HW5 EMA 471 Intermediate Problem Solving for Engineers

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Engineering Mechanics Department
University of Wisconsin, Madison

Instructor: Professor Robert J. Witt

By

Nasser M. Abbasi

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0.1 Problem 1

EP 471 – Homework #5
Due: Thursday, April 7th, 2016

(1) (8 pts) It is stated without proof in Exercise 15 that Gaussian quadrature of order N produces an exact result when applied to the integration of a polynomial of order $2N - 1$. Consider the following polynomials in the interval $0 \le x \le 4$:

(a) $f_1(x) = x^5$

(b) $f_2(x) = x^7$

Integrate (a) and (b) over the interval $0 \le x \le 4$ using Gaussian quadrature of $N = 3$ and 4 respectively and compare your results to the analytical values. Does the quadrature of the appropriate order produce an exact match?

Figure 1: problem 1 description

A polynomial $f(x)$ of order p is integrated exactly with the Gaussian quadrature method using $\frac{p+1}{2}$ number of Gaussian points.

Hence $f_1(x) = x^5$ needs $\frac{5+1}{2} = 3$ Gaussian points and $f_2(x) = x^7$ needs $\frac{7+1}{2} = 4$ Gaussian points for exact result.

The integral $\int_a^b f(x)\,dx$ is first converted to be in the domain $\{-1, +1\}$ as follows

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{+1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) dt$$

For a polynomial $f(x)$ of order $p = 3$, two Gaussian points are needed to evaluate the above integral exactly. Therefore the above integral simplifies to

$$\int_a^b f(x)\,dx = A\left(w_1 f(At_1 + B) + w_2 f(At_2 + B)\right)$$

Where

$$A = \frac{b-a}{2}$$
$$B = \frac{b+a}{2}$$

And w_i is the weight at location t_i. The weights and location of the weights are obtained from tables. For higher order polynomials, more points and weights are needed.

In general, using N points the integral is

$$\begin{aligned}\int_a^b f(x)\,dx &= \frac{b-a}{2}\sum_{i=1}^{N} w_i f\left(\frac{b-a}{2}t_i + \frac{a+b}{2}\right)\\ &= A\sum_{i=1}^{N} w_i f(At_i + B)\end{aligned} \tag{1}$$

The program nma_EMA_471_HW5_problem_1.m integrates the above two polynomials $f_1(x), f_2(x)$ using Gaussian quadrature method (1) using $N = 3$ and $N = 4$ points respectively and compares the result of each to the analytical solution. The following table shows the result.

Table 1: HW5, problem 1 result

function	analytical result	N	Gaussian quadrature result
$f_1(x) = x^4$	$\int_0^4 x^4\,dx = \frac{2048}{3} = 682.666666666667$	3	682.666666666667
$f_1(x) = x^7$	$\int_0^4 x^7\,dx = 8192$	4	8192

The result is exact. Note: The above Matlab program used the exact weights and points for Gaussian quadrature as given in `https://en.wikipedia.org/wiki/Gaussian_quadrature`

```
function nma_EMA_471_HW5_problem_1()
%Solves problem 1, HW5, EMA 471
%Nasser M. Abbasi

%reference  https://en.wikipedia.org/wiki/Gaussian_quadrature for points
%and weights. These are exact.

gauss_3_points=[-sqrt(3/5) , 5/9;    %point, weight   per row
                0          , 8/9;
                sqrt(3/5)  , 5/9];

gauss_4_points=[-sqrt(3/7+ 2/7*sqrt(6/5)) , (18-sqrt(30))/36;
                -sqrt(3/7- 2/7*sqrt(6/5)) , (18+sqrt(30))/36;
                 sqrt(3/7- 2/7*sqrt(6/5)) , (18+sqrt(30))/36;
                 sqrt(3/7+ 2/7*sqrt(6/5)) , (18-sqrt(30))/36];

f1=@(x) x.^5;
integrate(f1,0,4,gauss_3_points)

f1=@(x) x.^7;
integrate(f1,0,4,gauss_4_points)

end
%====================================
function the_sum = integrate(f,from,to,g)
%INPUT:   f  is handle to function to integrate
%         from,to these are lower and upper integral bounds
%         g  This is matrix of Gaussian quadrature. first column is points
%            second column is corresponding weights

A = (to-from)/2;
B = (to+from)/2;
i = 1:size(g,1);
the_sum = A * sum( g(i,2) .* f(A*g(i,1)+B) );   %vectored sum
end
```

0.2 Problem 2

(2) (12 pts) In one of our exercises we evaluated the following function by Gaussian quadrature:

$$I = \int_0^{2.5} \frac{\sin^2 x \cdot \sin(x^2)}{(1+x^2)^2} dx$$

using 2, 3 and 4 gauss points. In this case, the domain is large enough and the function changes sharply enough that we don't get very good agreement with Matlab's `quad` utility even for 4 gauss points. To get better agreement, we could use a larger number of Gauss points, or break the domain into pieces, $0 \le x \le 1.25$ and $1.25 \le x \le 2.5$, for example.

The following extends the table in Exercise 15 to 5, 6, 7 and 8 gauss points:

Number of terms, N	Values of t (t_i)	Weighting factor	Valid up to degree
5	0	0.56888889	9
	±0.53846931	0.47862867	
	±0.90617985	0.23692689	
6	±0.23861918	0.46791393	11
	±0.66120939	0.36076157	
	±0.93246951	0.17132449	
7	0	0.41795918	13
	±0.40584515	0.38183005	
	±0.74153119	0.27970539	
	±0.94910791	0.12948497	
8	±0.18343464	0.36268378	15
	±0.52553241	0.31370665	
	±0.79666648	0.22238103	
	±0.96028986	0.10122854	

(a) Keeping the original domain, $0 \le x \le 2.5$, how does the agreement with Matlab's `quad` utility improve with 5, 6, 7 and 8 point Gaussian quadrature?
(b) Breaking the domain into two pieces, $0 \le x \le 1.25$ and $1.25 \le x \le 2.5$, evaluate the integral using 5 and 6 gauss points in each subdomain. How does the agreement compare with results from Matlab's `quad` utility now?

Figure 2: problem 2 description

0.2.1 part a

The program `nma_EMA_471_HW5_problem_2_part_a.m` implements the first part of this problem.

The following table shows the result of the computation. It shows the result of the integral using Gaussian quadrature for different number of points with the relative error against Matlab's Quad (integral) command.

Number of points	relative error (percentage)	value of integral
2	79.845442	0.023377470178383
3	23.892753	0.143704430262801
4	3.060319	0.112441294428254
5	4.692010	0.121433298541329
6	0.019015	0.116013045399658
7	0.011820	0.116004700249995
8	0.001535	0.115989208171974

Table 2: Gaussian quadrature using different points $\int_0^{2.5} \frac{\sin^2 x \sin x^2}{(1+x^2)^2}\, dx$. Compared with Matlab Quad result of 0.115990989197426 for same integral

The following is a plot of the above data

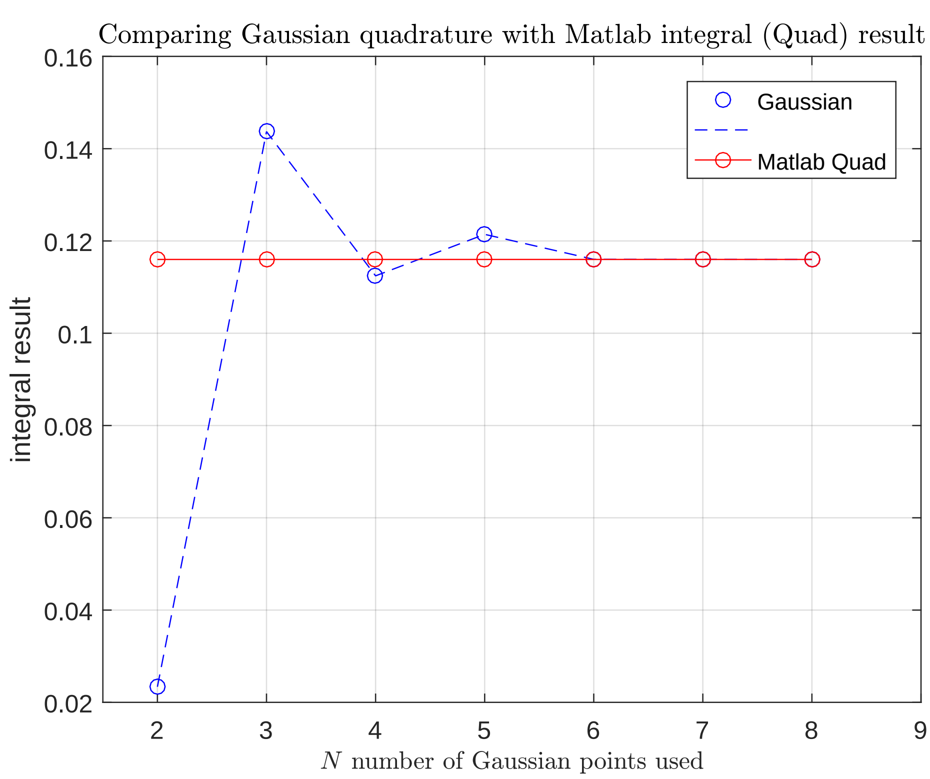


Figure 3: Comparing Gaussian quadrature with Matlab's integral result

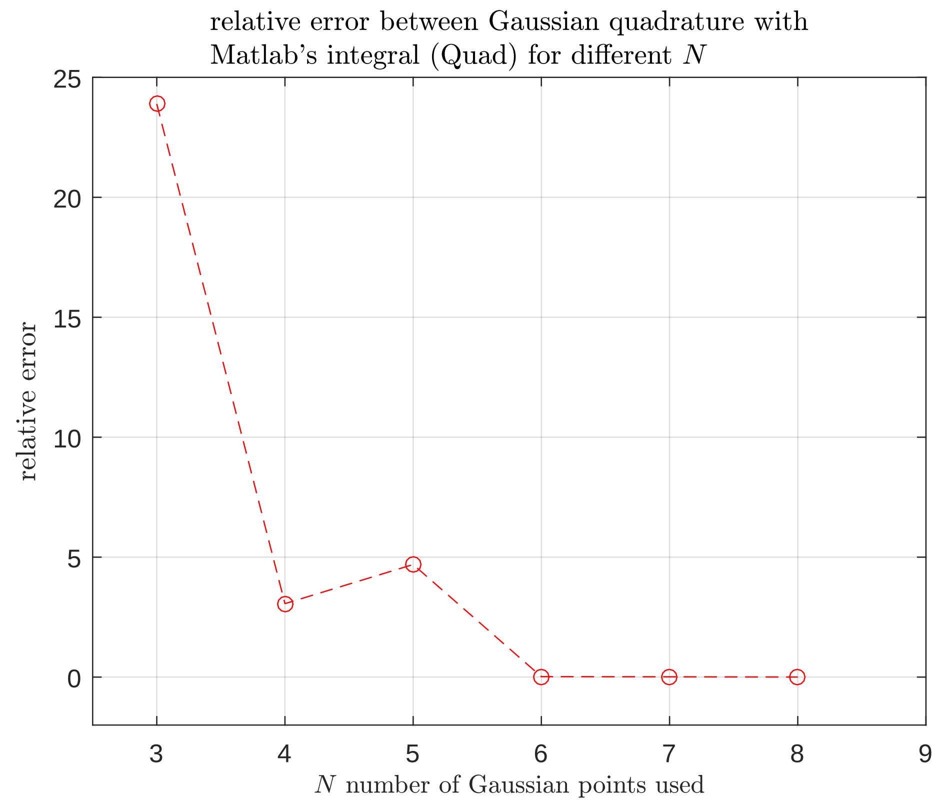


Figure 4: Relative error for different N values

0.2.2 part b

The program `nma_EMA_471_HW5_problem_2_part_b.m` implements the second part of this problem. By breaking the domain into 2 parts, the following table shows the result of the computation. It shows the result of the integral using Gaussian quadrature for 5 and 6 points with the relative error against Matlab's Quad (integral) command. The integration was done on each subdomain and the results added.

