i,1} - 2T_{i,0}}{h^2} - 20T_{i,0} &= -200 \qquad (2)
\end{aligned}$$

The above is what we will use on the bottom edge, for $i = 1 \cdots N$ where $N$ is the number of internal nodes. Now that we found the PDE on the left and on the right edge, we write the PDE on the internal nodes, which is the standard form

$$\begin{aligned}
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - 20T &= -200 \\
\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{h^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{h^2} - 20T_{i,j} &= -200 \qquad (3)
\end{aligned}$$

Using (1,2,3) equations, we now find the $Ax = f$ form. Let us assume that $N = 3$, so our grid is the following

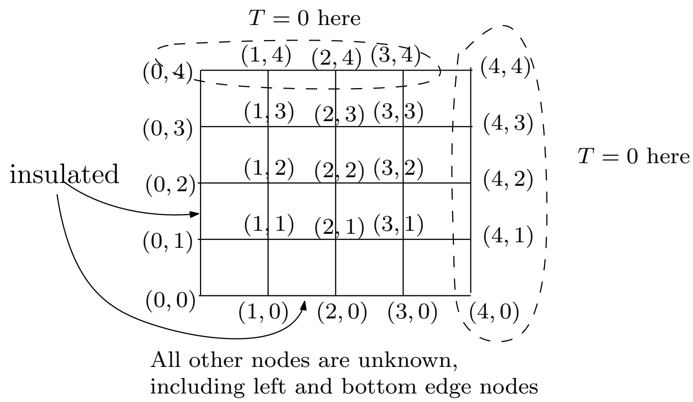



Figure 9: Example grid used to find the $A$ matrix

For $i = 0, j = 0$, this is special node. We can either use the condition for the left edge to handle it, or the condition for the bottom edge, or use the average of the adjacent nodes. Let us use the bottom edge condition for it. Hence at this node the PDE is from (2)

$$\begin{aligned}
\frac{T_{1,0} - 2T_{0,0} + T_{-1,0}}{h^2} + \frac{2T_{0,1} - 2T_{0,0}}{h^2} - 20T_{0,0} &= -200 \\
\frac{2T_{1,0} - 2T_{0,0}}{h^2} + \frac{2T_{0,1} - 2T_{0,0}}{h^2} - 20T_{0,0} &= -200 \\
2T_{1,0} - 2T_{0,0} + 2T_{0,1} - 2T_{0,0} - 20h^2T_{0,0} &= -200h^2 \\
2T_{1,0} + T_{0,0}\left(-4 - 20h^2\right) + 2T_{0,1} &= -200h^2 \qquad (0,0)
\end{aligned}$$

For $i = 1, j = 0$, from (2)

$$\begin{aligned}
\frac{T_{2,0} - 2T_{1,0} + T_{0,0}}{h^2} + \frac{2T_{1,1} - 2T_{1,0}}{h^2} - 20T_{1,0} &= -200 \\
T_{2,0} + T_{1,0}\left(-4 - 20h^2\right) + T_{0,0} + 2T_{1,1} &= -200h^2 \qquad (1,0)
\end{aligned}$$

And for $i = 2, j = 0$

$$\begin{aligned}
\frac{T_{3,0} - 2T_{2,0} + T_{1,0}}{h^2} + \frac{2T_{2,1} - 2T_{2,0}}{h^2} - 20T_{2,0} &= -200 \\
T_{3,0} - 4T_{2,0} + T_{1,0} + 2T_{2,1} - 20h^2T_{2,0} &= -200h^2 \\
T_{3,0} + T_{2,0}\left(-4 - 20h^2\right) + T_{1,0} + 2T_{2,1} &= -200h^2 \qquad (2,0)
\end{aligned}$$

And for $i = 3, j = 0$

$$\begin{aligned}
\frac{T_{4,0} - 2T_{3,0} + T_{2,0}}{h^2} + \frac{2T_{3,1} - 2T_{3,0}}{h^2} - 20T_{3,0} &= -200 \\
T_{4,0} - 4T_{3,0} + T_{2,0} + 2T_{3,1} - 20h^2T_{3,0} &= -200h^2 \\
T_{4,0} + T_{3,0}\left(-4 - 20h^2\right) + T_{2,0} + 2T_{3,1} &= -200h^2
\end{aligned}$$

But $T_{4,0}$ is know, since it is on the right edge and is zero there. Hence the above becomes

$$T_{3,0}(-4-40h)+T_{2,0}+2T_{3,1}=-200h^2 \tag{3,0}$$

We now move to the second grid line from the bottom. On the first node, which is $i=0, j=1$, we use the left edge PDE, which is (1). Hence

$$\begin{aligned}\frac{2T_{1,1}-2T_{0,1}}{h^2}+\frac{T_{0,2}-2T_{0,1}+T_{0,0}}{h^2}-20T_{0,1}&=-200\\ 2T_{1,1}-4T_{0,1}+T_{0,2}+T_{0,0}-20h^2T_{0,1}&=-200h^2\\ 2T_{1,1}+T_{0,1}\left(-4-20h^2\right)+T_{0,2}+T_{0,0}&=-200h^2 \end{aligned}\tag{0,1}$$

And for $i=1, j=1$, this is an internal node, so we use (3)

$$\begin{aligned}\frac{T_{2,1}-2T_{1,1}+T_{0,1}}{h^2}+\frac{T_{1,2}-2T_{1,1}+T_{1,0}}{h^2}-20T_{1,1}&=-200\\ T_{2,1}-4T_{1,1}+T_{0,1}+T_{1,2}+T_{1,0}-20h^2T_{1,1}&=-200h^2\\ T_{2,1}+T_{1,1}\left(-4-20h^2\right)+T_{0,1}+T_{1,2}+T_{1,0}&=-200h^2 \end{aligned}\tag{1,1}$$

And for $i=2, j=1$, this is an internal node, so we use (3)

$$\begin{aligned}\frac{T_{3,1}-2T_{2,1}+T_{1,1}}{h^2}+\frac{T_{2,2}-2T_{2,1}+T_{2,0}}{h^2}-20T_{2,1}&=-200\\ T_{3,1}+T_{2,1}\left(-4-20h^2\right)+T_{1,1}+T_{2,2}+T_{2,0}&=-200h^2 \end{aligned}\tag{2,1}$$

And for $i=3, j=1$, this is an internal node, so we use (3) and set $T_{4,1}=0$ since known

$$\begin{aligned}\frac{T_{4,1}-2T_{3,1}+T_{2,1}}{h^2}+\frac{T_{3,2}-2T_{3,1}+T_{3,0}}{h^2}-20T_{3,1}&=-200\\ T_{3,1}\left(-4-20h^2\right)+T_{2,1}+T_{3,2}+T_{3,0}&=-200h^2 \end{aligned}\tag{3,1}$$

For node $i=4, j=1$, this is a known value for $T$ there, so we skip it. Going to the next grid row above, for $i=0, j=2$, this is a left edge node, so we use (1)

$$\begin{aligned}\frac{2T_{1,2}-2T_{0,2}}{h^2}+\frac{T_{0,3}-2T_{0,2}+T_{0,1}}{h^2}-20T_{0,2}&=-200\\ 2T_{1,2}+T_{0,2}\left(-4-20h^2\right)+T_{0,3}+T_{0,1}&=-200h^2 \end{aligned}\tag{0,2}$$

For $i=1, j=2$, this is an internal node, so we use (3)

$$\begin{aligned}\frac{T_{2,2}-2T_{1,2}+T_{0,2}}{h^2}+\frac{T_{1,3}-2T_{1,2}+T_{1,1}}{h^2}-20T_{1,2}&=-200\\ T_{2,2}+T_{1,2}\left(-4-20h^2\right)+T_{0,2}+T_{1,3}+T_{1,1}&=-200h^2 \end{aligned}\tag{1,2}$$

For $i=2, j=2$, this is an internal node, so we use (3)

$$
\begin{aligned}
\frac{T_{3,2}-2T_{2,2}+T_{1,2}}{h^2}+\frac{T_{2,3}-2T_{2,2}+T_{2,1}}{h^2}-20T_{2,2} &= -200\\
T_{3,2}+T_{2,2}\left(-4-20h^2\right)+T_{1,2}+T_{2,3}+T_{2,1} &= -200h^2 \qquad (2,2)
\end{aligned}
$$

And for $i=3, j=2$, this is an internal node, so we use (3) and set $T_{4,2}=0$ since known

$$
\begin{aligned}
\frac{T_{4,2}-2T_{3,2}+T_{2,2}}{h^2}+\frac{T_{3,3}-2T_{3,2}+T_{3,1}}{h^2}-20T_{3,2} &= -200\\
T_{3,2}\left(-4-20h^2\right)+T_{2,2}+T_{3,3}+T_{3,1} &= -200h^2 \qquad (3,2)
\end{aligned}
$$

For node $i=4, j=2$, this is a known value for $T$ there, so we skip it. Going to the next grid row above, for $i=0, j=3$, this is a left edge node, so we use (1)

$$
\begin{aligned}
\frac{2T_{1,3}-2T_{0,3}}{h^2}+\frac{T_{0,4}-2T_{0,3}+T_{0,2}}{h^2}-20T_{0,3} &= -200\\
2T_{1,3}+T_{0,3}\left(-4-20h^2\right)+T_{0,4}+T_{0,2} &= -200h^2
\end{aligned}
$$

But $T_{0,4}$ is on the top edge, which is known and is zero, therefore the above is

$$
2T_{1,3}+T_{0,3}\left(-4-20h^2\right)+T_{0,2}=-200h^2 \qquad (0,3)
$$

For node $i=1, j=3$, this is an internal node, so we use (3)

$$
\begin{aligned}
\frac{T_{2,3}-2T_{1,3}+T_{0,3}}{h^2}+\frac{T_{1,4}-2T_{1,3}+T_{1,2}}{h^2}-20T_{1,3} &= -200\\
T_{2,3}+T_{1,3}\left(-4-20h^2\right)+T_{0,3}+T_{1,4}+T_{1,2} &= -200h^2
\end{aligned}
$$

But $T_{1,4}$ is on the top edge, which is known and is zero, therefore the above is

$$
T_{2,3}+T_{1,3}\left(-4-20h^2\right)+T_{0,3}+T_{1,2}=-200h^2 \qquad (1,3)
$$

And on node $i=2, j=3$, this is an internal node, so we use (3)

$$
\begin{aligned}
\frac{T_{3,3}-2T_{2,3}+T_{1,3}}{h^2}+\frac{T_{2,4}-2T_{2,3}+T_{2,2}}{h^2}-20T_{2,3} &= -200\\
T_{3,3}+T_{2,3}\left(-4-20h^2\right)+T_{1,3}+T_{2,4}+T_{2,2} &= -200h^2
\end{aligned}
$$

But $T_{2,4}$ is on the top edge, which is known and is zero, therefore the above is

$$
T_{3,3}+T_{2,3}\left(-4-20h^2\right)+T_{1,3}+T_{2,2}=-200h^2 \qquad (2,3)
$$

Finally, for node $i=3, j=3$

$$
\begin{aligned}
\frac{T_{4,3}-2T_{3,3}+T_{2,3}}{h^2}+\frac{T_{3,4}-2T_{3,3}+T_{3,2}}{h^2}-20T_{3,3} &= -200\\
T_{4,3}+T_{3,3}\left(-4-20h^2\right)+T_{2,3}+T_{3,4}+T_{3,2} &= -200h^2
\end{aligned}
$$

But $T_{3,4}$ and $T_{4,3}$ are known and is zero, therefore the above is

$$T_{3,3}\left(-4-20h^2\right)+T_{2,3}+T_{3,2}=-200h^2 \tag{3,3}$$

Now we able to see the form of the matrix $A$ by writing the equations for this small grid. There are 16 unknowns. Let $\left(-4-20h^2\right)=\alpha$ then

$$\begin{array}{c} \text{bottom edge}\\ \text{bottom edge}\\ \text{bottom edge}\\ \text{bottom edge}\\ \text{left edge}\\ \text{internal nodes}\\ \text{internal nodes}\\ \text{internal nodes}\\ \text{left edge}\\ \text{internal nodes}\\ \text{internal nodes}\\ \text{internal nodes}\\ \text{left edge}\\ \text{row below top edge}\\ \text{row below top edge}\\ \text{row below top edge} \end{array} \begin{bmatrix} \alpha & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & \alpha & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & \alpha & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & \alpha & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \alpha & 2 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \alpha & 2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \alpha & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \alpha & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \alpha \end{bmatrix} \begin{bmatrix} T_{0,0}\\ T_{1,0}\\ T_{2,0}\\ T_{3,0}\\ T_{0,1}\\ T_{1,1}\\ T_{2,1}\\ T_{3,1}\\ T_{0,2}\\ T_{1,2}\\ T_{2,2}\\ T_{3,2}\\ T_{0,3}\\ T_{1,3}\\ T_{2,3}\\ T_{3,3} \end{bmatrix} = \begin{bmatrix} -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2\\ -200h^2 \end{bmatrix}$$

The above was implemented in the matlab program `nma_EMA_471_HW6_problem_2.m`. The maximum value of $T$ found was

$$9.577$$

Here is the 3D plot and a contour plot.

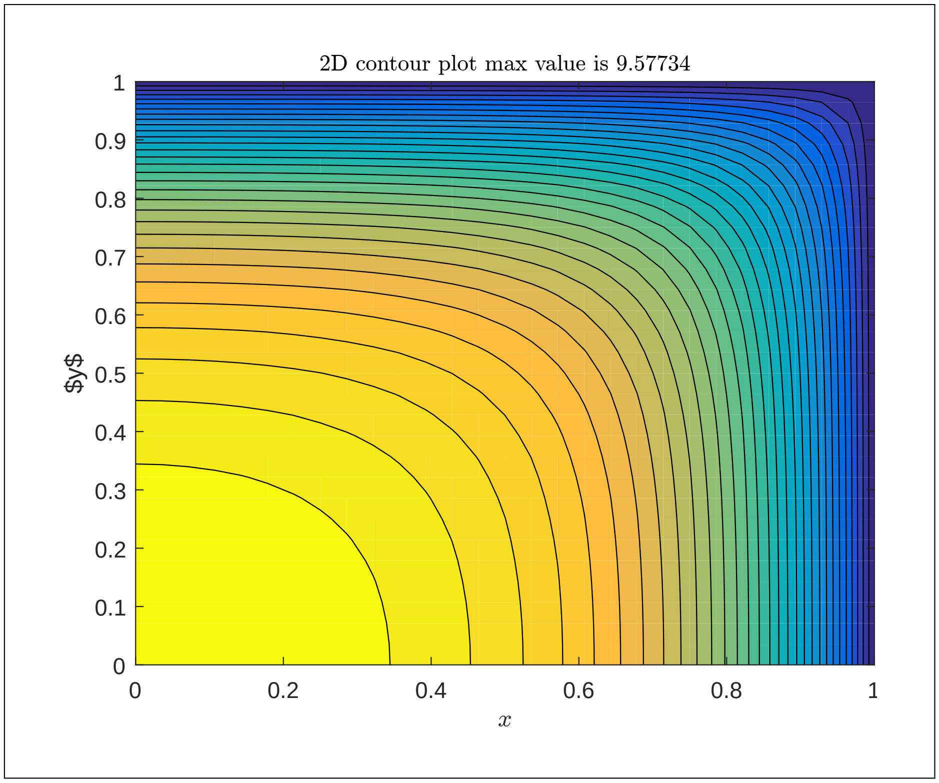



Figure 10: contour plot

And this is the 3D plot.

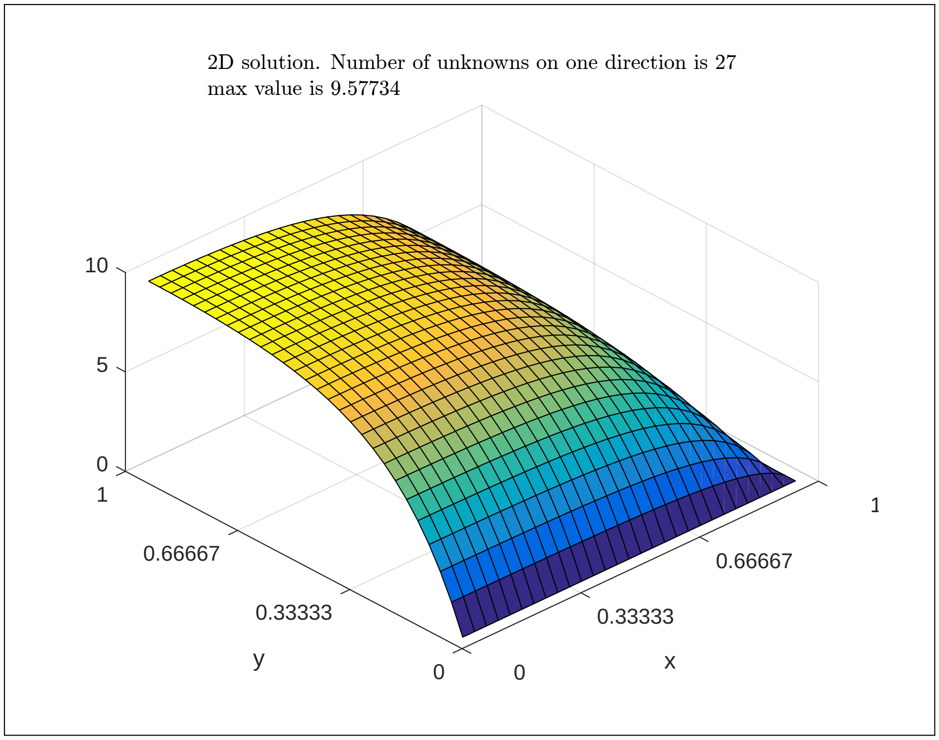



Figure 11: 3D plot

I have also solved this in Mathematica, using the finite element solver build into NDSOlve. I also obtained the same result. Here is the contour plot and the 3D plot.

```
Clear[u, x, y];
r = NDSolveValue[{
   D[u[x, y], {x, 2}] + D[u[x, y], {y, 2}] - 20*u[x, y] == - 200 +
     NeumannValue[0, x == 0] + NeumannValue[0, y == 0],
      DirichletCondition[u[x, y] == 0, x == 1],
   DirichletCondition[u[x, y] == 0, y == 1 ]}, u, {x, 0, 1}, {y, 0, 1},
    Method -> {"FiniteElement",
    "MeshOptions" -> {"BoundaryMeshGenerator" -> "Continuation"}}]

ContourPlot[r[x, y], {x, 0, 1}, {y, 0, 1}]
Plot3D[r[x, y], {x, 0, 1}, {y, 0, 1}, Mesh -> All]
```

.