Number of points	relative error (percentage)	value of integral using Gaussian quadrature
5	1.231392	0.114562685084637
6	0.000303	0.115990636860983

Table 3: Gaussian quadrature using 5 and 6 points $\int_0^{1.25} \frac{\sin^2 x \sin x^2}{(1+x^2)^2}\,dx + \int_{1.25}^{2.5} \frac{\sin^2 x \sin x^2}{(1+x^2)^2}\,dx$. Compared with Matlab integral result of $\int_0^{2.5} \frac{\sin^2 x \sin x^2}{(1+x^2)^2}\,dx = 0.115990989197426$

The above shows clearly that by breaking the domain into two smaller parts, and adding each result, the final result of Gaussian quadrature improved compared to part(a) where one large domain was used. This makes sense. Because we have effectively used more sampling points in part(b) compared to part(a) when looking at the whole domain.

This shows that, to obtain more accuracy using Gaussian quadrature, and still use the same number of points N, then we can break the domain into smaller regions, and use N on each region, and add the result obtained from each region.

To see the difference between part(a) and (b) more clearly, the following table shows the result for 5 and 6 points side by side from part(a) and part(b). The table below shows the relative error is much smaller for part(b).

Number of points	relative error part(b)	relative error part(a)
5	1.231392	4.692010
6	0.000303	0.019015

Table 4: Gaussian quadrature using 5 and 6 points. Comparing part(a) and part(b) relative error against Matlab's Quad

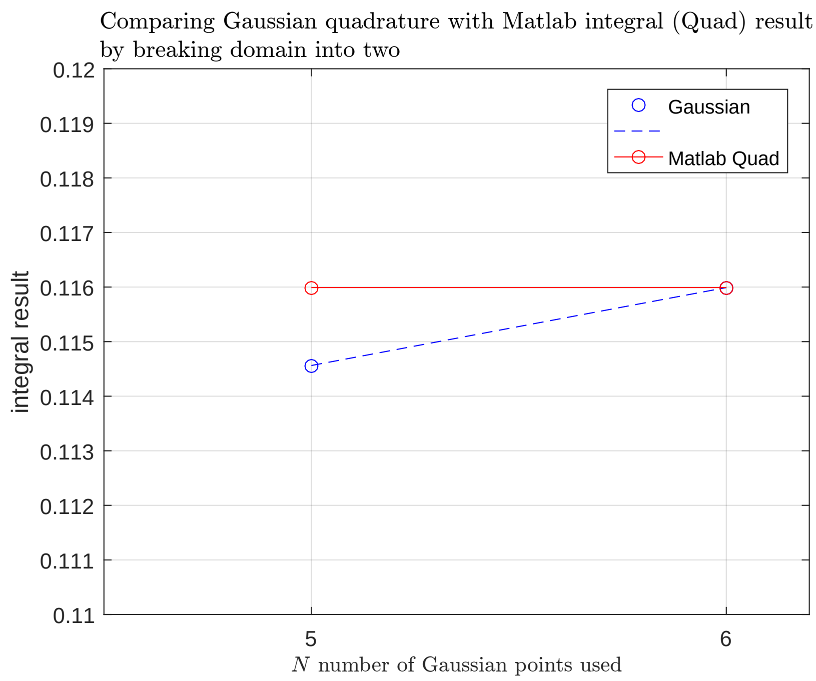


Figure 5: Comparing Gaussian quadrature with Matlab's integral result, part(b)

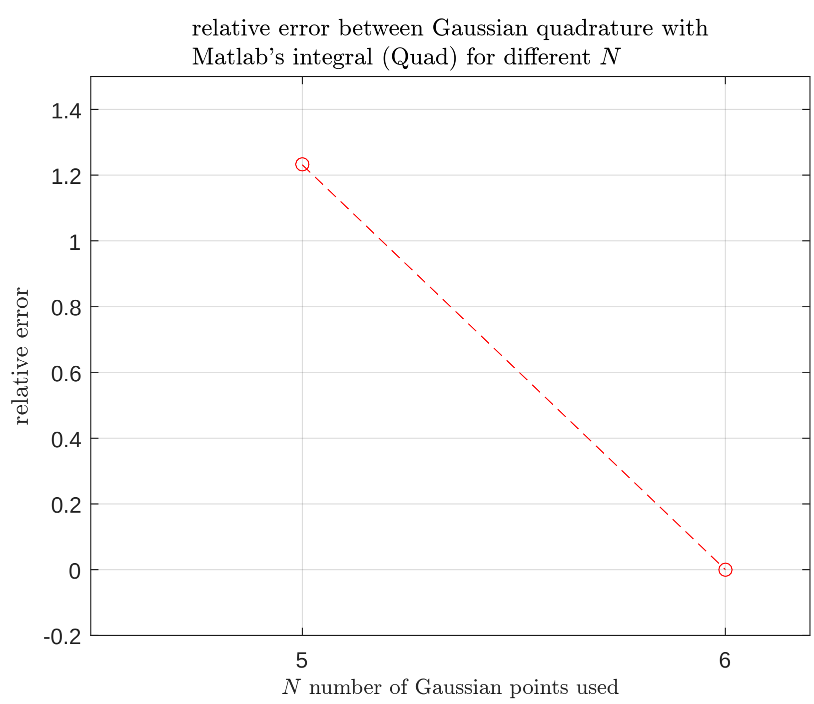


Figure 6: Relative error for different N values, part(b)

```
function nma_EMA_471_HW5_problem_2_part_a()
%Solves problem 2, part(a) HW5, EMA 471
%Nasser M. Abbasi

close all; clc;

gauss_2_points=[-0.57735027 , 1;    %point, weight   per row
                 0.57735027  , 1
                ];
```

```
gauss_3_points=[-sqrt(3/5) , 5/9;    %point, weight   per row
                0          , 8/9;
                sqrt(3/5)  , 5/9];

gauss_4_points=[-sqrt(3/7+ 2/7*sqrt(6/5)) , (18-sqrt(30))/36;
                -sqrt(3/7- 2/7*sqrt(6/5)) , (18+sqrt(30))/36;
                 sqrt(3/7- 2/7*sqrt(6/5)) , (18+sqrt(30))/36;
                 sqrt(3/7+ 2/7*sqrt(6/5)) , (18-sqrt(30))/36];

gauss_5_points=[0                           , 128/225;
            -(1/3)*sqrt(5-2*sqrt(10/6)) , (322+13*sqrt(70))/900;
             (1/3)*sqrt(5-2*sqrt(10/6)) , (322+13*sqrt(70))/900;
            -(1/3)*sqrt(5+2*sqrt(10/6)) , (322-13*sqrt(70))/900;
             (1/3)*sqrt(5+2*sqrt(10/6)) , (322-13*sqrt(70))/900];

gauss_6_points=[0.238619186083197 , 0.467913934572691;
               -0.238619186083197 , 0.467913934572691;
                0.661209386466265 , 0.360761573048139;
               -0.661209386466265 , 0.360761573048139;
                0.932469514203152 , 0.171324492379170;
               -0.932469514203152 , 0.171324492379170];

gauss_7_points=[0                 , 0.417959183673469;
                0.405845151377397 , 0.381830050505119;
               -0.405845151377397 , 0.381830050505119;
                0.741531185599394 , 0.279705391489277;
               -0.741531185599394 , 0.279705391489277;
                0.949107912342759 , 0.129484966168870;
               -0.949107912342759 , 0.129484966168870];


gauss_8_points=[0.183434642495650 , 0.362683783378361;
               -0.183434642495650 , 0.362683783378361;
                0.525532409916329 , 0.313706645877887;
               -0.525532409916329 , 0.313706645877887;
                0.796666477413627 , 0.222381034453374;
               -0.796666477413627 , 0.222381034453374;
                0.960289856497536 , 0.101228536290376;
               -0.960289856497536 , 0.101228536290376];

f=@(x) sin(x).^2 .* sin(x.^2) ./ (1+x.^2).^2 ;
x_min     = 0;
x_max     = 2.5;
data      = zeros(7,4);
chk       =  integral(f,x_min,x_max);
data(:,1) = chk;

for i=1:size(data,1)
    switch i
      case 1
           data(i,2) = integrate(f,x_min,x_max,gauss_2_points);
           data(i,3) = 100*abs(chk - data(i,2))/abs(chk);
           data(i,4) = 2;
      case 2
           data(i,2) = integrate(f,x_min,x_max,gauss_3_points);
           data(i,3) = 100*abs(chk - data(i,2))/abs(chk);
           data(i,4) = 3;
      case 3
           data(i,2) = integrate(f,x_min,x_max,gauss_4_points);
           data(i,3) = 100*abs(chk - data(i,2))/abs(chk);
           data(i,4) = 4;
      case 4
```

```
            data(i,2) = integrate(f,x_min,x_max,gauss_5_points);
            data(i,3) = 100*abs(chk - data(i,2))/abs(chk);
            data(i,4) = 5;
        case 5
            data(i,2) = integrate(f,x_min,x_max,gauss_6_points);
            data(i,3) = 100*abs(chk - data(i,2))/abs(chk);
            data(i,4) = 6;
        case 6
            data(i,2) = integrate(f,x_min,x_max,gauss_7_points);
            data(i,3) = 100*abs(chk - data(i,2))/abs(chk);
            data(i,4) = 7;
        case 7
            data(i,2) = integrate(f,x_min,x_max,gauss_8_points);
            data(i,3) = 100*abs(chk - data(i,2))/abs(chk);
            data(i,4) = 8;
    end
end

figure;
plot(data(:,4),data(:,2),'bo',data(:,4),data(:,2),'b--');
hold on;
plot(data(:,4),data(:,1),'r-o');
xlim([1.5,9]);
xlabel('$N$ number of Gaussian points used', ...
        'interpreter','Latex','Fontsize',10);
ylabel('integral result');
title('Comparing Gaussian quadrature with Matlab integral (Quad) result',...
       'interpreter','Latex');
legend('Gaussian','','Matlab Quad');
grid;

figure;
plot(data(2:end,4),data(2:end,3),'ro',data(2:end,4),...
       data(2:end,3),'r--');
xlabel('$N$ number of Gaussian points used','interpreter',...
        'Latex','Fontsize',10);
ylabel('relative error','interpreter','Latex');
title({'relative error between Gaussian quadrature with',...
        'Matlab''s integral (Quad) for different $N$'},...
         'interpreter','Latex');
grid;
xlim([2.5,9]);
ylim([-2,25]);



end
%==================================
function the_sum = integrate(f,from,to,g)
%INPUT:   f  is handle to function to integrate
% from,to these are lower and upper integral bounds
% g  This is matrix of Gaussian quadrature. first column is points
% second column is corresponding weights

A = (to-from)/2;
B = (to+from)/2;
i = 1:size(g,1);
the_sum = A * sum( g(i,2) .* f(A*g(i,1)+B) );   %vectored sum
end
```

```
function nma_EMA_471_HW5_problem_2_part_b()
%Solves problem 2, part(b) HW5, EMA 471
%Nasser M. Abbasi