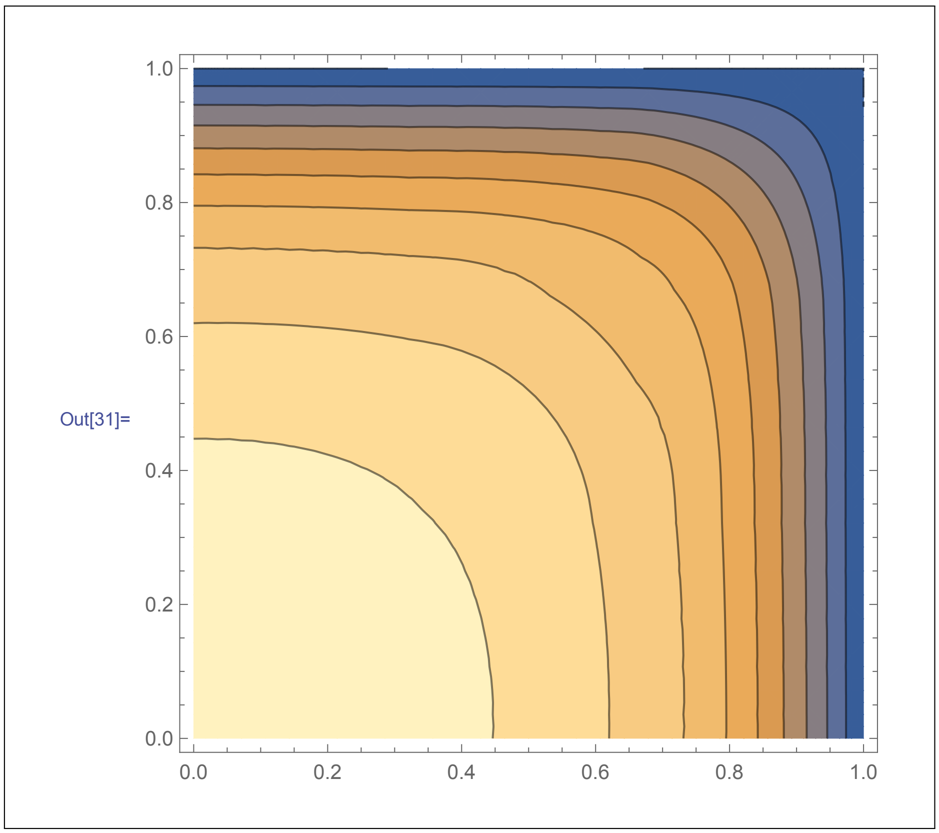



Figure 12: Contour plot using Mathematica

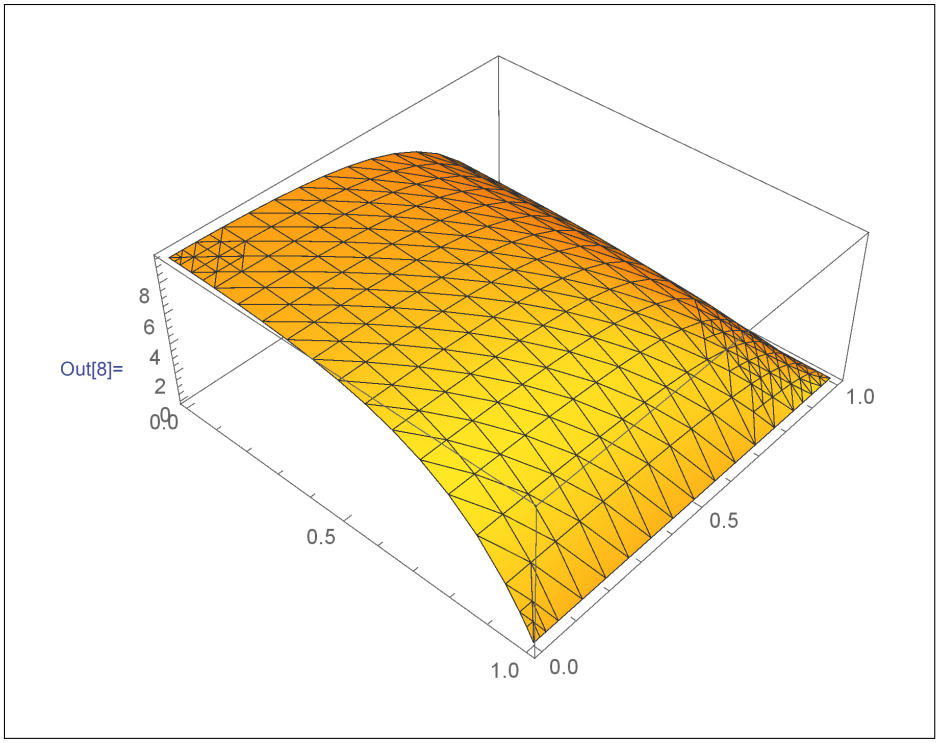



Figure 13: 3D plot using Mathematica

```
function nma_EMA_471_HW6_problem_2
%solution to probnlem 2, HW6, EMA 471

close all;
n  = 27;  %number of internal nodes in one direction.
%change grid size as needed
h  = 1/(n+1); %grid spacing
A  = make_A_matrix(h,n);
f  = -200*h^2*ones((n+1)^2,1);  %RHS
u  = A\f; %direct solver

u  = reshape(u,n+1,n+1)'; %change it for natural setting
u  = [u;zeros(1,n+1)];    %add zero boundary conditions
u  = [u zeros(n+2,1)];
u  = flipud(u); %so it looks right
figure;
surf(u);

title({sprintf('2D solution. Number of unknowns on one direction is %d',n),  ...
    sprintf('max value is $%6.5f$',max(u(:)))},...
    'Interpreter', 'latex','fontsize',10);

%relable ticks for 0..1 in both directions
r = get(gca,'XTickLabel');
set(gca,'XTickLabel',num2str((0:1/(length(r)-1):1)'));
r = get(gca,'YTickLabel');
set(gca,'YTickLabel',num2str((0:1/(length(r)-1):1)'));
xlabel('$x$','Interpreter', 'latex');
ylabel('$y$','Interpreter', 'latex');

%found a bug in Matlab!
```

```
%set(gca,'TickLabelInterpreter', 'Latex','fontsize',8);

%do contour plot also
x     = 0:h:1;
y     = 0:h:1;
[X,Y] = meshgrid(x,y);

figure
contourf(X,Y,flipud(u),30);
xlabel('$x$', 'Interpreter', 'latex');
ylabel('$y$','Interpreter', 'latex');
title({sprintf('2D contour plot max value is $%6.5f$',...
       max(u(:)))},...
    'Interpreter', 'latex','fontsize',10);
end
%==================================
function A = make_A_matrix(h,n)
%h is grid spacing.
%n is number of internal nodes on any one grid direction.
%please see report for details.

a = -4-20*h^2;
A = zeros((n+1)^2,(n+1)^2);

%make T(0,0), lower corner node
A(1,1)   = a;
A(1,2)   = 2;
A(1,n+2) = 2;

%rest of bottom edge nodes. These are node on lower edge, where
%it is insulated
for i = 2:n
    A(i,i)     = a;
    A(i,i-1)   = 1;
    A(i,i+1)   = 1;
    A(i,i+n+1) = 2;
end
%last node on botton edge. special handling
i          = i+1;
A(i,i-1)   = 1;
A(i,i)     = a;
A(i,i+n+1) = 2;

%now make left edge rows. First special
for i = n+2:n+1:((n+1)^2-n)
    A(i,i-(n+1)) = 1;

    if i==n+2
       A(i,i)     = a;
       A(i,i+1)   = 2;
       A(i,i+n+1) = 1;
    else
      A(i,i)      = a;
      A(i,i+1)    = 2;
      if i<((n+1)^2-n)
          A(i,i+n+1) = 1;
      end
    end
end
```

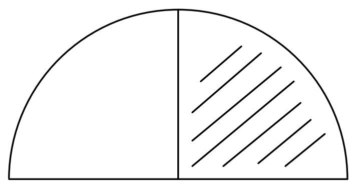
```

%now make middle rows
for i = n+3: n+1: ((n+1)^2-2*n)
   for k = i:i+n-1
            A(k,k-(n+1)) = 1;
            A(k,k-1)     = 1;
            A(k,k)       = a;
            if k<(i+n-1)
               A(k,k+1) = 1;
            end
            A(k,k+n+1) = 1;
   end
end

%now do the top edge
for i = (n+1)^2-(n-1):(n+1)^2
    A(i,i-(n+1))  = 1;
    A(i,i-1)      = 1;
    A(i,i)        = a;
    if i<(n+1)^2
        A(i,i+1) = 1;
    end
end
end
```

### 0.3 Problem 3

(3) (15 pts) The Laplacian operator can be represented in other coordinate systems. In a cylindrical system, it takes the form:

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}$$

Solve the problem, $\nabla^2 T = -q'''/k$, over a semicircular domain that is defined by $0 \le r \le 1$ and $0 \le \theta \le \pi$ where the right hand side of the equation is −200 when $0 \le \theta \le \pi/2$ and is equal to 0 when $\pi/2 < \theta \le \pi$. The heat source exists over just half the plate as shown below:



The extension of finite difference methods to this problem is straightforward at all places except the origin where $r \to 0$. Instead of using the Laplacian operator directly at the origin,

we can instead write an energy balance based on energy produced in a small control volume around the origin. When we do this, the result is:

$$T_0 = \frac{0.5T_1 + T_2 + ... + T_{N-1} + 0.5T_N}{N-1} + \frac{q'''\Delta r^2}{8k}$$

Here, $N$ is the total number of circumferential nodes one radial mesh spacing from the origin, with $T_1$ and $T_N$ being the nodal temperatures at $\theta = 0$ and $\theta = \pi$ respectively, and the other $T_i$'s being nodal temperatures between these limits. $T_0$ is the temperature at $r = 0$. For example, if $N = 5$, this corresponds to an angular mesh spacing of $\pi/4$. (I'm not recommending an angular mesh spacing this coarse; it's just for illustration.) In the limit that the radial mesh spacing goes to zero, this states that the temperature at the origin is a kind of simple average of its nearest neighbors, where the temperatures at the edges receive a half-weighting relative to those in the interior. Since the mesh spacing isn't actually infinitely fine, we'll have the term on the far right. Since, for this specific problem, $q'''/k$ is 200, the term on the far right is actually $25\Delta r^2$. For all nodes other than the one at the origin, we write finite-difference equations based on the Laplacian operator.

One other issue to consider is the treatment of those nodes at $\theta = \pi/2$, a line that straddles the region where there is a heat source and where there is none. The term on the right hand side of the heat conduction equation should be −100 for those nodes.

Solve this problem over the domain subject to the conditions $T = 0$ at $r = 1$ and an insulated condition $\partial T/\partial\theta = 0$ at $\theta = 0$ and $\theta = \pi$. Report the temperature at the origin, $T_0$.

---

**Reminder**: If you have not done so already, send me an email about what you are planning to do for your project. Take some time to think about it and make sure you run it past me if not one of the default projects. You don't want to be in the position of leaving this to the last minute. We will allocate some in-class time for you to work on these as well.

Figure 14: problem 3 description

The grid numbering used is the following

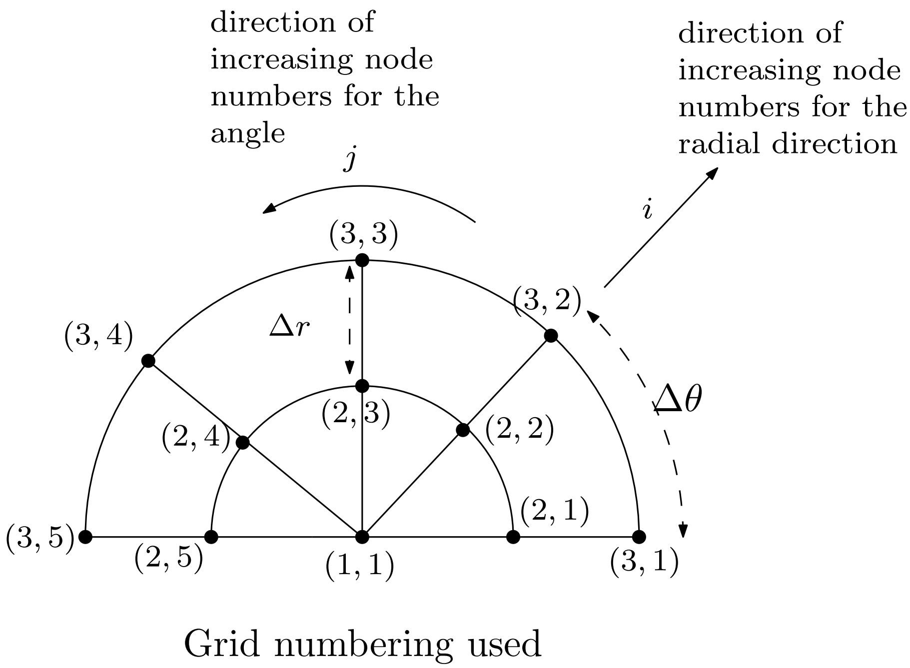



Figure 15: Grid numbering for problem 3

The PDE is, for $0 \leq \theta \leq \frac{\pi}{2}$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = -200$$

And for $\frac{\pi}{2} < \theta \leq \pi$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

For the nodes with $\theta = \frac{\pi}{2}$ we will use

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = -100$$

Using centered difference, and using $T_{ij}$ to mean $T_{r_i,\theta j}$ then the above can be written as

$$\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} = -200 \tag{1}$$

Where $i = 1 \cdots N_r$ where $N_r$ is the number of grid points in the radial direction, which is 3 in the diagram above. The following diagram shows the boundary conditions to use.

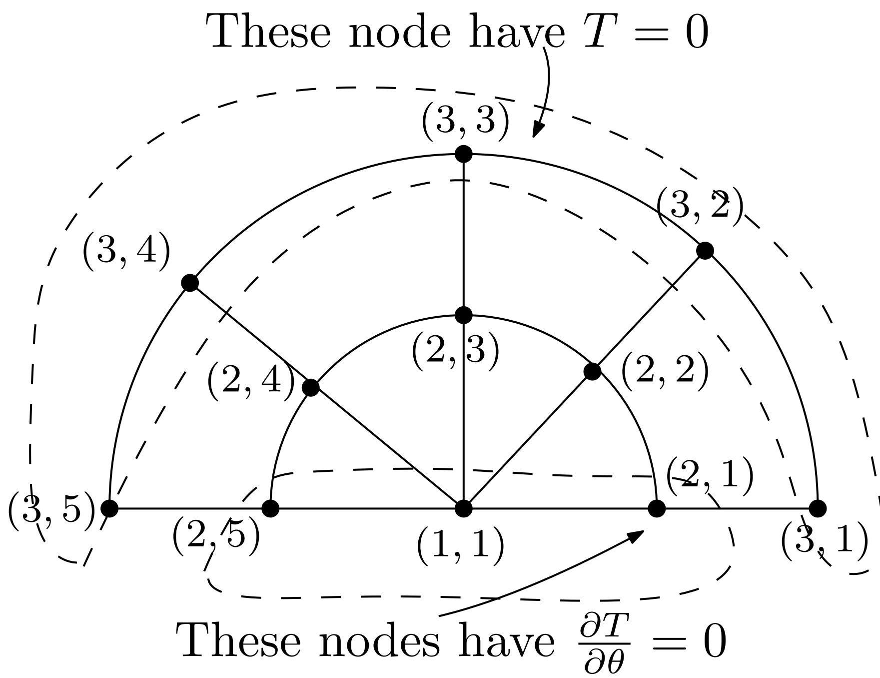



Figure 16: boundary conditions for problem 3

We now find the $A$ matrix in order to solve the problem using the direct method. We assume $N_\theta = 7$ and $N_r = 5$ for the purpose of seeing what the $A$ structure is

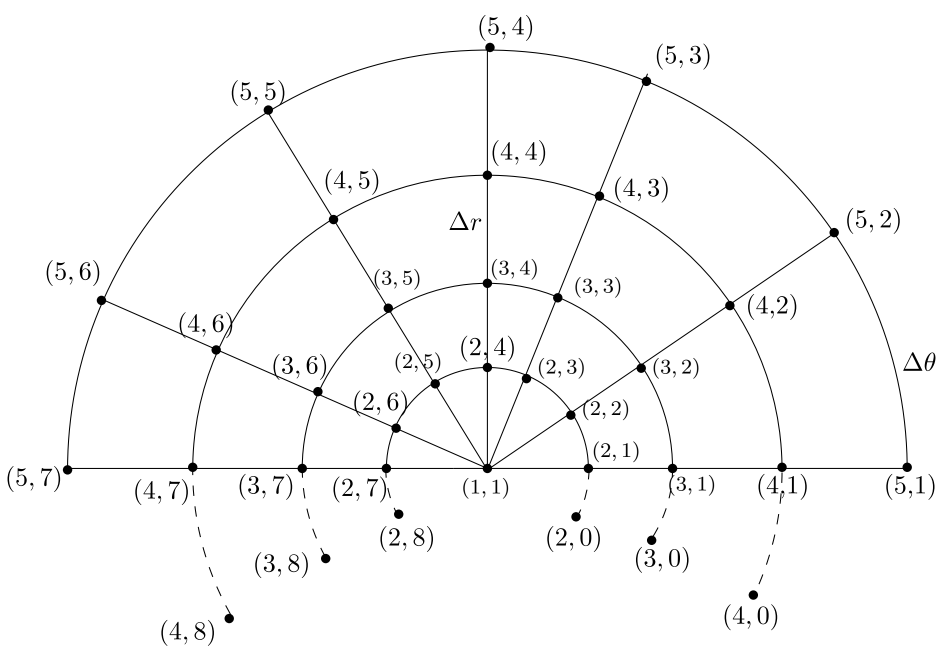



Figure 17: Example grid using $N_r = 5$ and $N_\theta = 7$ problem 3

For node (1,1), this is the special node. which is

$$T_{1,1} = \frac{0.5T_{2,1} + T_{2,2} + T_{2,3} + \cdots + T_{2,N-1} + 0.5T_{2,N_\theta}}{N_\theta - 1} + 25\Delta r^2$$

Where $N_\theta$ is the number of grid point in the angular direction. For example, in the diagram given above, $N_\theta = 7$, hence

$$T_{1,1} - \frac{0.5T_{2,1} + T_{2,2} + T_{2,3} + T_{2,4} + T_{2,5} + T_{2,6} + 0.5T_{2,7}}{6} = 25\Delta r^2$$

$$T_{1,1} - \frac{1}{12}T_{2,1} - \frac{1}{6}T_{2,2} - \frac{1}{6}T_{2,3} - \frac{1}{6}T_{2,4} - \frac{1}{6}T_{2,5} - \frac{1}{6}T_{2,6} - \frac{1}{12}T_{2,7} = 25\Delta r^2 \tag{1,1}$$

For node (2,1), this is an insulated node. Hence by introducing an imaginary node as shown above, then on this node, the PDE is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = -200$$

$$\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta\theta^2} = -200$$

$$\frac{T_{1,1} - 2T_{21} + T_{3,1}}{\Delta r^2} + \frac{1}{\Delta r}\frac{T_{3,1} - T_{1,1}}{2\Delta r} + \frac{1}{\Delta r^2}\frac{T_{2,0} - 2T_{2,1} + T_{2,2}}{\Delta\theta^2} = -200$$

But due to insulation, then $\frac{\partial T}{\partial \theta} = \frac{T_{i,j+1} - T_{i,j-1}}{2\Delta\theta} = 0$ which means $T_{i,j+1} = T_{i,j-1}$, or at this node $T_{2,0} = T_{2,2}$, hence the above becomes

$$\frac{T_{1,1} - 2T_{21} + T_{3,1}}{\Delta r^2} + \frac{1}{\Delta r}\frac{T_{3,1} - T_{1,1}}{2\Delta r} + \frac{1}{\Delta r^2}\frac{2T_{2,2} - 2T_{2,1}}{\Delta\theta^2} = -200$$

Collecting terms

$$T_{1,1}\left(\frac{1}{\Delta r^2} - \frac{1}{2\Delta r^2}\right) + T_{2,1}\left(\frac{-2}{\Delta r^2} - \frac{2}{\Delta r^2\Delta\theta^2}\right) + T_{3,1}\left(\frac{1}{\Delta r^2} + \frac{1}{2\Delta r^2}\right) + T_{2,2}\left(\frac{2}{\Delta r^2\Delta\theta^2}\right) = -200$$

$$T_{1,1}\left(\frac{1}{2\Delta r^2}\right) + T_{2,1}\left(\frac{-2}{\Delta r^2} - \frac{2}{\Delta r^2\Delta\theta^2}\right) + T_{3,1}\left(\frac{3}{2\Delta r^2}\right) + T_{2,2}\left(\frac{2}{\Delta r^2\Delta\theta^2}\right) = -200 \tag{2,1}$$

At node (3,1)