```

```
close all; clc;


gauss_5_points=[0                            , 128/225;
                -(1/3)*sqrt(5-2*sqrt(10/6)) , (322+13*sqrt(70))/900;
                 (1/3)*sqrt(5-2*sqrt(10/6)) , (322+13*sqrt(70))/900;
                -(1/3)*sqrt(5+2*sqrt(10/6)) , (322-13*sqrt(70))/900;
                 (1/3)*sqrt(5+2*sqrt(10/6)) , (322-13*sqrt(70))/900];

gauss_6_points=[0.238619186083197 , 0.467913934572691;
               -0.238619186083197 , 0.467913934572691;
                0.661209386466265 , 0.360761573048139;
               -0.661209386466265 , 0.360761573048139;
                0.932469514203152 , 0.171324492379170;
               -0.932469514203152 , 0.171324492379170];


f=@(x) sin(x).^2 .* sin(x.^2) ./ (1+x.^2).^2 ;
data      = zeros(2,4);
chk       =  integral(f,0,2.5);
data(:,1) = chk;

data(1,2) = integrate(f,0,1.25,gauss_5_points)+integrate(f,1.25,2.5,gauss_5_points);
data(1,3) = 100*abs(chk - data(1,2))/abs(chk);
data(1,4) = 5;

data(2,2) = integrate(f,0,1.25,gauss_6_points)+integrate(f,1.25,2.5,gauss_6_points);
data(2,3) = 100*abs(chk - data(2,2))/abs(chk);
data(2,4) = 6;

figure;
plot(data(:,4),data(:,2),'bo',data(:,4),data(:,2),'b--');
hold on;
plot(data(:,4),data(:,1),'r-o');
xlim([1.5,9]);
xlabel('$N$ number of Gaussian points used','interpreter','Latex','Fontsize',10);
ylabel('integral result');
title({'Comparing Gaussian quadrature with Matlab integral (Quad) result', ...
       'by breaking domain into two'},'interpreter','Latex');
legend('Gaussian','','Matlab Quad');
grid;
xlim([4.5,6.2]);
ylim([.11,.12]);
ax = gca;
ax.XTick = [5 6];


figure;
plot(data(:,4),data(:,3),'ro',data(:,4),data(:,3),'r--');
xlabel('$N$ number of Gaussian points used','interpreter','Latex','Fontsize',10);
ylabel('relative error','interpreter','Latex');
title({'relative error between Gaussian quadrature with',...
        'Matlab''s integral (Quad) for different $N$'},'interpreter','Latex');
grid;
xlim([4.5,6.2]);
ylim([-.2,1.5]);
ax = gca;
ax.XTick = [5 6];



end
%====================================
```

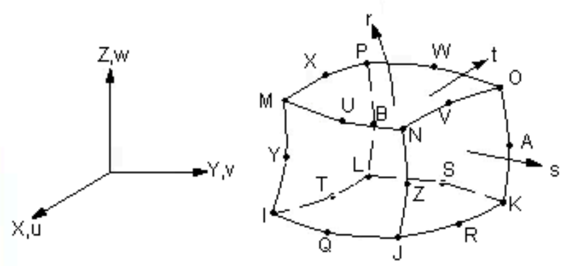
```
function the_sum = integrate(f,from,to,g)
%INPUT:   f  is handle to function to integrate
%         from,to these are lower and upper integral bounds
%         g  This is matrix of Gaussian quadrature. first column is points
%            second column is corresponding weights

A = (to-from)/2;
B = (to+from)/2;
i = 1:size(g,1);
the_sum = A * sum( g(i,2) .* f(A*g(i,1)+B) );   %vectored sum
end
```

0.3 Problem 3

(3) (20 pts) When using commercial software such as ANSYS, one can find the interpolation functions used for various element types in the Shape Functions section of the on-line Theory Manual. The most general 3D continuum elements are 20-node brick elements, and this figure from ANSYS lists the interpolation scheme:

Figure 12.17: 20-Node Brick Element



These shape functions are used for 20-node solid elements such as SOLID90:

$$
\begin{aligned}
u = \frac{1}{8}(&u_I(1-s)(1-t)(1-r)(-s-t-r-2) + u_J(1+s)(1-t)(1-r)(s-t-r-2) \\
&+ u_K(1+s)(1+t)(1-r)(s+t-r-2) + u_L(1-s)(1+t)(1-r)(-s+t-r-2) \\
&+ u_M(1-s)(1-t)(1+r)(-s-t+r-2) + u_N(1+s)(1-t)(1+r)(s-t+r-2) \\
&+ u_O(1+s)(1+t)(1+r)(s+t+r-2) + u_P(1-s)(1+t)(1+r)(-s+t+r-2)) \\
&+ \frac{1}{4}(u_Q(1-s^2)(1-t)(1-r) + u_R(1+s)(1-t^2)(1-r) \\
&+ u_S(1-s^2)(1+t)(1-r) + u_T(1-s)(1-t^2)(1-r) \\
&+ u_U(1-s^2)(1-t)(1+r) + u_V(1+s)(1-t^2)(1+r) \\
&+ u_W(1-s^2)(1+t)(1+r) + u_X(1-s)(1-t^2)(1+r) \\
&+ u_Y(1-s)(1-t)(1-r^2) + u_Z(1+s)(1-t)(1-r^2) \\
&+ u_A(1+s)(1+t)(1-r^2) + u_B(1-s)(1+t)(1-r^2))
\end{aligned}
$$

Note that in this representation, "*r*", "*s*" and "*t*" have replaced "xi" (ξ), "eta" (η) and "zeta" (ζ) as the natural coordinate system variables. The interpolation shown above is for displacement degree-of-freedom *u*, but this same interpolation holds for the other degrees-of-freedom as well as for the coordinates (*x*,*y*,*z*) within the element domain. Calculation of the volume of this element would be accomplished through:

Figure 7: problem 3 description

03/28/16

$$V = \int dx\,dy\,dz = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \frac{\partial(x,y,z)}{\partial(r,s,t)}\; dr\; ds\; dt \cong \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w_i w_j w_k \det J(r_i, s_j, t_k)$$

In the event that we need to find the response of a 3D structure to its own weight, we have to convert the continuously distributed weight density into a series of 20 discrete weights at each of the nodes. The work-equivalent finite element result is that the force at node i (i = I through B in the figure above) is found from:

$$F_i = \int \gamma(x,y,z) f_i(r,s,t)\, dx\, dy\, dz$$

$$= \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \gamma[x(r,s,t), y(r,s,t), z(r,s,t)] f_i(r,s,t) \frac{\partial(x,y,z)}{\partial(r,s,t)}\; dr\; ds\; dt$$

Here $\gamma = \rho g$ is the weight density and may be a function of position within the element. The f_i are the interpolation functions listed in the figure on the previous page. For example, for i = I,

$$f_i = f_I = \frac{1}{8}(1-r)(1-s)(1-t)(-r-s-t-2)$$

Aside: The interpolation functions are defined so that they go to one when their associated node is approached from inside the element. All the other interpolation functions go to zero at the same location, so that the interpolated quantity goes to the nodal quantity. In the case of node I, you'll notice from the figure that this corresponds to $r = s = t = -1$.

Given the nodal coordinates listed below for nodes I through B (taken from one element in a mesh prepared using ANSYS), and given the spatial dependence of the mass density, find the z-component forces to be applied to each of the nodes so that the weight is applied to the element in a work-equivalent fashion (assumes **g** points in the –z direction).

Nodal coordinates (x, y, z) are: (all values in units of cm):

Node I:	1	0	0
Node J:	1.1	0	0
Node K:	1.09066	0	−0.086305
Node L:	0.99692	0	−0.078459
Node M:	1.0077	0.16559	0
Node N:	1.1069	0.17203	0
Node O:	1.1035	0.17202	−0.08663
Node P:	1.0046	0.16557	−0.07882
Node Q:	1.05	0	0
Node R:	1.0992	0	−0.043186
Node S:	1.0468	0	−0.082382
Node T:	0.9923	0	−0.039260
Node U:	1.0573	0.16881	0
Node V:	1.1061	0.17202	−0.043328
Node W:	1.0540	0.16880	−0.082725
Node X:	1.0069	0.16558	−0.039418
Node Y:	1.0051	0.082737	0
Node Z:	1.1046	0.085964	0
Node A:	1.1012	0.085938	−0.086550
Node B:	1.0020	0.082709	−0.078731

03/28/16

The mass density (units of g/cm^3) is due to a functionally graded material and has the functional form:

$$\rho = \rho_o(x^2 + z^2) \qquad ; \qquad \rho_o = 1$$

Figure 8: more problem 3 description

0.3.1 shape functions

The following are the shape functions

$$
\begin{aligned}
f_I &= \frac{1}{8}(1-r)(1-s)(1-t)(-r-s-t-2)\\
f_J &= \frac{1}{8}(1-r)(s+1)(1-t)(-r+s-t-2)\\
f_K &= \frac{1}{8}(1-r)(s+1)(t+1)(-r+s+t-2)\\
f_L &= \frac{1}{8}(1-r)(1-s)(t+1)(-r-s+t-2)\\
f_M &= \frac{1}{8}(r+1)(1-s)(1-t)(r-s-t-2)\\
f_N &= \frac{1}{8}(r+1)(s+1)(1-t)(r+s-t-2)\\
f_O &= \frac{1}{8}(r+1)(s+1)(t+1)(r+s+t-2)\\
f_P &= \frac{1}{8}(r+1)(1-s)(t+1)(r-s+t-2)\\
f_Q &= \frac{1}{4}(1-r)\left(1-s^2\right)(1-t)\\
f_R &= \frac{1}{4}(1-r)(s+1)\left(1-t^2\right)\\
f_S &= \frac{1}{4}(1-r)\left(1-s^2\right)(t+1)\\
f_T &= \frac{1}{4}(1-r)(1-s)\left(1-t^2\right)\\
f_U &= \frac{1}{4}(r+1)\left(1-s^2\right)(1-t)\\
f_V &= \frac{1}{4}(r+1)(s+1)\left(1-t^2\right)\\
f_W &= \frac{1}{4}(r+1)\left(1-s^2\right)(t+1)\\
f_X &= \frac{1}{4}(r+1)(1-s)\left(1-t^2\right)\\
f_Y &= \frac{1}{4}\left(1-r^2\right)(1-s)(1-t)\\
f_Z &= \frac{1}{4}\left(1-r^2\right)(s+1)(1-t)\\
f_A &= \frac{1}{4}\left(1-r^2\right)(s+1)(t+1)\\
f_B &= \frac{1}{4}\left(1-r^2\right)(1-s)(t+1)
\end{aligned}
$$

To obtain the Jacobian, we need to obtain the determinant of

$$
\begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r}\\
\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s}\\
\frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}
\end{pmatrix}
$$

Where

$$
\begin{aligned}
x(r,s,t) &= \sum_{i=I}^{I=B} x_i f_i(r,s,t)\\
y(r,s,t) &= \sum_{i=I}^{I=B} y_i f_i(r,s,t)\\
z(r,s,t) &= \sum_{i=I}^{I=B} z_i f_i(r,s,t)
\end{aligned}
$$

Once we determine $x(r,s,t), y(r,s,t), z(r,s,t)$ from the above, then we can determine the Jacobian determinant at each Gaussian point.

0.3.2 $x(s,t,r)$ terms

From the above sum, expanding $x(r,s,t)$ gives

$x_= x_I f_I + x_J f_J + x_K f_K + x_L f_L + x_M f_M + x_N f_N + x_O f_O + x_P f_P + x_Q f_Q + x_R f_R + x_S f_S + x_T f_T + x_U f_U + x_V f_V$

Substituting the values of f_i into the above results in

$$
\begin{aligned}
x(r,s,t) = & x_I \frac{1}{8}(1-r)(1-s)(1-t)(-r-s-t-2)+ \\
& x_J \frac{1}{8}(1-r)(s+1)(1-t)(-r+s-t-2)+ \\
& x_K \frac{1}{8}(1-r)(s+1)(t+1)(-r+s+t-2)+ \\
& x_L \frac{1}{8}(1-r)(1-s)(t+1)(-r-s+t-2)+ \\
& x_M \frac{1}{8}(r+1)(1-s)(1-t)(r-s-t-2)+ \\
& x_N \frac{1}{8}(r+1)(s+1)(1-t)(r+s-t-2)+ \\
& x_O \frac{1}{8}(r+1)(s+1)(t+1)(r+s+t-2)+ \\
& x_P \frac{1}{8}(r+1)(1-s)(t+1)(r-s+t-2)+ \\
& x_Q \frac{1}{4}(1-r)\left(1-s^2\right)(1-t)+ \\
& x_R \frac{1}{4}(1-r)(s+1)\left(1-t^2\right)+ \\
& x_S \frac{1}{4}(1-r)\left(1-s^2\right)(t+1)+ \\
& x_T \frac{1}{4}(1-r)(1-s)\left(1-t^2\right)+ \\
& x_U \frac{1}{4}(r+1)\left(1-s^2\right)(1-t)+ \\
& x_V \frac{1}{4}(r+1)(s+1)\left(1-t^2\right)+ \\
& x_W \frac{1}{4}(r+1)\left(1-s^2\right)(t+1)+ \\
& x_X \frac{1}{4}(r+1)(1-s)\left(1-t^2\right)+ \\
& x_Y \frac{1}{4}\left(1-r^2\right)(1-s)(1-t)+ \\
& x_Z \frac{1}{4}\left(1-r^2\right)(s+1)(1-t)+ \\
& x_A \frac{1}{4}\left(1-r^2\right)(s+1)(t+1)+ \\
& x_B \frac{1}{4}\left(1-r^2\right)(1-s)(t+1)
\end{aligned}
$$

Taking partial derivative of the above w.r.t. r, s, t in turn gives the following

$$
\begin{aligned}
\frac{\partial x}{\partial r} =& x_I\left(\frac{1}{8}(s-1)(t-1)(2r+s+t+1)\right)+ \\
& x_J\left(\frac{1}{8}(s+1)(t-1)(-2r+s-t-1)\right)+ \\
& x_K\left(-\frac{1}{8}(s+1)(t+1)(-2r+s+t-1)\right)+ \\
& x_L\left(-\frac{1}{8}(s-1)(t+1)(2r+s-t+1)\right)+ \\
& x_M\left(-\frac{1}{8}(s-1)(t-1)(-2r+s+t+1)\right)+ \\
& x_N\left(-\frac{1}{8}(s+1)(t-1)(2r+s-t-1)\right)+ \\
& x_O\left(\frac{1}{8}(s+1)(t+1)(2r+s+t-1)\right)+ \\
& x_P\left(\frac{1}{8}(s-1)(t+1)(-2r+s-t+1)\right)+ \\
& x_Q\left(-\frac{1}{4}\left(s^2-1\right)(t-1)\right)+ \\
& x_R\left(\frac{1}{4}(s+1)\left(t^2-1\right)\right)+ \\
& x_S\left(\frac{1}{4}\left(s^2-1\right)(t+1)\right)+ \\
& x_T\left(-\frac{1}{4}(s-1)\left(t^2-1\right)\right)+ \\
& x_U\left(\frac{1}{4}\left(s^2-1\right)(t-1)\right)+ \\
& x_V\left(-\frac{1}{4}(s+1)\left(t^2-1\right)\right)+ \\
& x_W\left(-\frac{1}{4}\left(s^2-1\right)(t+1)\right)+ \\
& x_X\left(\frac{1}{4}(s-1)\left(t^2-1\right)\right)+ \\
& x_Y\left(-\frac{1}{2}r(s-1)(t-1)\right)+ \\
& x_Z\left(\frac{1}{2}r(s+1)(t-1)\right)+ \\
& x_A\left(-\frac{1}{2}r(s+1)(t+1)\right)+ \\
& x_B\left(\frac{1}{2}r(s-1)(t+1)\right)
\end{aligned}
$$

In the Matlab implementation,the terms r, s, t in the above expression are the Gaussian integration points along the three directions. Similarly, we now find $\frac{\partial x}{\partial s}$ as above. This results in

$$
\begin{aligned}
\frac{\partial x}{\partial s} =& x_I\left(\frac{1}{8}(r-1)(t-1)(r+2s+t+1)\right)+\\
& x_J\left(-\frac{1}{8}(r-1)(t-1)(r-2s+t+1)\right)+\\
& x_K\left(\frac{1}{8}(r-1)(t+1)(r-2s-t+1)\right)+\\
& x_L\left(-\frac{1}{8}(r-1)(t+1)(r+2s-t+1)\right)+\\
& x_M\left(\frac{1}{8}(r+1)(t-1)(r-2s-t-1)\right)+\\
& x_N\left(-\frac{1}{8}(r+1)(t-1)(r+2s-t-1)\right)+\\
& x_O\left(\frac{1}{8}(r+1)(t+1)(r+2s+t-1)\right)+\\
& x_P\left(-\frac{1}{8}(r+1)(t+1)(r-2s+t-1)\right)+\\
& x_Q\left(-\frac{1}{2}(r-1)s(t-1)\right)+\\
& x_R\left(\frac{1}{4}(r-1)\left(t^2-1\right)\right)+\\
& x_S\left(\frac{1}{2}(r-1)s(t+1)\right)+\\
& x_T\left(-\frac{1}{4}(r-1)\left(t^2-1\right)\right)+\\
& x_U\left(\frac{1}{2}(r+1)s(t-1)\right)+\\
& x_V\left(-\frac{1}{4}(r+1)\left(t^2-1\right)\right)+\\
& x_W\left(-\frac{1}{2}(r+1)s(t+1)\right)+\\
& x_X\left(\frac{1}{4}(r+1)\left(t^2-1\right)\right)+\\
& x_Y\left(-\frac{1}{4}\left(r^2-1\right)(t-1)\right)+\\
& x_Z\left(\frac{1}{4}\left(r^2-1\right)(t-1)\right)+\\
& x_A\left(-\frac{1}{4}\left(r^2-1\right)(t+1)\right)+\\
& x_B\left(\frac{1}{4}\left(r^2-1\right)(t+1)\right)
\end{aligned}
$$

Similarly, we now find $\frac{\partial x}{\partial t}$ as above. This results in

$$
\begin{aligned}
\frac{\partial x}{\partial t} =& x_I\left(\frac{1}{8}(r-1)(s-1)(r+s+2t+1)\right)+ \\
& x_J\left(-\frac{1}{8}(r-1)(s+1)(r-s+2t+1)\right)+ \\
& x_K\left(\frac{1}{8}(r-1)(s+1)(r-s-2t+1)\right)+ \\
& x_L\left(-\frac{1}{8}(r-1)(s-1)(r+s-2t+1)\right)+ \\
& x_M\left(\frac{1}{8}(r+1)(s-1)(r-s-2t-1)\right)+ \\
& x_N\left(-\frac{1}{8}(r+1)(s+1)(r+s-2t-1)\right)+ \\
& x_O\left(\frac{1}{8}(r+1)(s+1)(r+s+2t-1)\right)+ \\
& x_P\left(-\frac{1}{8}(r+1)(s-1)(r-s+2t-1)\right)+ \\
& x_Q\left(-\frac{1}{4}(r-1)\left(s^2-1\right)\right)+ \\
& x_R\left(\frac{1}{2}(r-1)(s+1)t\right)+ \\
& x_S\left(\frac{1}{4}(r-1)\left(s^2-1\right)\right)+ \\
& x_T\left(-\frac{1}{2}(r-1)(s-1)t\right)+ \\
& x_U\left(\frac{1}{4}(r+1)\left(s^2-1\right)\right)+ \\
& x_V\left(-\frac{1}{2}(r+1)(s+1)t\right)+ \\
& x_W\left(-\frac{1}{4}(r+1)\left(s^2-1\right)\right)+ \\
& x_X\left(\frac{1}{2}(r+1)(s-1)t\right)+ \\
& x_Y\left(-\frac{1}{4}\left(r^2-1\right)(s-1)\right)+ \\
& x_Z\left(\frac{1}{4}\left(r^2-1\right)(s+1)\right)+ \\
& x_A\left(-\frac{1}{4}\left(r^2-1\right)(s+1)\right)+ \\
& x_B\left(\frac{1}{4}\left(r^2-1\right)(s-1)\right)
\end{aligned}
$$

0.3.3 $y(r,s,t)$ terms

We now repeat all the above to find $\frac{\partial y}{\partial r}, \frac{\partial y}{\partial s}$ and $\frac{\partial y}{\partial t}$. We first need to expand $y(r,s,t) = \sum_{i=I}^{I=B} y_i f_i(r,s,t)$, which gives

$$y = y_I f_I + y_J f_J + y_K f_K + y_L f_L + y_M f_M + y_N f_N + y_O f_O + y_P f_P + y_Q f_Q + y_R f_R + y_S f_S + y_T f_T + y_U f_U + y_V f_V$$

Expanding the above gives

$$\begin{aligned}
y(r,s,t) =& y_I \frac{1}{8}(1-r)(1-s)(1-t)(-r-s-t-2)+\\
& y_J \frac{1}{8}(1-r)(s+1)(1-t)(-r+s-t-2)+\\
& y_K \frac{1}{8}(1-r)(s+1)(t+1)(-r+s+t-2)+\\
& y_L \frac{1}{8}(1-r)(1-s)(t+1)(-r-s+t-2)+\\
& y_M \frac{1}{8}(r+1)(1-s)(1-t)(r-s-t-2)+\\
& y_N \frac{1}{8}(r+1)(s+1)(1-t)(r+s-t-2)+\\
& y_O \frac{1}{8}(r+1)(s+1)(t+1)(r+s+t-2)+\\
& y_P \frac{1}{8}(r+1)(1-s)(t+1)(r-s+t-2)+\\
& y_Q \frac{1}{4}(1-r)\left(1-s^2\right)(1-t)+\\
& y_R \frac{1}{4}(1-r)(s+1)\left(1-t^2\right)+\\
& y_S \frac{1}{4}(1-r)\left(1-s^2\right)(t+1)+\\
& y_T \frac{1}{4}(1-r)(1-s)\left(1-t^2\right)+\\
& y_U \frac{1}{4}(r+1)\left(1-s^2\right)(1-t)+\\
& y_V \frac{1}{4}(r+1)(s+1)\left(1-t^2\right)+\\
& y_W \frac{1}{4}(r+1)\left(1-s^2\right)(t+1)+\\
& y_X \frac{1}{4}(r+1)(1-s)\left(1-t^2\right)+\\
& y_Y \frac{1}{4}\left(1-r^2\right)(1-s)(1-t)+\\
& y_Z \frac{1}{4}\left(1-r^2\right)(s+1)(1-t)+\\
& y_A \frac{1}{4}\left(1-r^2\right)(s+1)(t+1)+\\
& y_B \frac{1}{4}\left(1-r^2\right)(1-s)(t+1)
\end{aligned}$$

Taking partial derivatives of the above w.r.t. r, s, t in turn, we see that it gives similar results to earlier ones, but the only difference is in the multipliers now being the y_i values of coordinates instead of the x_i coordinates. This is reproduced again for completion

$$
\begin{aligned}
\frac{\partial y}{\partial r} =& y_I\left(\frac{1}{8}(s-1)(t-1)(2r+s+t+1)\right)+\\
& y_J\left(\frac{1}{8}(s+1)(t-1)(-2r+s-t-1)\right)+\\
& y_K\left(-\frac{1}{8}(s+1)(t+1)(-2r+s+t-1)\right)+\\
& y_L\left(-\frac{1}{8}(s-1)(t+1)(2r+s-t+1)\right)+\\
& y_M\left(-\frac{1}{8}(s-1)(t-1)(-2r+s+t+1)\right)+\\
& y_N\left(-\frac{1}{8}(s+1)(t-1)(2r+s-t-1)\right)+\\
& y_O\left(\frac{1}{8}(s+1)(t+1)(2r+s+t-1)\right)+\\
& y_P\left(\frac{1}{8}(s-1)(t+1)(-2r+s-t+1)\right)+\\
& y_Q\left(-\frac{1}{4}\left(s^2-1\right)(t-1)\right)+\\
& y_R\left(\frac{1}{4}(s+1)\left(t^2-1\right)\right)+\\
& y_S\left(\frac{1}{4}\left(s^2-1\right)(t+1)\right)+\\
& y_T\left(-\frac{1}{4}(s-1)\left(t^2-1\right)\right)+\\
& y_U\left(\frac{1}{4}\left(s^2-1\right)(t-1)\right)+\\
& y_V\left(-\frac{1}{4}(s+1)\left(t^2-1\right)\right)+\\
& y_W\left(-\frac{1}{4}\left(s^2-1\right)(t+1)\right)+\\
& y_X\left(\frac{1}{4}(s-1)\left(t^2-1\right)\right)+\\
& y_Y\left(-\frac{1}{2}r(s-1)(t-1)\right)+\\
& y_Z\left(\frac{1}{2}r(s+1)(t-1)\right)+\\
& y_A\left(-\frac{1}{2}r(s+1)(t+1)\right)+\\
& y_B\left(\frac{1}{2}r(s-1)(t+1)\right)
\end{aligned}
$$