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = -200$$

$$\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta\theta^2} = -200$$

$$\frac{T_{2,1} - 2T_{3,1} + T_{4,1}}{\Delta r^2} + \frac{1}{2(\Delta r)}\frac{T_{4,1} - T_{2,1}}{2\Delta r} + \frac{1}{[2(\Delta r)]^2}\frac{T_{3,0} - 2T_{3,1} + T_{3,2}}{\Delta\theta^2} = -200$$

But due to insulation, then $\frac{\partial T}{\partial \theta} = \frac{T_{i,j+1} - T_{i,j-1}}{2\Delta\theta} = 0$ which means $T_{i,j+1} = T_{i,j-1}$, or at this node $T_{3,0} = T_{3,2}$, hence the above becomes

$$\frac{T_{2,1}-2T_{3,1}+T_{4,1}}{\Delta r^2}+\frac{1}{2\,(\Delta r)}\frac{T_{4,1}-T_{2,1}}{2\Delta r}+\frac{1}{[2\,(\Delta r)]^2}\frac{2T_{3,2}-2T_{3,1}}{\Delta\theta^2}=-200$$

Collecting terms

$$T_{2,1}\left(\frac{1}{\Delta r^2}-\frac{1}{4\Delta r^2}\right)+T_{3,1}\left(-\frac{2}{\Delta r^2}-\frac{2}{[2\,(\Delta r)]^2\,\Delta\theta^2}\right)+T_{4,1}\left(\frac{1}{\Delta r^2}+\frac{1}{4\Delta r^2}\right)+T_{3,2}\left(\frac{2}{[2\,(\Delta r)]^2\,\Delta\theta^2}\right)=-200$$

$$T_{2,1}\left(\frac{3}{4\Delta r^2}\right)+T_{3,1}\left(-\frac{2}{\Delta r^2}-\frac{1}{2\Delta r^2\Delta\theta^2}\right)+T_{4,1}\left(\frac{5}{4\Delta r^2}\right)+T_{3,2}\left(\frac{1}{2\Delta r^2\Delta\theta^2}\right)=-200 \qquad (3,1)$$

At node $(4,1)$

$$\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2T}{\partial\theta^2}=-200$$

$$\frac{T_{i-1,j}-2T_{ij}+T_{i+1,j}}{\Delta r^2}+\frac{1}{(i-1)\,(\Delta r)}\frac{T_{i+1,j}-T_{i-1,j}}{2\Delta r}+\frac{1}{[(i-1)\,(\Delta r)]^2}\frac{T_{i,j-1}-2T_{ij}+T_{i,j+1}}{\Delta\theta^2}=-200$$

$$\frac{T_{3,1}-2T_{4,1}+T_{5,1}}{\Delta r^2}+\frac{1}{3\Delta r}\frac{T_{5,1}-T_{3,1}}{2\Delta r}+\frac{1}{9\Delta r^2}\frac{T_{4,0}-2T_{4,1}+T_{4,2}}{\Delta\theta^2}=-200$$

But due to insulation, then $\frac{\partial T}{\partial\theta}=\frac{T_{i,j+1}-T_{i,j-1}}{2\Delta\theta}=0$ which means $T_{i,j+1}=T_{i,j-1}$, or at this node $T_{4,0}=T_{4,2}$, hence the above becomes (and also $T_{5,1}=0$ since on boundary)

$$\frac{T_{3,1}-2T_{4,1}}{\Delta r^2}+\frac{1}{3\Delta r}\frac{-T_{3,1}}{2\Delta r}+\frac{1}{9\Delta r^2}\frac{-2T_{4,1}+2T_{4,2}}{\Delta\theta^2}=-200$$

Collecting terms

$$T_{3,1}\left(\frac{5}{6\Delta r^2}\right)+T_{4,1}\left(-\frac{2}{\Delta r^2}-\frac{2}{9\Delta r^2\Delta\theta^2}\right)+T_{4,2}\left(\frac{2}{9\Delta r^2\Delta\theta^2}\right)=-200 \qquad (4,1)$$

The above *completes half of the bottom grid row.* Now we move to the next grid at, one $\Delta\theta$ above.

At node $(2,2)$, this is an internal node.

$$\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2T}{\partial\theta^2}=-200$$

$$\frac{T_{1,2}-2T_{22}+T_{3,2}}{\Delta r^2}+\frac{1}{\Delta r}\frac{T_{3,2}-T_{1,2}}{2\Delta r}+\frac{1}{\Delta r^2}\frac{T_{2,1}-2T_{22}+T_{2,3}}{\Delta\theta^2}=-200$$

Collecting terms

$$T_{1,2}\left(\frac{1}{\Delta r^2}-\frac{1}{2\Delta r^2}\right)+T_{22}\left(\frac{-2}{\Delta r^2}-\frac{2}{\Delta r^2\Delta\theta^2}\right)+T_{3,2}\left(\frac{1}{\Delta r^2}+\frac{1}{2\Delta r^2}\right)+T_{2,1}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right)+T_{2,3}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right)=-200$$

But $T_{1,2}=T_{1,1}$. This is the same node. Hence the above becomes

$$T_{1,1}\left(\frac{1}{2\Delta r^2}\right)+T_{22}\left(\frac{-2}{\Delta r^2}-\frac{2}{\Delta r^2\Delta\theta^2}\right)+T_{3,2}\left(\frac{3}{2\Delta r^2}\right)+T_{2,1}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right)+T_{2,3}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right)=-200 \qquad (2,2)$$

At node $(3,2)$, this is an internal node

$$
\begin{aligned}
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} &= -200\\
\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta\theta^2} &= -200\\
\frac{T_{2,2} - 2T_{3,2} + T_{4,2}}{\Delta r^2} + \frac{1}{2\Delta r}\frac{T_{4,2} - T_{2,2}}{2\Delta r} + \frac{1}{[2(\Delta r)]^2}\frac{T_{3,1} - 2T_{3,2} + T_{3,3}}{\Delta\theta^2} &= -200\\
T_{2,2}\left(\frac{3}{4\Delta r^2}\right) + T_{3,2}\left(\frac{-2}{\Delta r^2} - \frac{1}{2\Delta r^2\Delta\theta^2}\right) + T_{4,2}\left(\frac{5}{4\Delta r^2}\right) + T_{3,1}\left(\frac{1}{4\Delta r^2\Delta\theta^2}\right) + T_{3,3}\left(\frac{1}{4\Delta r^2\Delta\theta^2}\right) &= -200 \qquad (3,2)
\end{aligned}
$$

At node $(4,2)$ ,this is an internal node

$$
\begin{aligned}
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} &= -200\\
\frac{T_{3,2} - 2T_{4,2} + T_{5,2}}{\Delta r^2} + \frac{1}{3\Delta r}\frac{T_{5,2} - T_{3,2}}{2\Delta r} + \frac{1}{[3(\Delta r)]^2}\frac{T_{4,1} - 2T_{4,2} + T_{4,3}}{\Delta\theta^2} &= -200
\end{aligned}
$$

But $T_{5,2} = 0$ since on boundary, hence

$$
\begin{aligned}
\frac{T_{3,2} - 2T_{4,2}}{\Delta r^2} + \frac{1}{3\Delta r}\frac{-T_{3,2}}{2\Delta r} + \frac{1}{[3(\Delta r)]^2}\frac{T_{4,1} - 2T_{4,2} + T_{4,3}}{\Delta\theta^2} &= -200\\
T_{3,2}\left(\frac{1}{\Delta r^2} - \frac{1}{6\Delta r^2}\right) + T_{4,2}\left(\frac{-2}{\Delta r^2} - \frac{2}{9\Delta r^2\Delta\theta^2}\right) + T_{4,1}\left(\frac{1}{9\Delta r^2\Delta\theta^2}\right) + T_{4,3}\left(\frac{1}{9\Delta r^2\Delta\theta^2}\right) &= -200\\
T_{3,2}\left(\frac{5}{6\Delta r^2}\right) + T_{4,2}\left(\frac{-2}{\Delta r^2} - \frac{2}{9\Delta r^2\Delta\theta^2}\right) + T_{4,1}\left(\frac{1}{9\Delta r^2\Delta\theta^2}\right) + T_{4,3}\left(\frac{1}{9\Delta r^2\Delta\theta^2}\right) &= -200 \qquad (4,2)
\end{aligned}
$$

This *completes first internal grid line on the right half*. Now we move another $\Delta\theta$ anti-clock wise and process the central line.

At node $(2,3)$

$$
\begin{aligned}
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} &= -200\\
\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta\theta^2} &= -200\\
\frac{T_{1,3} - 2T_{2,3} + T_{3,3}}{\Delta r^2} + \frac{1}{\Delta r}\frac{T_{3,3} - T_{1,3}}{2\Delta r} + \frac{1}{\Delta r^2}\frac{T_{2,2} - 2T_{2,3} + T_{2,4}}{\Delta\theta^2} &= -200\\
T_{1,1}\left(\frac{1}{2\Delta r^2}\right) + T_{2,3}\left(\frac{-2}{\Delta r^2} - \frac{2}{\Delta r^2\Delta\theta^2}\right) + T_{3,3}\left(\frac{3}{2\Delta r^2}\right) + T_{2,2}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right) + T_{2,4}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right) &= -200 \qquad (2,3)
\end{aligned}
$$

At node $(3,3)$

$$
\begin{aligned}
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} &= -200\\
\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} &= -200\\
\frac{T_{2,3} - 2T_{3,3} + T_{4,3}}{\Delta r^2} + \frac{1}{2\Delta r}\frac{T_{4,3} - T_{2,3}}{2\Delta r} + \frac{1}{4\Delta r^2}\frac{T_{3,2} - 2T_{3,3} + T_{3,4}}{\Delta \theta^2} &= -200\\
T_{2,3}\left(\frac{3}{4\Delta r^2}\right) + T_{3,3}\left(\frac{-2}{\Delta r^2} - \frac{1}{2\Delta r^2\Delta\theta^2}\right) + T_{4,3}\left(\frac{5}{4\Delta r^2}\right) + T_{3,2}\left(\frac{1}{4\Delta r^2\Delta\theta^2}\right) + T_{3,4}\left(\frac{1}{4\Delta r^2\Delta\theta^2}\right) &= -200 \qquad (3,3)
\end{aligned}
$$