Similarly, we now find $\frac{\partial y}{\partial s}$ as above. This results in

$$
\begin{aligned}
\frac{\partial y}{\partial s} =& y_I\left(\frac{1}{8}(r-1)(t-1)(r+2s+t+1)\right)+\\
& y_J\left(-\frac{1}{8}(r-1)(t-1)(r-2s+t+1)\right)+\\
& y_K\left(\frac{1}{8}(r-1)(t+1)(r-2s-t+1)\right)+\\
& y_L\left(-\frac{1}{8}(r-1)(t+1)(r+2s-t+1)\right)+\\
& y_M\left(\frac{1}{8}(r+1)(t-1)(r-2s-t-1)\right)+\\
& y_N\left(-\frac{1}{8}(r+1)(t-1)(r+2s-t-1)\right)+\\
& y_O\left(\frac{1}{8}(r+1)(t+1)(r+2s+t-1)\right)+\\
& y_P\left(-\frac{1}{8}(r+1)(t+1)(r-2s+t-1)\right)+\\
& y_Q\left(-\frac{1}{2}(r-1)s(t-1)\right)+\\
& y_R\left(\frac{1}{4}(r-1)\left(t^2-1\right)\right)+\\
& y_S\left(\frac{1}{2}(r-1)s(t+1)\right)+\\
& y_T\left(-\frac{1}{4}(r-1)\left(t^2-1\right)\right)+\\
& y_U\left(\frac{1}{2}(r+1)s(t-1)\right)+\\
& y_V\left(-\frac{1}{4}(r+1)\left(t^2-1\right)\right)+\\
& y_W\left(-\frac{1}{2}(r+1)s(t+1)\right)+\\
& y_X\left(\frac{1}{4}(r+1)\left(t^2-1\right)\right)+\\
& y_Y\left(-\frac{1}{4}\left(r^2-1\right)(t-1)\right)+\\
& y_Z\left(\frac{1}{4}\left(r^2-1\right)(t-1)\right)+\\
& y_A\left(-\frac{1}{4}\left(r^2-1\right)(t+1)\right)+\\
& y_B\left(\frac{1}{4}\left(r^2-1\right)(t+1)\right)
\end{aligned}
$$

We now find $\frac{\partial y}{\partial t}$ as above. This results in

$$\begin{aligned}
\frac{\partial y}{\partial t} =& y_I\left(\frac{1}{8}(r-1)(s-1)(r+s+2t+1)\right)+\\
& y_J\left(-\frac{1}{8}(r-1)(s+1)(r-s+2t+1)\right)+\\
& y_K\left(\frac{1}{8}(r-1)(s+1)(r-s-2t+1)\right)+\\
& y_L\left(-\frac{1}{8}(r-1)(s-1)(r+s-2t+1)\right)+\\
& y_M\left(\frac{1}{8}(r+1)(s-1)(r-s-2t-1)\right)+\\
& y_N\left(-\frac{1}{8}(r+1)(s+1)(r+s-2t-1)\right)+\\
& y_O\left(\frac{1}{8}(r+1)(s+1)(r+s+2t-1)\right)+\\
& y_P\left(-\frac{1}{8}(r+1)(s-1)(r-s+2t-1)\right)+\\
& y_Q\left(-\frac{1}{4}(r-1)\left(s^2-1\right)\right)+\\
& y_R\left(\frac{1}{2}(r-1)(s+1)t\right)+\\
& y_S\left(\frac{1}{4}(r-1)\left(s^2-1\right)\right)+\\
& y_T\left(-\frac{1}{2}(r-1)(s-1)t\right)+\\
& y_U\left(\frac{1}{4}(r+1)\left(s^2-1\right)\right)+\\
& y_V\left(-\frac{1}{2}(r+1)(s+1)t\right)+\\
& y_W\left(-\frac{1}{4}(r+1)\left(s^2-1\right)\right)+\\
& y_X\left(\frac{1}{2}(r+1)(s-1)t\right)+\\
& y_Y\left(-\frac{1}{4}\left(r^2-1\right)(s-1)\right)+\\
& y_Z\left(\frac{1}{4}\left(r^2-1\right)(s+1)\right)+\\
& y_A\left(-\frac{1}{4}\left(r^2-1\right)(s+1)\right)+\\
& y_B\left(\frac{1}{4}\left(r^2-1\right)(s-1)\right)
\end{aligned}$$

0.3.4 $z(r,s,t)$ terms

We now repeat all the above to find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$. These produce similar results to the above, but will have z_i as multipliers. We first need to expand $z(r,s,t) = \sum_{i=I}^{I=B} z_i f_i(r,s,t)$, which gives

$$z = z_I f_I + z_J f_J + z_K f_K + z_L f_L + z_M f_M + z_N f_N + z_O f_O + z_P f_P + z_Q f_Q + z_R f_R + z_S f_S + z_T f_T + z_U f_U + z_V f_V +$$

Expanding the above gives

$$
\begin{aligned}
z(r,s,t) = & z_I \frac{1}{8}(1-r)(1-s)(1-t)(-r-s-t-2)+ \\
& z_J \frac{1}{8}(1-r)(s+1)(1-t)(-r+s-t-2)+ \\
& z_K \frac{1}{8}(1-r)(s+1)(t+1)(-r+s+t-2)+ \\
& z_L \frac{1}{8}(1-r)(1-s)(t+1)(-r-s+t-2)+ \\
& z_M \frac{1}{8}(r+1)(1-s)(1-t)(r-s-t-2)+ \\
& z_N \frac{1}{8}(r+1)(s+1)(1-t)(r+s-t-2)+ \\
& z_O \frac{1}{8}(r+1)(s+1)(t+1)(r+s+t-2)+ \\
& z_P \frac{1}{8}(r+1)(1-s)(t+1)(r-s+t-2)+ \\
& z_Q \frac{1}{4}(1-r)\left(1-s^2\right)(1-t)+ \\
& z_R \frac{1}{4}(1-r)(s+1)\left(1-t^2\right)+ \\
& z_S \frac{1}{4}(1-r)\left(1-s^2\right)(t+1)+ \\
& z_T \frac{1}{4}(1-r)(1-s)\left(1-t^2\right)+ \\
& z_U \frac{1}{4}(r+1)\left(1-s^2\right)(1-t)+ \\
& z_V \frac{1}{4}(r+1)(s+1)\left(1-t^2\right)+ \\
& z_W \frac{1}{4}(r+1)\left(1-s^2\right)(t+1)+ \\
& z_X \frac{1}{4}(r+1)(1-s)\left(1-t^2\right)+ \\
& z_Y \frac{1}{4}\left(1-r^2\right)(1-s)(1-t)+ \\
& z_Z \frac{1}{4}\left(1-r^2\right)(s+1)(1-t)+ \\
& z_A \frac{1}{4}\left(1-r^2\right)(s+1)(t+1)+ \\
& z_B \frac{1}{4}\left(1-r^2\right)(1-s)(t+1)
\end{aligned}
$$

Taking partial derivative of the above w.r.t. r, s, t in turns gives the following

$$
\begin{aligned}
\frac{\partial z}{\partial r} =& z_I \left(\frac{1}{8}(s-1)(t-1)(2r+s+t+1)\right) + \\
& z_J \left(\frac{1}{8}(s+1)(t-1)(-2r+s-t-1)\right) + \\
& z_K \left(-\frac{1}{8}(s+1)(t+1)(-2r+s+t-1)\right) + \\
& z_L \left(-\frac{1}{8}(s-1)(t+1)(2r+s-t+1)\right) + \\
& z_M \left(-\frac{1}{8}(s-1)(t-1)(-2r+s+t+1)\right) + \\
& z_N \left(-\frac{1}{8}(s+1)(t-1)(2r+s-t-1)\right) + \\
& z_O \left(\frac{1}{8}(s+1)(t+1)(2r+s+t-1)\right) + \\
& z_P \left(\frac{1}{8}(s-1)(t+1)(-2r+s-t+1)\right) + \\
& z_Q \left(-\frac{1}{4}\left(s^2-1\right)(t-1)\right) + \\
& z_R \left(\frac{1}{4}(s+1)\left(t^2-1\right)\right) + \\
& z_S \left(\frac{1}{4}\left(s^2-1\right)(t+1)\right) + \\
& z_T \left(-\frac{1}{4}(s-1)\left(t^2-1\right)\right) + \\
& z_U \left(\frac{1}{4}\left(s^2-1\right)(t-1)\right) + \\
& z_V \left(-\frac{1}{4}(s+1)\left(t^2-1\right)\right) + \\
& z_W \left(-\frac{1}{4}\left(s^2-1\right)(t+1)\right) + \\
& z_X \left(\frac{1}{4}(s-1)\left(t^2-1\right)\right) + \\
& z_Y \left(-\frac{1}{2}r(s-1)(t-1)\right) + \\
& z_Z \left(\frac{1}{2}r(s+1)(t-1)\right) + \\
& z_A \left(-\frac{1}{2}r(s+1)(t+1)\right) + \\
& z_B \left(\frac{1}{2}r(s-1)(t+1)\right)
\end{aligned}
$$

Similarly, $\frac{\partial z}{\partial s}$ results in

$$
\begin{aligned}
\frac{\partial z}{\partial s} =& z_I\left(\frac{1}{8}(r-1)(t-1)(r+2s+t+1)\right)+\\
& z_J\left(-\frac{1}{8}(r-1)(t-1)(r-2s+t+1)\right)+\\
& z_K\left(\frac{1}{8}(r-1)(t+1)(r-2s-t+1)\right)+\\
& z_L\left(-\frac{1}{8}(r-1)(t+1)(r+2s-t+1)\right)+\\
& z_M\left(\frac{1}{8}(r+1)(t-1)(r-2s-t-1)\right)+\\
& z_N\left(-\frac{1}{8}(r+1)(t-1)(r+2s-t-1)\right)+\\
& z_O\left(\frac{1}{8}(r+1)(t+1)(r+2s+t-1)\right)+\\
& z_P\left(-\frac{1}{8}(r+1)(t+1)(r-2s+t-1)\right)+\\
& z_Q\left(-\frac{1}{2}(r-1)s(t-1)\right)+\\
& z_R\left(\frac{1}{4}(r-1)\left(t^2-1\right)\right)+\\
& z_S\left(\frac{1}{2}(r-1)s(t+1)\right)+\\
& z_T\left(-\frac{1}{4}(r-1)\left(t^2-1\right)\right)+\\
& z_U\left(\frac{1}{2}(r+1)s(t-1)\right)+\\
& z_V\left(-\frac{1}{4}(r+1)\left(t^2-1\right)\right)+\\
& z_W\left(-\frac{1}{2}(r+1)s(t+1)\right)+\\
& z_X\left(\frac{1}{4}(r+1)\left(t^2-1\right)\right)+\\
& z_Y\left(-\frac{1}{4}\left(r^2-1\right)(t-1)\right)+\\
& z_Z\left(\frac{1}{4}\left(r^2-1\right)(t-1)\right)+\\
& z_A\left(-\frac{1}{4}\left(r^2-1\right)(t+1)\right)+\\
& z_B\left(\frac{1}{4}\left(r^2-1\right)(t+1)\right)
\end{aligned}
$$

And $\frac{\partial z}{\partial t}$ gives

$$\begin{aligned}
\frac{\partial z}{\partial t} =& z_I\left(\frac{1}{8}(r-1)(s-1)(r+s+2t+1)\right)+\\
& z_J\left(-\frac{1}{8}(r-1)(s+1)(r-s+2t+1)\right)+\\
& z_K\left(\frac{1}{8}(r-1)(s+1)(r-s-2t+1)\right)+\\
& z_L\left(-\frac{1}{8}(r-1)(s-1)(r+s-2t+1)\right)+\\
& z_M\left(\frac{1}{8}(r+1)(s-1)(r-s-2t-1)\right)+\\
& z_N\left(-\frac{1}{8}(r+1)(s+1)(r+s-2t-1)\right)+\\
& z_O\left(\frac{1}{8}(r+1)(s+1)(r+s+2t-1)\right)+\\
& z_P\left(-\frac{1}{8}(r+1)(s-1)(r-s+2t-1)\right)+\\
& z_Q\left(-\frac{1}{4}(r-1)\left(s^2-1\right)\right)+\\
& z_R\left(\frac{1}{2}(r-1)(s+1)t\right)+\\
& z_S\left(\frac{1}{4}(r-1)\left(s^2-1\right)\right)+\\
& z_T\left(-\frac{1}{2}(r-1)(s-1)t\right)+\\
& z_U\left(\frac{1}{4}(r+1)\left(s^2-1\right)\right)+\\
& z_V\left(-\frac{1}{2}(r+1)(s+1)t\right)+\\
& z_W\left(-\frac{1}{4}(r+1)\left(s^2-1\right)\right)+\\
& z_X\left(\frac{1}{2}(r+1)(s-1)t\right)+\\
& z_Y\left(-\frac{1}{4}\left(r^2-1\right)(s-1)\right)+\\
& z_Z\left(\frac{1}{4}\left(r^2-1\right)(s+1)\right)+\\
& z_A\left(-\frac{1}{4}\left(r^2-1\right)(s+1)\right)+\\
& z_B\left(\frac{1}{4}\left(r^2-1\right)(s-1)\right)
\end{aligned}$$

Finally now we can determine the Jacobian and its determinant using the above expressions. This is done in the Matlab code provided. The following Jacobian Matrix is evaluated at each Gaussian integration point then its determinant is found using `det()` command.

$$\begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r}\\
\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s}\\
\frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}
\end{pmatrix}$$

0.3.5 results

The first step was to obtain estimate of the volume in order to verify that the volume calculation was valid and that the Jacobian was correct.

An independent small piece of code was written to plot the 3D shape and obtain its volume using a build-in function in the computer algebra program Mathematica. This is a plot of the 3D shape generated and below it is the code used to generate the plot, with the volume found shown in the title.

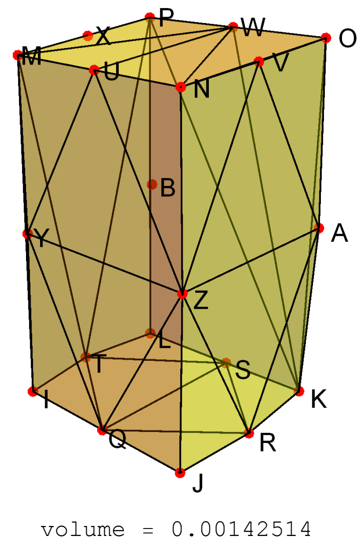


Figure 9: 3D plot of the volume in physical coordinates

```
xI=1;xJ=1.1;xK=1.09066;xL=0.99692;xM=1.0077;
xN=1.1069;xO=1.1035;xP=1.0046;xQ=1.05;xR=1.0992;
xS=1.0468;xT=0.9923;xU=1.0573;xV=1.1061;xW=1.0540;
xX=1.0069;xY=1.0051;xZ=1.1046;xA=1.1012;xB=1.0020;
xCoordinates={xI,xJ,xK,xL,xM,xN,xO,xP,xQ,xR,xS,xT,xU,xV,xW,xX,xY,xZ,xA,xB};
yI=0;yJ=0;yK=0;yL=0;yM=0.16559;
yN=0.17203;yO=0.17202;yP=0.16557;yQ=0;yR=0;
yS=0;yT=0;yU=0.16881;yV=0.17202;yW=0.16880;
yX=0.16558;yY=0.082737;yZ=0.085964;yA=0.085938;yB=0.082709;
yCoordinates={yI,yJ,yK,yL,yM,yN,yO,yP,yQ,yR,yS,yT,yU,yV,yW,yX,yY,yZ,yA,yB};
zI=0;zJ=0;zK=-0.086305;zL=-0.078459;zM=0;
zN=0;zO=-0.08663;zP=-0.07882;zQ=0;zR=-0.043186;
zS=-0.082382;zT=-0.039260;zU=0;zV=-0.043328;zW=-0.082725;
zX=-0.039418;zY=0;zZ=0;zA=-0.086550;zB=-0.078731;
zCoordinates={zI,zJ,zK,zL,zM,zN,zO,zP,zQ,zR,zS,zT,zU,zV,zW,zX,zY,zZ,zA,zB};
data3D=Table[{xCoordinates[[i]],yCoordinates[[i]],
          zCoordinates[[i]]},{i,1,Length@yCoordinates}];
nodes={"I","J","K","L","M","N","O","P","Q",
       "R","S","T","U","V","W","X","Y","Z","A","B"};
Needs["TetGenLink`"];
{pts,surface}=TetGenConvexHull[data3D];
c=First@Last@Reap@Do[Sow@{nodes[[i]],data3D[[i]]},{i,1,Length[nodes]}];
Labeled[Graphics3D[{
   {Red,PointSize[0.02],Point[data3D]},
   {Yellow,Opacity[.4],EdgeForm[{Thin,Black}],
    GraphicsComplex[data3D,Polygon[surface]]},
   {Text[Style[#[[1]],14],1.01*#[[2]]]&/@c}
},Boxed->False,Axes->False,SphericalRegion->True,
    ImageSize->300,ImageMargins->5],
    Row[{"volume = ",RegionMeasure@ConvexHullMesh[data3D]}]]
```

.

We now know the volume should be 0.0042514cm^3 from the above independent verification. The Matlab code was now implemented, and the volume was verified to be the same. Also, a separate test was run to verify that $\|J\| = 1$ for a test 3D volume which was aligned along the same orientation as the natural coordinates as shown below.

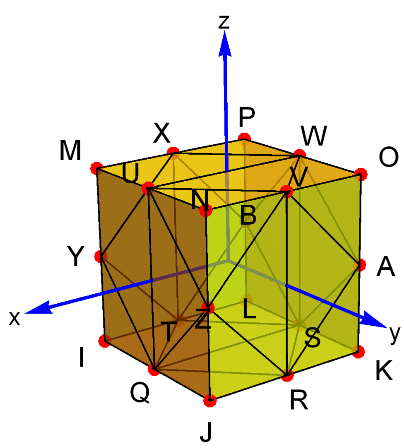


Figure 10: 3D plot of the aligned volume used for verification

The code used to plot the above is

```
xI=1;xJ=1.1;xK=1.09066;xL=0.99692;xM=1.0077; xN=1.1069;xO=1.1035;xP=1.0046;
xQ=1.05;xR=1.0992;xS=1.0468;xT=0.9923;xU=1.0573;xV=1.1061;xW=1.0540;
xX=1.0069;xY=1.0051;xZ=1.1046;xA=1.1012;xB=1.0020;
xCoordinates={xI,xJ,xK,xL,xM,xN,xO,xP,xQ,xR,xS,xT,xU,xV,xW,xX,xY,xZ,xA,xB};
yI=0;yJ=0;yK=0;yL=0;yM=0.16559; yN=0.17203;yO=0.17202;yP=0.16557;yQ=0;yR=0;
yS=0;yT=0;yU=0.16881;yV=0.17202;yW=0.16880; yX=0.16558;yY=0.082737;yZ=0.085964;
yA=0.085938;yB=0.082709;
yCoordinates={yI,yJ,yK,yL,yM,yN,yO,yP,yQ,yR,yS,yT,yU,yV,yW,yX,yY,yZ,yA,yB};
zI=0;zJ=0;zK=-0.086305;zL=-0.078459;zM=0; zN=0;zO=-0.08663;zP=-0.07882;zQ=0;
zR=-0.043186; zS=-0.082382;zT=-0.039260;zU=0;zV=-0.043328;zW=-0.082725;
zX=-0.039418;zY=0;zZ=0;zA=-0.086550;zB=-0.078731;
zCoordinates={zI,zJ,zK,zL,zM,zN,zO,zP,zQ,zR,zS,zT,zU,zV,zW,zX,zY,zZ,zA,zB};
data3D=Table[{xCoordinates[[i]],yCoordinates[[i]],zCoordinates[[i]]},{i,1,
    Length@yCoordinates}];
nodes={"I","J","K","L","M","N","O","P","Q",
       "R","S","T","U","V","W","X","Y","Z","A","B"};
Needs["TetGenLink`"];
{pts,surface}=TetGenConvexHull[data3D];
c=First@Last@Reap@Do[Sow@{nodes[[i]],data3D[[i]]},{i,1,Length[nodes]}];
Labeled[Graphics3D[{
   {Red,PointSize[0.02],Point[data3D]},
   {Yellow,Opacity[.4],EdgeForm[{Thin,Black}],
    GraphicsComplex[data3D,Polygon[surface]]},
   {Text[Style[#[[1]],14],1.01*#[[2]]]&/@c}
},Boxed->False,Axes->False,SphericalRegion->True,
    ImageSize->300,ImageMargins->5],
    Row[{"volume = ",RegionMeasure@ConvexHullMesh[data3D]}]]
```

.

This test also passed in Matlab and gave a volume of $8 cm^3$ as expected. Here is the small code segment in Matlab which verifies the above.

```
wt(1) = 5/9;         wt(2) = 8/9;   wt(3) = 5/9;
gs(1) = -sqrt(3/5); gs(2) = 0;      gs(3) = sqrt(3/5) ;

%set nodal coordinates as cube of side 2, centered
%with natural coordinates origin
xI=1;       xJ=1;      xK=-1;        xL=-1;       xM=1;
```

```
xN=1;       xO=-1;      xP=-1;        xQ=1;        xR=0;
xS=-1;      xT=0;     xU=1;         xV=0;        xW=-1;
xX=0;       xY=1;       xZ=1;         xA=-1;       xB=-1;

xCoordinates=[xI,xJ,xK,xL,xM,xN,xO,xP,xQ,xR,xS,xT,xU,xV,...
             xW,xX,xY,xZ,xA,xB];
yI=-1;       yJ=1;         yK=1;           yL=-1;       yM=-1;
yN=1;        yO=1;         yP=-1;          yQ=0;        yR=1;
yS=0;        yT=-1;        yU=0;           yV=1;        yW=0;
yX=-1;       yY=-1;        yZ=1;           yA=1;        yB=-1;
yCoordinates=[yI,yJ,yK,yL,yM,yN,yO,yP,yQ,yR,yS,yT,yU,yV,yW,...
             yX,yY,yZ,yA,yB];

zI=-1;         zJ=-1;     zK=-1;    zL=-1;     zM=1;
zN=1;          zO=1;      zP=1;     zQ=-1;     zR=-1;
zS=-1;         zT=-1;     zU=1;     zV=1;      zW=1;
zX=1;          zY=0;      zZ=0;     zA=0;      zB=0;
zCoordinates=[zI,zJ,zK,zL,zM,zN,zO,zP,zQ,zR,zS,zT,zU,zV,...
              zW,zX,zY,zZ,zA,zB];

%to collect sum of integrals at each node
the_sum = zeros(20,1);

%find the volume first, to use for verification.
for i=1:3
    s = gs(i);
    for j=1:3
        t = gs(j);
        for k=1:3
          r = gs(k);
          J = get_jacobian(r,s,t,xCoordinates,yCoordinates,zCoordinates);
          detJ = det(J);
          fprintf('|J|= %3.3f at Gaussian point [r=%3.3f,s=%3.3f,t=%3.3f]\n',
   detJ,r,s,t);
          for ii=1:length(xCoordinates)
              the_sum(ii)=the_sum(ii)+ wt(i)*wt(j)*wt(k)*f(ii,r,s,t)*detJ;
          end
        end
    end
end

fprintf('volume for test is [%3.3f] (is it 8?)\n',sum(the_sum));
end
```

.