At node $(4,3)$

$$
\begin{aligned}
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} &= -200\\
\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} &= -200\\
\frac{T_{3,3} - 2T_{4,3} + T_{5,3}}{\Delta r^2} + \frac{1}{3\Delta r}\frac{T_{5,3} - T_{3,3}}{2\Delta r} + \frac{1}{9\Delta r^2}\frac{T_{4,2} - 2T_{4,3} + T_{4,4}}{\Delta \theta^2} &= -200
\end{aligned}
$$

But $T_{5,3} = 0$ since at boundary, hence

$$
\begin{aligned}
\frac{T_{3,3} - 2T_{4,3}}{\Delta r^2} + \frac{1}{3\Delta r}\frac{-T_{3,3}}{2\Delta r} + \frac{1}{9\Delta r^2}\frac{T_{4,2} - 2T_{4,3} + T_{4,4}}{\Delta \theta^2} &= -200\\
T_{3,3}\left(\frac{5}{6\Delta r^2}\right) + T_{4,3}\left(\frac{-2}{\Delta r^2} - \frac{2}{9\Delta r^2\Delta\theta^2}\right) + T_{4,2}\left(\frac{1}{9\Delta r^2\Delta\theta^2}\right) + T_{4,4}\left(\frac{1}{9\Delta r^2\Delta\theta^2}\right) &= -200 \qquad (4,3)
\end{aligned}
$$

Now we move to the central line. On this line the PDE is $\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = -100$.

Hence at node $(2,4)$ we have

$$
\begin{aligned}
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} &= -100\\
\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} &= -100\\
\frac{T_{1,4} - 2T_{2,4} + T_{3,4}}{\Delta r^2} + \frac{1}{\Delta r}\frac{T_{3,4} - T_{1,4}}{2\Delta r} + \frac{1}{\Delta r^2}\frac{T_{2,3} - 2T_{2,4} + T_{2,5}}{\Delta \theta^2} &= -100\\
T_{1,4}\left(\frac{1}{\Delta r^2} - \frac{1}{2\Delta r^2}\right) + T_{2,4}\left(-\frac{2}{\Delta r^2} - \frac{2}{\Delta r^2\Delta\theta^2}\right) + T_{3,4}\left(\frac{1}{\Delta r^2} + \frac{1}{2\Delta r^2}\right) + T_{2,3}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right) + T_{2,5}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right) &= -100
\end{aligned}
$$

But $T_{1,4} = T_{1,1}$. This is the same node. Hence the above becomes

$$
T_{1,1}\left(\frac{1}{2\Delta r^2}\right) + T_{2,4}\left(-\frac{2}{\Delta r^2} - \frac{2}{\Delta r^2\Delta\theta^2}\right) + T_{3,4}\left(\frac{3}{2\Delta r^2}\right) + T_{2,3}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right) + T_{2,5}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right) = -100 \qquad (2,4)
$$

At node $(3,4)$

$$\begin{aligned}
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} &= -100 \\
\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} &= -100 \\
\frac{T_{2,4} - 2T_{3,4} + T_{4,4}}{\Delta r^2} + \frac{1}{2\Delta r}\frac{T_{4,4} - T_{2,4}}{2\Delta r} + \frac{1}{4\Delta r^2}\frac{T_{3,3} - 2T_{3,4} + T_{3,5}}{\Delta \theta^2} &= -100 \\
T_{2,4}\left(\frac{3}{4\Delta r^2}\right) + T_{3,4}\left(\frac{-2}{\Delta r^2} - \frac{1}{2\Delta r^2\Delta\theta^2}\right) + T_{4,4}\left(\frac{5}{4\Delta r^2}\right) + T_{3,3}\left(\frac{1}{4\Delta r^2\Delta\theta^2}\right) + T_{3,5}\left(\frac{1}{4\Delta r^2\Delta\theta^2}\right) &= -100 \qquad (3,4)
\end{aligned}$$

At node $(4,4)$

$$\begin{aligned}
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} &= -100 \\
\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} &= -100 \\
\frac{T_{3,4} - 2T_{4,4} + T_{5,4}}{\Delta r^2} + \frac{1}{3\Delta r}\frac{T_{5,4} - T_{3,4}}{2\Delta r} + \frac{1}{9\Delta r^2}\frac{T_{4,3} - 2T_{4,4} + T_{4,5}}{\Delta \theta^2} &= -100
\end{aligned}$$

But $T_{5,4} = 0$ since at B.C. hence

$$\begin{aligned}
\frac{T_{3,4} - 2T_{4,4}}{\Delta r^2} + \frac{1}{3\Delta r}\frac{-T_{3,4}}{2\Delta r} + \frac{1}{9\Delta r^2}\frac{T_{4,3} - 2T_{4,4} + T_{4,5}}{\Delta \theta^2} &= -100 \\
T_{3,4}\left(\frac{5}{6\Delta r^2}\right) + T_{4.4}\left(\frac{-2}{\Delta r^2} + \frac{-2}{9\Delta r^2\Delta\theta^2}\right) + T_{4,3}\left(\frac{1}{9\Delta r^2\Delta\theta^2}\right) + T_{4,5}\left(\frac{1}{9\Delta r^2\Delta\theta^2}\right) &= -100 \qquad (4,4)
\end{aligned}$$

This completes the central line, now we move $\Delta\theta$ anti-clock wise and process the next grid line.

Node $(2,5)$ is in the left side, where the PDE is $\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$. Hence

$$\begin{aligned}
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} &= 0 \\
\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} &= 0 \\
\frac{T_{1,5} - 2T_{2,5} + T_{3,5}}{\Delta r^2} + \frac{1}{\Delta r}\frac{T_{3,5} - T_{1,5}}{2\Delta r} + \frac{1}{\Delta r^2}\frac{T_{2,4} - 2T_{2,5} + T_{2,6}}{\Delta \theta^2} &= 0 \\
T_{1,1}\left(\frac{1}{2\Delta r^2}\right) + T_{2,5}\left(\frac{-2}{\Delta r^2} - \frac{2}{\Delta r^2\Delta\theta^2}\right) + T_{3,5}\left(\frac{3}{2\Delta r^2}\right) + T_{2,4}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right) + T_{2,6}\left(\frac{1}{\Delta r^2\Delta\theta^2}\right) &= 0 \qquad (2,5)
\end{aligned}$$

At node $(3,5)$

$$\begin{aligned}
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} &= 0 \\
\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} &= 0 \\
\frac{T_{2,5} - 2T_{3,5} + T_{4,5}}{\Delta r^2} + \frac{1}{2\Delta r}\frac{T_{4,5} - T_{2,5}}{2\Delta r} + \frac{1}{4\Delta r^2}\frac{T_{3,4} - 2T_{3,5} + T_{3,6}}{\Delta \theta^2} &= 0 \\
T_{2,5}\left(\frac{3}{4\Delta r^2}\right) + T_{3,5}\left(\frac{-2}{\Delta r^2} - \frac{1}{2\Delta r^2\Delta\theta^2}\right) + T_{4,5}\left(\frac{5}{4\Delta r^2}\right) + T_{3,4}\left(\frac{1}{4\Delta r^2\Delta\theta^2}\right) + T_{3,6}\left(\frac{1}{4\Delta r^2\Delta\theta^2}\right) &= 0 \qquad (3,5)
\end{aligned}$$

At node $(4,5)$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

$$\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} = 0$$

$$\frac{T_{3,5} - 2T_{4,5} + T_{5,5}}{\Delta r^2} + \frac{1}{3\Delta r}\frac{T_{5,5} - T_{3,5}}{2\Delta r} + \frac{1}{9\Delta r^2}\frac{T_{4,4} - 2T_{4,5} + T_{4,6}}{\Delta \theta^2} = 0$$

But $T_{5,5} = 0$ since at boundary

$$\frac{T_{3,5} - 2T_{4,5}}{\Delta r^2} + \frac{1}{3\Delta r}\frac{-T_{3,5}}{2\Delta r} + \frac{1}{9\Delta r^2}\frac{T_{4,4} - 2T_{4,5} + T_{4,6}}{\Delta \theta^2} = 0$$

$$T_{3,5}\left(\frac{5}{6\Delta r^2}\right) + T_{4,5}\left(\frac{-2}{\Delta r^2} - \frac{2}{9\Delta r^2 \Delta \theta^2}\right) + T_{4,4}\left(\frac{1}{9\Delta r^2 \Delta \theta^2}\right) + T_{4,6}\left(\frac{1}{9\Delta r^2 \Delta \theta^2}\right) = 0 \qquad (4,5)$$

At node $(2,6)$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

$$\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} = 0$$

$$\frac{T_{1,6} - 2T_{2,6} + T_{3,6}}{\Delta r^2} + \frac{1}{\Delta r}\frac{T_{3,6} - T_{1,6}}{2\Delta r} + \frac{1}{\Delta r^2}\frac{T_{2,5} - 2T_{2,6} + T_{2,7}}{\Delta \theta^2} = 0$$

$$T_{1,1}\left(\frac{1}{2\Delta r^2}\right) + T_{2,6}\left(\frac{-2}{\Delta r^2} - \frac{2}{\Delta r^2 \Delta \theta^2}\right) + T_{3,6}\left(\frac{3}{2\Delta r^2}\right) + T_{2,5}\left(\frac{1}{\Delta r^2 \Delta \theta^2}\right) + T_{2,7}\left(\frac{1}{\Delta r^2 \Delta \theta^2}\right) = 0 \qquad (2,6)$$

At node $(3,6)$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

$$\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} = 0$$

$$\frac{T_{2,6} - 2T_{3,6} + T_{4,6}}{\Delta r^2} + \frac{1}{2\Delta r}\frac{T_{4,6} - T_{2,6}}{2\Delta r} + \frac{1}{4\Delta r^2}\frac{T_{3,5} - 2T_{3,6} + T_{3,7}}{\Delta \theta^2} = 0$$

$$T_{2,6}\left(\frac{3}{4\Delta r^2}\right) + T_{3,6}\left(\frac{-2}{\Delta r^2} - \frac{1}{2\Delta r^2 \Delta \theta^2}\right) + T_{4,6}\left(\frac{5}{4\Delta r^2}\right) + T_{3,5}\left(\frac{1}{4\Delta r^2 \Delta \theta^2}\right) + T_{3,7}\left(\frac{1}{4\Delta r^2 \Delta \theta^2}\right) = 0 \qquad (3,6)$$

At node $(4,6)$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

$$\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{\Delta r^2} + \frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{\Delta \theta^2} = 0$$

$$\frac{T_{3,6} - 2T_{4,6} + T_{5,6}}{\Delta r^2} + \frac{1}{3\Delta r}\frac{T_{5,6} - T_{3,6}}{2\Delta r} + \frac{1}{9\Delta r^2}\frac{T_{4,5} - 2T_{4,6} + T_{4,7}}{\Delta \theta^2} = 0$$

But $T_{5,6} = 0$ since at boundary hence

$$\frac{T_{3,6}-2T_{4,6}}{\Delta r^2}+\frac{1}{3\Delta r}\frac{-T_{3,6}}{2\Delta r}+\frac{1}{9\Delta r^2}\frac{T_{4,5}-2T_{4,6}+T_{4,7}}{\Delta\theta^2}=0$$

$$T_{3,6}\left(\frac{5}{6\Delta r^2}\right)+T_{4,6}\left(\frac{-2}{\Delta r^2}-\frac{2}{9\Delta r^2\Delta\theta^2}\right)+T_{4,5}\left(\frac{1}{9\Delta r^2\Delta\theta^2}\right)+T_{4,7}\left(\frac{1}{9\Delta r^2\Delta\theta^2}\right)=0 \tag{4,6}$$

We now move to the bottom grid line at $\theta=\pi$, where it is insulated.

At node $(2,7)$

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}=0$$

$$\frac{T_{1,7}-2T_{2,7}+T_{3,7}}{\Delta r^2}+\frac{1}{\Delta r}\frac{T_{3,7}-T_{1,7}}{2\Delta r}+\frac{1}{\Delta r^2}\frac{T_{2,6}-2T_{2,7}+T_{2,8}}{\Delta\theta^2}=0$$

But $T_{2,8}=T_{2,5}$ due to insulation and $T_{1,7}=T_{1,1}$ since same node, hence above becomes

$$\frac{T_{1,1}-2T_{2,7}+T_{3,7}}{\Delta r^2}+\frac{1}{\Delta r}\frac{T_{3,7}-T_{1,1}}{2\Delta r}+\frac{1}{\Delta r^2}\frac{2T_{2,6}-2T_{2,7}}{\Delta\theta^2}=0$$

$$T_{1,1}\left(\frac{1}{2\Delta r^2}\right)+T_{2,7}\left(\frac{-2}{\Delta r^2}-\frac{2}{\Delta r^2\Delta\theta^2}\right)+T_{3,7}\left(\frac{3}{2\Delta r^2}\right)+T_{2,6}\left(\frac{2}{\Delta r^2\Delta\theta^2}\right)=0 \tag{2,7}$$

At node $(3,7)$

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}=0$$

$$\frac{T_{i-1,j}-2T_{ij}+T_{i+1,j}}{\Delta r^2}+\frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j}-T_{i-1,j}}{2\Delta r}+\frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1}-2T_{ij}+T_{i,j+1}}{\Delta\theta^2}=0$$

$$\frac{T_{2,7}-2T_{3,7}+T_{4,7}}{\Delta r^2}+\frac{1}{2\Delta r}\frac{T_{4,7}-T_{2,7}}{2\Delta r}+\frac{1}{4\Delta r^2}\frac{T_{3,6}-2T_{3,7}+T_{3,8}}{\Delta\theta^2}=0$$

But $T_{3,8}=T_{3,6}$ due to insulation, hence

$$\frac{T_{2,7}-2T_{3,7}+T_{4,7}}{\Delta r^2}+\frac{1}{2\Delta r}\frac{T_{4,7}-T_{2,7}}{2\Delta r}+\frac{1}{4\Delta r^2}\frac{2T_{3,6}-2T_{3,7}}{\Delta\theta^2}=0$$

$$T_{2,7}\left(\frac{3}{4\Delta r^2}\right)+T_{3,7}\left(\frac{-2}{\Delta r^2}-\frac{1}{2\Delta r^2\Delta\theta^2}\right)+T_{4,7}\left(\frac{5}{4\Delta r^2}\right)+T_{3,6}\left(\frac{1}{2\Delta r^2\Delta\theta^2}\right)=0 \tag{3,7}$$

At node $(4,7)$

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}=0$$

$$\frac{T_{i-1,j}-2T_{i,j}+T_{i+1,j}}{\Delta r^2}+\frac{1}{(i-1)(\Delta r)}\frac{T_{i+1,j}-T_{i-1,j}}{2\Delta r}+\frac{1}{[(i-1)(\Delta r)]^2}\frac{T_{i,j-1}-2T_{ij}+T_{i,j+1}}{\Delta\theta^2}=0$$

$$\frac{T_{3,7}-2T_{4,7}+T_{5,7}}{\Delta r^2}+\frac{1}{3\Delta r}\frac{T_{5,7}-T_{3,7}}{2\Delta r}+\frac{1}{9\Delta r^2}\frac{T_{4,6}-2T_{4,7}+T_{4,8}}{\Delta\theta^2}=0$$

But $T_{4,8}=T_{4,6}$ due to insulation and $T_{5,7}=0$ since on boundary, hence

$$
\begin{aligned}
\frac{T_{3,7}-2T_{4,7}}{\Delta r^2}+\frac{1}{3\Delta r}\frac{-T_{3,7}}{2\Delta r}+\frac{1}{9\Delta r^2}\frac{2T_{4,6}-2T_{4,7}}{\Delta\theta^2} &= 0\\
T_{3,7}\left(\frac{5}{6\Delta r^2}\right)+T_{4,7}\left(\frac{-2}{\Delta r^2}-\frac{2}{9\Delta r^2\Delta\theta^2}\right)+T_{4,6}\left(\frac{2}{9\Delta r^2\Delta\theta^2}\right) &= 0
\end{aligned}
\tag{4,7}
$$

Now we are able to see the $A$ matrix structure. The number of unknowns is 22. Let $\alpha_1=\left(\frac{-2}{\Delta r^2}-\frac{1}{2\Delta r^2\Delta\theta^2}\right)$, $\alpha_2=\left(\frac{-2}{\Delta r^2}-\frac{2}{\Delta r^2\Delta\theta^2}\right)$, $\alpha_3=\left(\frac{-2}{\Delta r^2}-\frac{2}{9\Delta r^2\Delta\theta^2}\right)$, $\beta=\frac{1}{\Delta r^2}$, $\gamma=\frac{1}{\Delta r^2\Delta\theta^2}$ then the above equations can now be written as $Ax=f$

$$
\begin{pmatrix}
1 & -\frac{1}{12} & 0 & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{12} & 0 & 0\\
\frac{1}{2}\beta & \alpha_2 & \frac{3}{2}\beta & 0 & 2\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{3}{4}\beta & \alpha_1 & \frac{5}{4}\beta & 0 & \frac{1}{2}\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{5}{6}\beta & \alpha_3 & 0 & 0 & \frac{2}{9}\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2}\beta & \gamma & 0 & 0 & \alpha_2 & \frac{3}{2}\beta & 0 & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{4}\gamma & 0 & \frac{3}{4}\beta & \alpha_1 & \frac{5}{4}\beta & 0 & \frac{1}{4}\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{9}\gamma & 0 & \frac{5}{4}\beta & \alpha_3 & 0 & 0 & \frac{1}{9}\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2}\beta & 0 & 0 & 0 & \gamma & 0 & 0 & \alpha_2 & \frac{3}{2}\beta & 0 & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{4}\gamma & 0 & \frac{3}{4}\beta & \alpha_1 & \frac{5}{4}\beta & 0 & \frac{1}{4}\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9}\gamma & 0 & \frac{5}{6}\beta & \alpha_3 & 0 & 0 & \frac{1}{9}\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2}\beta & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 & 0 & \alpha_2 & \frac{3}{2}\beta & 0 & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}\gamma & 0 & \frac{3}{4}\beta & \alpha_1 & \frac{5}{4}\beta & 0 & \frac{1}{4}\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9}\gamma & 0 & \frac{5}{6}\beta & \alpha_3 & 0 & 0 & \frac{1}{9}\gamma & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2}\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 & 0 & \alpha_3 & \frac{3}{2}\beta & 0 & \gamma & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}\gamma & 0 & \frac{3}{4}\beta & \alpha_1 & \frac{5}{4}\beta & 0 & \frac{1}{4}\gamma & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9}\gamma & 0 & \frac{5}{6}\beta & \alpha_3 & 0 & 0 & \frac{1}{9}\gamma & 0 & 0 & 0\\
\frac{1}{2}\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 & \alpha_3 & \frac{3}{2}\beta & 0 & \gamma & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}\gamma & 0 & \frac{3}{4}\beta & \alpha_1 & \frac{5}{4}\beta & 0 & \frac{1}{4}\gamma & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9}\gamma & 0 & \frac{5}{6}\beta & \alpha_3 & 0 & 0 & \frac{1}{9}\gamma\\
\frac{1}{2}\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\gamma & 0 & 0 & \alpha_2 & \frac{3}{2}\beta & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\gamma & 0 & \frac{3}{4}\beta & \alpha_1 & \frac{5}{4}\beta\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{9}\gamma & 0 & \frac{5}{6}\beta & \alpha_3
\end{pmatrix}
\begin{pmatrix}
T_{11}\\ T_{21}\\ T_{31}\\ T_{41}\\ T_{22}\\ T_{32}\\ T_{42}\\ T_{23}\\ T_{33}\\ T_{43}\\ T_{24}\\ T_{34}\\ T_{44}\\ T_{25}\\ T_{35}\\ T_{45}\\ T_{26}\\ T_{36}\\ T_{46}\\ T_{27}\\ T_{37}\\ T_{47}
\end{pmatrix}
=
\begin{pmatrix}
25\Delta r^2\\ -200\\ -200\\ -200\\ -200\\ -200\\ -200\\ -200\\ -200\\ -200\\ -100\\ -100\\ -100\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0
\end{pmatrix}
$$

We now can see the pattern to use. The above is solved using $Ax=f$ in Matlab.