The output of the above is given below, with the rest of the program output.

The program nma_EMA_471_HW5_problem_3.m implements the main solution to this problem and included in the zip file. It runs both the Jacobian verification and the load calculations after that.

The main loop of the Matlab function iterates over three indices i, j, k from 1 to 3 each. In the inner most loop, it finds the determinant of the Jacobian, the mass density, and evaluates the shape function at the Gaussian points, then sums the result. At the end it prints the work-equivalent conversion for each of the twenty nodes. The following shows the main core of the program

```
for i=1:3
    s = gs(i);
    for j=1:3
        t = gs(j);
        for k=1:3
            r = gs(k);
```

```
            J = get_jacobian(r,s,t,xCoordinates,...
                            yCoordinates,zCoordinates);
            detJ = det(J);
            mg = find_mass_density(r,s,t,xCoordinates,zCoordinates);

            for ii=1:length(xCoordinates)
                the_sum(ii)=the_sum(ii)+...
                        wt(i)*wt(j)*wt(k)*mg*g*f(ii,r,s,t)*detJ;
            end
        end
    end
end
```

.

The final result is in the following table

Corner	work-equivalent load (z direction) $\frac{\text{gram-cm}}{\text{sec}^2}$	Newton units
I	−0.1097835	−0.000001934
J	−0.1336398	−0.000001990
K	−0.1159590	−0.000002
L	−0.1143340	−0.000001928
M	−0.1373666	−0.000001927
N	−0.1611689	−0.000001984
O	−0.1418730	−0.00000198
P	−0.1160910	−0.000001924
Q	0.1701952	0.000002614
R	0.1812529	0.000002739
S	0.0808510	0.000002592
T	0.0741811	0.000002513
U	0.2242564	0.000002589
V	0.2374809	0.000002728
W	0.1905987	0.000002589
X	0.1802555	0.000002476
Y	0.1723161	0.000002482
Z	0.2334560	0.000002734
A	0.1929484	0.000002710
B	0.0986489	0.000002487

Table 5: work-equivalent conversion at each corner, problem 3

We also see that the load on the corners is negative while on the middle nodes it is positive. This agrees with what one would expect as per class notes on the 8-node element. Only difference is that this is a 3D element.