The solution below. The temperature at origin was found to be 4.737 degrees and the maximum was 6.666 degrees.

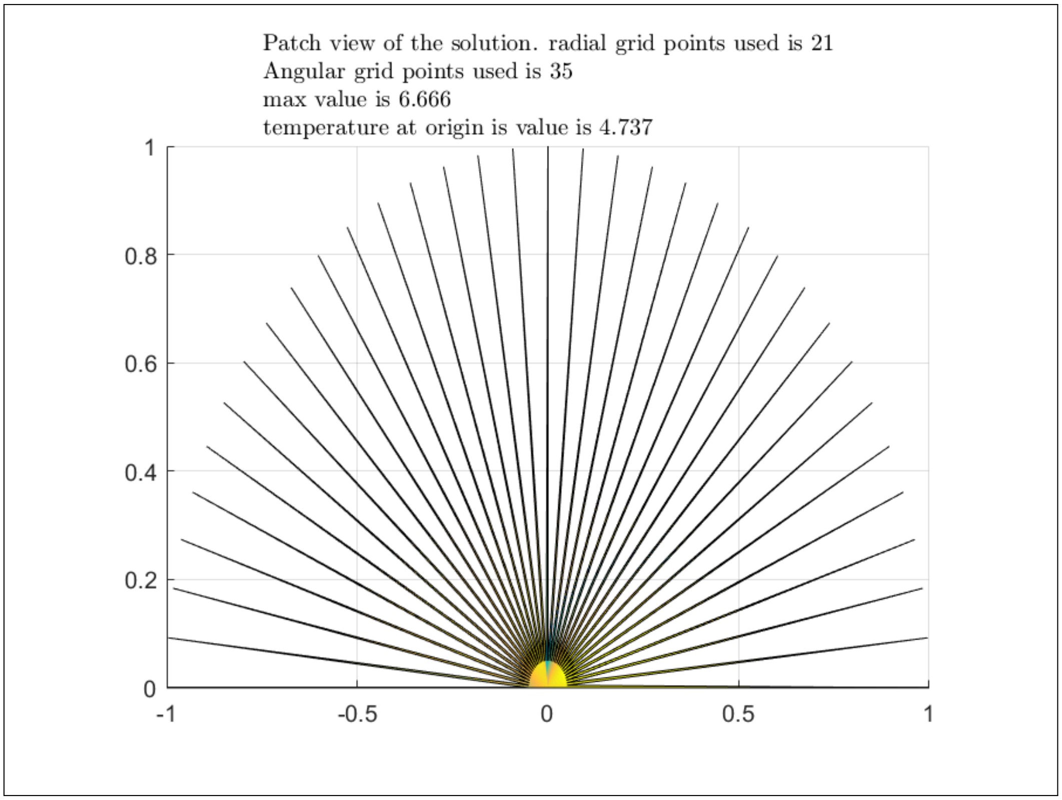



Figure 18: problem 3 solution using Matlab patch command

```
function nma_EMA_471_HW6_problem_3
%solution to problem 3, HW6, EMA 471
%EMA 471
%
%Please see report for derivation of the A matrix for
%this problem This uses the direct method, not iterative. Was
%very time consuming to make the A matrix. An iterative method
%would have been simpler
%
%version completed after midnight. 4/21/2016

close all; clc;
Nr         = 21;  %grid points in radial direction
Na         = 35;  %grid points in angular direction
n          = (Nr-2)*Na+1;  %number of unknowns
delR       = 1/(Nr-1);  %grid spacing radial direction
delAngle   = pi/(Na-1); %grid spacing angular direction in radians
A          = make_A_matrix(Nr,Na,n,delR,delAngle);
f          = ones(n,1);     %RHS.
f(1)       = 25*delR^2;     %this where the origin maps to
f(2:(Nr-2)*(Na-1)/2+1)  = -200;  %right side of disk
f((Nr-2)*(Na-1)/2:(Nr-2)*(Na-1)/2+(Nr-2))  = -100;  %center line

%left side of disk
f((Nr-2)*(Na-1)/2:(Nr-2)*(Na-1)/2+(Nr-2)+1:end) = 0;

u = A\f;  %direct (All this hard work, just to make this one call)
```

```

fprintf('T_0 = %6.4f\n',u(1));

%find X,Y coordinates for patch command. I could use cart2pol()
%also, but need to patch the coordinates in the process to add
%the boundary conditions, which has known solution of zero,
%and loop seemed easier

X   = zeros((Nr-1)*Na+1,1);  %X coordinates
Y   =   X;
sol = X;
for i=1:Nr
    if i==1   %this is origin
        sol(i) = u(i);
        X(i)   = 0;
        Y(i)   = 0;
    else
         for j = 0:Na-1

             %watch out. coordinate index is different
             %than solution index, since we are
             %adding boundary conditions
             coord_idx = i+j*(Nr-1);

             if i==Nr
                 sol_idx = i+j*(Nr-1);
                 sol(sol_idx)=0;  %B.C.
             else
                 sol_idx = i+j*(Nr-2);
                 sol(sol_idx)=u(sol_idx);
             end
             X(coord_idx)=(i-1)*delR*cos(j*delAngle);
             Y(coord_idx)=(i-1)*delR*sin(j*delAngle);
         end
     end
end

%make patch plot and print the solution
patch(X,Y,sol);
grid;

title({sprintf('Patch view of the solution. radial grid points used is %d',Nr),  ...
    sprintf('Angular grid points used is %d',Na),...
    sprintf('max value is $%6.3f$',max(sol)),...
    sprintf('temperature at origin is value is $%6.3f$',sol(1))},...
    'Interpreter', 'latex','fontsize',10);

end
%==================================
function A = make_A_matrix(Nr,Na,n,delR,delAngle)
%please see report for details. This is complicated A matrix due to
%the geometry

a1    = (-2/delR^2-1/(2*delR^2*delAngle^2));
a2    = (-2/delR^2-2/(delR^2*delAngle^2));
a3    = (-2/delR^2-2/(9*delR^2*delAngle^2));
b     =  1/delR^2;
gamma = 1/(delR^2*delAngle^2);
z     = 1;  %this is used to count the angular jumps.
```

```
            %Needed to sync with

A = zeros(n);

%make first line. This is the bottom edge, right size of disk
A(1,1)   = 1;
A(1,2)   = -1/(2*(Na-1));
for j = Nr:(Nr-2):(Nr+(Nr-2)*(Na-3))
    A(1,j) = -1/(Na-1);
end
A(1,(Nr-2)*Na-1) = -1/(2*(Na-1));


for i=2:(Nr-2):2+(Nr-2)*(Na-1)
    if i==2 || i==2+(Nr-2)*(Na-1)   %these are the lower edges
        if i == 2 %first edge, at theta=0
            zz = 1; %to help me find where I am in the matrix
            for k=i:i+Nr-3 %process each radial line
                           %(internal nodes)
                if k==i %first in block
                    A(k,1)=(1/2)*b;
                    A(k,2)=a2;
                    A(k,3)=(3/2)*b;
                    A(k,Nr)=2*gamma;
                elseif k==i+Nr-3 %last
                    A(k,Nr-2)=(5/6)*b;
                    A(k,Nr-1)=a3;
                    A(k,Nr-1+Nr-2)=2/9*gamma;
                else
                    zz=zz+1;
                    A(k,zz)=(3/5)*b;
                    A(k,zz+1)=a1;
                    A(k,zz+2)=5/4*b;
                    A(k,zz+Nr-2)=1/2*gamma;
                end
            end
        else %last edge, at theta=pi
            zz=0;
            for k=i:i+Nr-3 %process each radial line (internal)
                z0=2+(Nr-2)*(Na-1); %where last edge node starts
                if k==i %first in block
                    A(k,z0)=a2;
                    A(k,z0-Nr-2)=2*gamma;
                    A(k,z0+1)=3/2*b;
                elseif k==i+Nr-3 %last
                    zz=zz+1;
                    A(k,z0+zz)=a3;
                    A(k,z0+zz-1)=5/6*b;
                    A(k,z0+zz-(Nr-2))=2/9*gamma;
                else
                    zz=zz+1;
                    A(k,z0+zz)=a1;
                    A(k,z0+zz-1)=(3/5)*b;
                    A(k,z0+zz+1)=5/4*b;
                    A(k,z0+zz-(Nr-2))=1/2*gamma;
                end
            end
        end
    else  %internal radial lines
```

```
        for k=i:i+Nr-3 %process each radial line (internal nodes)
            if k==i %first in block
                z = z + 1;
                A(k,1)=(1/2)*b;
                A(k,z)=gamma;
                A(k,z+(Nr-2))=a2;
                A(k,z+1+(Nr-2))=3/2*b;
                A(k,z+2*(Nr-2))=gamma;
            elseif k==i+Nr-3 %last
                z = z +1;
                A(k,z)=1/9*gamma;
                A(k,z+(Nr-3))=(5/4)*b;
                A(k,z+1+(Nr-3))=a3;
                A(k,z+2*(Nr-2))=1/8*gamma;
            else
                z=z+1; %to tag where entries start
                A(k,z)=(1/4)*gamma;
                A(k,z+(Nr-3))=(3/4)*b;
                A(k,z+1+(Nr-3))=a1;
                A(k,z+2+(Nr-3))=5/4*b;
                A(k,z+2*(Nr-2))=(1/4)*gamma;
            end
        end
    end
end
end
```