The following is the console output from running the above program. It is implemented using Matlab 2016a. It starts with the Jacobian verification then it will run the main task next only if the verification passes.

```
>>nma_EMA_471_HW5_problem_3()

starting verification of Jacobian....
|J|= 1.000 at Gaussian point [r=-0.775,s=-0.775,t=-0.775]
```

```
|J|= 1.000 at Gaussian point [r=0.000,s=-0.775,t=-0.775]
|J|= 1.000 at Gaussian point [r=0.775,s=-0.775,t=-0.775]
|J|= 1.000 at Gaussian point [r=-0.775,s=-0.775,t=0.000]
|J|= 1.000 at Gaussian point [r=0.000,s=-0.775,t=0.000]
|J|= 1.000 at Gaussian point [r=0.775,s=-0.775,t=0.000]
|J|= 1.000 at Gaussian point [r=-0.775,s=-0.775,t=0.775]
|J|= 1.000 at Gaussian point [r=0.000,s=-0.775,t=0.775]
|J|= 1.000 at Gaussian point [r=0.775,s=-0.775,t=0.775]
|J|= 1.000 at Gaussian point [r=-0.775,s=0.000,t=-0.775]
|J|= 1.000 at Gaussian point [r=0.000,s=0.000,t=-0.775]
|J|= 1.000 at Gaussian point [r=0.775,s=0.000,t=-0.775]
|J|= 1.000 at Gaussian point [r=-0.775,s=0.000,t=0.000]
|J|= 1.000 at Gaussian point [r=0.000,s=0.000,t=0.000]
|J|= 1.000 at Gaussian point [r=0.775,s=0.000,t=0.000]
|J|= 1.000 at Gaussian point [r=-0.775,s=0.000,t=0.775]
|J|= 1.000 at Gaussian point [r=0.000,s=0.000,t=0.775]
|J|= 1.000 at Gaussian point [r=0.775,s=0.000,t=0.775]
|J|= 1.000 at Gaussian point [r=-0.775,s=0.775,t=-0.775]
|J|= 1.000 at Gaussian point [r=0.000,s=0.775,t=-0.775]
|J|= 1.000 at Gaussian point [r=0.775,s=0.775,t=-0.775]
|J|= 1.000 at Gaussian point [r=-0.775,s=0.775,t=0.000]
|J|= 1.000 at Gaussian point [r=0.000,s=0.775,t=0.000]
|J|= 1.000 at Gaussian point [r=0.775,s=0.775,t=0.000]
|J|= 1.000 at Gaussian point [r=-0.775,s=0.775,t=0.775]
|J|= 1.000 at Gaussian point [r=0.000,s=0.775,t=0.775]
|J|= 1.000 at Gaussian point [r=0.775,s=0.775,t=0.775]
volume for test is [8.000] (is it 8?)
!! passed Jacobian test. Will run main program now
volume is 0.001426 cm^3

load at corner I = -0.1934002 [gram-cm/sec^2] = -0.000001934 N
load at corner J = -0.1990333 [gram-cm/sec^2] = -0.000001990 N
load at corner K = -0.2000363 [gram-cm/sec^2] = -0.000002000 N
load at corner L = -0.1928213 [gram-cm/sec^2] = -0.000001928 N
load at corner M = -0.1927113 [gram-cm/sec^2] = -0.000001927 N
load at corner N = -0.1984265 [gram-cm/sec^2] = -0.000001984 N
load at corner O = -0.1980087 [gram-cm/sec^2] = -0.000001980 N
load at corner P = -0.1923718 [gram-cm/sec^2] = -0.000001924 N
load at corner Q = 0.2614284 [gram-cm/sec^2] = 0.000002614 N
load at corner R = 0.2739156 [gram-cm/sec^2] = 0.000002739 N
load at corner S = 0.2592383 [gram-cm/sec^2] = 0.000002592 N
load at corner T = 0.2513418 [gram-cm/sec^2] = 0.000002513 N
load at corner U = 0.2589067 [gram-cm/sec^2] = 0.000002589 N
load at corner V = 0.2727979 [gram-cm/sec^2] = 0.000002728 N
load at corner W = 0.2588535 [gram-cm/sec^2] = 0.000002589 N
load at corner X = 0.2475648 [gram-cm/sec^2] = 0.000002476 N
load at corner Y = 0.2481743 [gram-cm/sec^2] = 0.000002482 N
load at corner Z = 0.2733760 [gram-cm/sec^2] = 0.000002734 N
load at corner A = 0.2710102 [gram-cm/sec^2] = 0.000002710 N
load at corner B = 0.2487254 [gram-cm/sec^2] = 0.000002487 N
>>
```

```
function nma_EMA_471_HW5_problem_3()
%Solves problem 3, HW5, EMA 471
%Nasser M. Abbasi

close all; clc;

status = do_jacobian_test();
if ~status
```

```
    error('failed jacobian test. Internal code error\n');
else
   fprintf('!! passed Jacobian test. Will run main program now\n\n');
   do_main_program();
end

end
%======================================
function status = do_jacobian_test()
%This function checks that |J|=1 at each Gaussian point.
%This verifies the code is ok before
%running the main program. This also checkes that volume is
% 2*2*2=8 cm^3 since we are using a cube with nodal coordinates
% with side length = 2 cm and it is aligned along the natural
%coordinates and centered at the natural coordinates origin also.
%

status = true;

wt(1) = 5/9;        wt(2) = 8/9;   wt(3) = 5/9;
gs(1) = -sqrt(3/5); gs(2) = 0;     gs(3) = sqrt(3/5) ;

%set nodal coordinates as cube of side 2,
%centered with natural coordinates origin
xI=1;       xJ=1;      xK=-1;        xL=-1;       xM=1;
xN=1;       xO=-1;      xP=-1;        xQ=1;        xR=0;
xS=-1;      xT=0;     xU=1;          xV=0;        xW=-1;
xX=0;       xY=1;      xZ=1;          xA=-1;       xB=-1;

xCoordinates=[xI,xJ,xK,xL,xM,xN,xO,xP,xQ,xR,xS,xT,xU,xV, ...
              xW,xX,xY,xZ,xA,xB];

yI=-1;       yJ=1;        yK=1;            yL=-1;       yM=-1;
yN=1;        yO=1;        yP=-1;           yQ=0;        yR=1;
yS=0;        yT=-1;       yU=0;            yV=1;        yW=0;
yX=-1;       yY=-1;       yZ=1;            yA=1;        yB=-1;
yCoordinates=[yI,yJ,yK,yL,yM,yN,yO,yP,yQ,yR,yS,yT,yU,yV,yW,...
              yX,yY,yZ,yA,yB];

zI=-1;          zJ=-1;     zK=-1;    zL=-1;     zM=1;
zN=1;           zO=1;      zP=1;     zQ=-1;     zR=-1;
zS=-1;          zT=-1;     zU=1;     zV=1;      zW=1;
zX=1;           zY=0;      zZ=0;     zA=0;      zB=0;
zCoordinates=[zI,zJ,zK,zL,zM,zN,zO,zP,zQ,zR,zS,zT,zU,zV,zW,...
              zX,zY,zZ,zA,zB];

the_sum = zeros(20,1);  %to collect sum of integrals at each node

%find the volume first, to use for verification.
format short;
format compact;
fprintf('starting verification of Jacobian....\n');
for i=1:3
    s = gs(i);

    for j=1:3
        t = gs(j);

        for k=1:3
            r = gs(k);

            J = get_jacobian(r,s,t,xCoordinates,...
                             yCoordinates,zCoordinates);
```

```
            detJ = det(J);
            fprintf('|J|= %3.3f at Gaussian point [r=%3.3f,s=%3.3f,t=%3.3f]\n',...
                detJ,r,s,t);
            if detJ <=0
                status = false;  %FAILED TEST
                return;
            end
            for ii=1:length(xCoordinates)
                the_sum(ii)=the_sum(ii)+ ...
                              wt(i)*wt(j)*wt(k)*f(ii,r,s,t)*detJ;
            end
        end
    end
end

fprintf('volume for test is [%3.3f] (is it 8?)\n',sum(the_sum));

end

%======================================
function do_main_program()
wt(1) = 5/9;        wt(2) = 8/9;   wt(3) = 5/9;
gs(1) = -sqrt(3/5); gs(2) = 0;     gs(3) = sqrt(3/5) ;

xI=1;      xJ=1.1;      xK=1.09066;    xL=0.99692;   xM=1.0077;
xN=1.1069; xO=1.1035;   xP=1.0046;     xQ=1.05;      xR=1.0992;
xS=1.0468; xT=0.9923;   xU=1.0573;     xV=1.1061;    xW=1.0540;
xX=1.0069; xY=1.0051;   xZ=1.1046;     xA=1.1012;    xB=1.0020;

xCoordinates=[xI,xJ,xK,xL,xM,xN,xO,xP,xQ,xR,xS,xT,xU,...
              xV,xW,xX,xY,xZ,xA,xB];

yI=0;        yJ=0;        yK=0;           yL=0;        yM=0.16559;
yN=0.17203;  yO=0.17202;  yP=0.16557;     yQ=0;        yR=0;
yS=0;        yT=0;        yU=0.16881;     yV=0.17202;  yW=0.16880;
yX=0.16558;  yY=0.082737; yZ=0.085964;    yA=0.085938; yB=0.082709;
yCoordinates=[yI,yJ,yK,yL,yM,yN,yO,yP,yQ,yR,yS,yT,yU,yV,yW,...
              yX,yY,yZ,yA,yB];

zI=0;         zJ=0;         zK=-0.086305;    zL=-0.078459;   zM=0;
zN=0;         zO=-0.08663;  zP=-0.07882;     zQ=0;           zR=-0.043186;
zS=-0.082382; zT=-0.039260; zU=0;            zV=-0.043328;   zW=-0.082725;
zX=-0.039418; zY=0;         zZ=0;            zA=-0.086550;   zB=-0.078731;
zCoordinates=[zI,zJ,zK,zL,zM,zN,zO,zP,zQ,zR,zS,zT,zU,zV,zW,zX,...
             zY,zZ,zA,zB];

the_sum = zeros(20,1);
g       = 9.81*100; %acceleration g in cm per sec^2

%find the volume first, to use for verification.
for i=1:3
    s = gs(i);

    for j=1:3
        t = gs(j);

        for k=1:3
            r = gs(k);

            J = get_jacobian(r,s,t,xCoordinates,...
                             yCoordinates,zCoordinates);
            detJ = det(J);
            if detJ <=0
```

```
                error('code internal error, invalid jacobian det. %7.6f\n',detJ);
            end
            for ii=1:length(xCoordinates)
                the_sum(ii)=the_sum(ii)+ ...
                            wt(i)*wt(j)*wt(k)*f(ii,r,s,t)*detJ;
            end
        end
    end
end

fprintf('volume is %7.6f cm^3\n', sum(the_sum));

for i=1:3
    s = gs(i);

    for j=1:3
        t = gs(j);

        for k=1:3
            r = gs(k);

            J = get_jacobian(r,s,t,xCoordinates,...
                                      yCoordinates,zCoordinates);
            detJ = det(J);
            if detJ <=0
                error('code internal error, invalid jacobian det. %7.6f\n',detJ);
            end
            mg = find_mass_density(r,s,t,xCoordinates,zCoordinates);

            for ii=1:length(xCoordinates)
                the_sum(ii)=the_sum(ii)+...
                          wt(i)*wt(j)*wt(k)*mg*g*f(ii,r,s,t)*detJ;
            end

        end
    end
end

map_node={'I','J','K','L','M','N','O','P','Q','R','S',...
                 'T','U','V',...
                 'W','X','Y','Z','A','B'};

for i=1:length(xCoordinates)
  fprintf('load at corner %c = %9.7f [gram-cm/sec^2] = %10.9f N\n',...
     map_node{i},the_sum(i),the_sum(i)*10^(-3)*10^(-2));
end
end
%===================================================
function mass_density = find_mass_density(r,s,t,...
                                      xCoordinates,zCoordinates)

        p0 = 1;
        X  = 0;
        for i=1:20
            X = X + xCoordinates(i)*f(i,r,s,t);
        end

        Z = 0;
        for i=1:20
            Z = Z + zCoordinates(i)*f(i,r,s,t);
        end

        mass_density = p0*(X^2+Z^2);
```

```
end
%=================================
function the_shape_function=f(idx,r,s,t)
switch idx
    case 1 %I
        the_shape_function=(1/8)*(1-r)*(1-s)*(1-t)*(-r-s-t-2);
    case 2 %J
        the_shape_function=(1/8)*(1-r)*(s+1)*(1-t)*(-r+s-t-2);
    case 3 %K
        the_shape_function=(1/8)*(1-r)*(s+1)*(t+1)*(-r+s+t-2);
    case 4 %L
        the_shape_function=(1/8)*(1-r)*(1-s)*(t+1)*(-r-s+t-2);
    case 5 %M
        the_shape_function=(1/8)*(r+1)*(1-s)*(1-t)*(r-s-t-2);
    case 6 %N
        the_shape_function=(1/8)*(r+1)*(s+1)*(1-t)*(r+s-t-2);
    case 7 %O
        the_shape_function=(1/8)*(r+1)*(s+1)*(t+1)*(r+s+t-2);
    case 8 %P
        the_shape_function=(1/8)*(r+1)*(1-s)*(t+1)*(r-s+t-2);
    case 9 %Q
        the_shape_function=(1/4)*(1-r)*(1-s^2)*(1-t);
    case 10 %R
        the_shape_function=(1/4)*(1-r)*(s+1)*(1-t^2);
    case 11 %S
        the_shape_function=(1/4)*(1-r)*(1-s^2)*(t+1);
    case 12 %T
        the_shape_function=(1/4)*(1-r)*(1-s)*(1-t^2);
    case 13 %U
        the_shape_function=(1/4)*(1+r)*(1-s^2)*(1-t);
    case 14 %V
        the_shape_function=(1/4)*(r+1)*(s+1)*(1-t^2);
    case 15 %W
        the_shape_function=(1/4)*(r+1)*(1-s^2)*(t+1);
    case 16 %X
        the_shape_function=(1/4)*(r+1)*(1-s)*(1-t^2);
    case 17 %Y
        the_shape_function=(1/4)*(1-r^2)*(1-s)*(1-t);
    case 18 %Z
        the_shape_function=(1/4)*(1-r^2)*(s+1)*(1-t);
    case 19 %A
        the_shape_function=(1/4)*(1-r^2)*(s+1)*(t+1);
    case 20 %B
        the_shape_function=(1/4)*(1-r^2)*(1-s)*(t+1);
end
end

%------------------------------- internal function
function the_result=dds(c,r,s,t)
%find dx/ds or dy/ds or dz/ds. These all have same
%form, except for the multiplier c, which is the nodal
%coordinates, passed in.
the_result=c(1)*((1/8)*(r-1)*(t-1)*(r+2*s+t+1))+...
    c(2)*(-(1/8)*(r-1)*(t-1)*(r-2*s+t+1))+...
    c(3)*((1/8)*(r-1)*(t+1)*(r-2*s-t+1))+...
    c(4)*(-(1/8)*(r-1)*(t+1)*(r+2*s-t+1))+...
    c(5)*((1/8)*(r+1)*(t-1)*(r-2*s-t-1))+...
    c(6)*(-(1/8)*(r+1)*(t-1)*(r+2*s-t-1))+...
    c(7)*((1/8)*(r+1)*(t+1)*(r+2*s+t-1))+...
    c(8)*(-(1/8)*(r+1)*(t+1)*(r-2*s+t-1))+...
    c(9)*(-(1/2)*(r-1)*s*(t-1))+...
    c(10)*((1/4)*(r-1)*(t^2-1))+...
    c(11)*((1/2)*(r-1)*s*(t+1))+...
```

```
    c(12)*(-(1/4)*(r-1)*(t^2-1))+...
    c(13)*((1/2)*(r+1)*s*(t-1))+...
    c(14)*(-(1/4)*(r+1)*(t^2-1))+...
    c(15)*(-(1/2)*(r+1)*s*(t+1))+...
    c(16)*((1/4)*(r+1)*(t^2-1))+...
    c(17)*(-(1/4)*(r^2-1)*(t-1))+...
    c(18)*((1/4)*(r^2-1)*(t-1))+...
    c(19)*(-(1/4)*(r^2-1)*(t+1))+...
    c(20)*((1/4)*(r^2-1)*(t+1));
end
%------------------------------- internal function
function the_result=ddt(c,r,s,t)
%find dx/dt or dy/dt or dz/dt. These all have same form,
%except for the multiplier c, which is the nodal coordinates,
%passed in.

the_result=c(1)*((1/8)*(r-1)*(s-1)*(r+s+2*t+1))+...
    c(2)*(-(1/8)*(r-1)*(s+1)*(r-s+2*t+1))+...
    c(3)*((1/8)*(r-1)*(s+1)*(r-s-2*t+1))+...
    c(4)*(-(1/8)*(r-1)*(s-1)*(r+s-2*t+1))+...
    c(5)*((1/8)*(r+1)*(s-1)*(r-s-2*t-1))+...
    c(6)*(-(1/8)*(r+1)*(s+1)*(r+s-2*t-1))+...
    c(7)*((1/8)*(r+1)*(s+1)*(r+s+2*t-1))+...
    c(8)*(-(1/8)*(r+1)*(s-1)*(r-s+2*t-1))+...
    c(9)*(-(1/4)*(r-1)*(s^2-1))+...
    c(10)*((1/2)*(r-1)*(s+1)*t)+...
    c(11)*((1/4)*(r-1)*(s^2-1))+...
    c(12)*(-(1/2)*(r-1)*(s-1)*t)+...
    c(13)*((1/4)*(r+1)*(s^2-1))+...
    c(14)*(-(1/2)*(r+1)*(s+1)*t)+...
    c(15)*(-(1/4)*(r+1)*(s^2-1))+...
    c(16)*((1/2)*(r+1)*(s-1)*t)+...
    c(17)*(-(1/4)*(r^2-1)*(s-1))+...
    c(18)*((1/4)*(r^2-1)*(s+1))+...
    c(19)*(-(1/4)*(r^2-1)*(s+1))+...
    c(20)*((1/4)*(r^2-1)*(s-1));
end
%------------------------------- internal function
function the_result=ddr(c,r,s,t)
%find dx/dr or dy/dr or dz/dr. These all have same form,
%except for the multiplier c, which is the nodal coordinates,
%passed in.

the_result=c(1)*((1/8)*(s-1)*(t-1)*(2*r+s+t+1))+...
    c(2)*((1/8)*(s+1)*(t-1)*(-2*r+s-t-1))+...
    c(3)*(-(1/8)*(s+1)*(t+1)*(-2*r+s+t-1))+...
    c(4)*(-(1/8)*(s-1)*(t+1)*(2*r+s-t+1))+...
    c(5)*(-(1/8)*(s-1)*(t-1)*(-2*r+s+t+1))+...
    c(6)*(-(1/8)*(s+1)*(t-1)*(2*r+s-t-1))+...
    c(7)*((1/8)*(s+1)*(t+1)*(2*r+s+t-1))+...
    c(8)*((1/8)*(s-1)*(t+1)*(-2*r+s-t+1))+...
    c(9)*(-(1/4)*(s^2-1)*(t-1))+...
    c(10)*((1/4)*(s+1)*(t^2-1))+...
    c(11)*((1/4)*(s^2-1)*(t+1))+...
    c(12)*(-(1/4)*(s-1)*(t^2-1))+...
    c(13)*((1/4)*(s^2-1)*(t-1))+...
    c(14)*(-(1/4)*(s+1)*(t^2-1))+...
    c(15)*(-(1/4)*(s^2-1)*(t+1))+...
    c(16)*((1/4)*(s-1)*(t^2-1))+...
    c(17)*(-(1/2)*r*(s-1)*(t-1))+...
    c(18)*((1/2)*r*(s+1)*(t-1))+...
    c(19)*(-(1/2)*r*(s+1)*(t+1))+...
    c(20)*((1/2)*r*(s-1)*(t+1));
```

```
end
%========================================
function J=get_jacobian(r,s,t,xCoordinates,yCoordinates,zCoordinates)

J = [ddr(xCoordinates,r,s,t),ddr(yCoordinates,r,s,t),...
     ddr(zCoordinates,r,s,t);
     dds(xCoordinates,r,s,t),dds(yCoordinates,r,s,t),...
     dds(zCoordinates,r,s,t);
     ddt(xCoordinates,r,s,t),ddt(yCoordinates,r,s,t),...
     ddt(zCoordinates,r,s,t)
     ];
end
```