EMA 548 HW1 Spring 2014, University of Wisconsin, Madison

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November 28, 2019

Contents

1 Problem 1 (BO section 3.1, 3.3,3.4)

1.1 problem 3.3 (c)

problem: Classify all the singular points (finite and infinite) of the following

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0$$

Answer:

Writing the DE in standard form

$$y'' + \frac{c-(a+b+1)x}{x(1-x)}y' - \frac{ab}{x(1-x)}y = 0$$
$$y'' + \overbrace{\left(\frac{c}{x(1-x)} - \frac{(a+b+1)}{(1-x)}\right)}^{p(x)}y' - \overbrace{\frac{ab}{x(1-x)}}^{q(x)}y = 0$$

$x=0,1$ are singular points. To classify what type of singularity, looking at $x=0$ then

$$y'' + \left(\frac{xc}{x(1-x)} - \frac{x(a+b+1)}{(1-x)}\right)y' - \frac{x^2ab}{x(1-x)}y = 0$$
$$y'' + \left(\frac{c}{(1-x)} - \frac{x(a+b+1)}{(1-x)}\right)y' - \frac{xab}{(1-x)}y = 0$$

Hence, $\lim_{x\to 0} xp(x) \to c$ and $\lim_{x\to 0} x^2 q(x) \to 0$, therefore the singularity at $x=0$ is removable, hence $x=0$ is a reqular singular point.

Now, looking at $x=1$.

$$y'' + \left(\frac{(x-1)c}{x(1-x)} - \frac{(x-1)(a+b+1)}{(1-x)}\right)y' - \frac{(x-1)^2ab}{x(1-x)}y = 0$$
$$y'' + \left(-\frac{c}{x} + (a+b+1)\right)y' + \frac{(x-1)ab}{x}y = 0$$

$\lim_{x\to 1}(x-1)p(x) \to (-c+(a+b+1))$ and $\lim_{x\to 1}(x-1)^2 q(x) \to 0$, therefore the singularity at $x=1$ is also removable, hence $x=1$ is a reqular singular point.

To check the type of singularity, if any, at $x=\infty$, the DE is first transformed using

$$x = \frac{1}{t} \tag{1}$$

This uses

$$\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = -t^2\frac{d}{dt} \tag{2}$$

and

$$\begin{aligned}
\frac{d^2}{dx^2} &= \frac{d}{dx}\left(\frac{d}{dx}\right) = \left(-t^2\frac{d}{dt}\right)\left(-t^2\frac{d}{dt}\right) = -t^2\frac{d}{dt}\left(-t^2\frac{d}{dt}\right) \\
&= -t^2\left(-2t\frac{d}{dt} - t^2\frac{d^2}{dt^2}\right) \\
&= 2t^3\frac{d}{dt} + t^4\frac{d^2}{dt^2} \qquad (3)
\end{aligned}$$

Sustituting eqs (1,2,3) into the original DE gives

$$\begin{aligned}
\frac{1}{t}\left(1-\frac{1}{t}\right)\left(2t^3\frac{d}{dt} + t^4\frac{d^2}{dt^2}\right)y + \left(c - (a+b+1)\frac{1}{t}\right)\left(-t^2\frac{d}{dt}\right)y - aby &= 0 \\
\frac{t-1}{t^2}\left(2t^3\frac{dy}{dt} + t^4\frac{d^2y}{dt^2}\right) + \left(-t^2c + (a+b+1)\,t\right)\frac{dy}{dt} - aby &= 0 \\
2t\,(t-1)\frac{dy}{dt} + t^2\,(t-1)\frac{d^2y}{dt^2} + \left(-t^2c + (a+b+1)\,t\right)\frac{dy}{dt} - aby &= 0 \\
t^2\,(t-1)\frac{d^2y}{dt^2} + \left(2t\,(t-1) - t^2c + (a+b+1)\,t\right)\frac{dy}{dt} - aby &= 0
\end{aligned}$$

Writing the above in standard form

$$\frac{d^2y}{dt^2} + \frac{\left(2t\,(t-1) - t^2c + (a+b+1)\,t\right)}{t^2\,(t-1)}\frac{dy}{dt} - \frac{ab}{t^2\,(t-1)}y = 0$$

Expanding

$$\frac{d^2y}{dt^2} + \overbrace{\left(\frac{2}{t} - \frac{c}{(t-1)} + \frac{(a+b+1)}{t\,(t-1)}\right)}^{p(t)}\frac{dy}{dt} - \overbrace{\frac{ab}{t^2\,(t-1)}}^{q(t)}y = 0$$

Hence at $t = 0$ there is a singularity (this means $x = \infty$). To find what type

$$\begin{aligned}
\lim_{t\to 0} tp\,(t) &= \lim_{t\to 0} t\left(\frac{2}{t} - \frac{c}{(t-1)} + \frac{(a+b+1)}{t\,(t-1)}\right) = \lim_{t\to 0}\left(2 - \frac{tc}{(t-1)} + \frac{(a+b+1)}{(t-1)}\right) \\
&= 2 - (a+b+1)
\end{aligned}$$

And

$$\lim_{t\to 0} t^2 q(t) = \lim_{t\to 0} t^2 \left(-\frac{ab}{t^2 (t-1)}\right) = \lim_{t\to 0} \left(\frac{-ab}{(t-1)}\right) = ab$$

Hence the singularity is removable. Therefore $x \to \infty$ is a regular singular point.

1.2 problem 3.3 (h)

problem: Classify all the singular points (finite and infinite) of the following

$$\left(1-x^2\right) y'' - 2xy' + \left(\lambda + 40\left(1-x^2\right) - \frac{\mu^2}{\left(1-x^2\right)}\right) y = 0$$

Answer:

Writing the DE in standard form

$$\left(1-x^2\right) y'' - 2xy' + \left(\lambda + 40\left(1-x^2\right) - \frac{\mu^2}{\left(1-x^2\right)}\right) y = 0$$

1.3 problem 3.4 (d)

problem: Classify $x = 0$ and $x = \infty$ of the following

$$x^2 y'' = y e^{\frac{1}{x}}$$

Answer:

1.4 problem 3.4 (f)

problem: Classify $x = 0$ and $x = \infty$ of the following

$$y'' = y \ln x$$

Answer:

2 Problem 2 (BO section 3.2, 3.7,3.8)

2.1 Problem 3.7

Problem: Estimate the number of terms in the Taylor series (3.2.1) and (3.2.2) that are necessary to compute

$Ai(x)$ and $Bi(x)$ correct to three decimal places at $x = \pm 1, \pm 100, \pm 10000$

Answer:

2.2 Problem 3.8

Problem: How many terms in the Taylor series solution to $y''' = x^3 y$ with $y(0) = 1, y'(0) = y''(0) = 0$ are needed to evaluate $\int_0^1 y(x)\,dx$ correct to three decimal places?

Answer:

3 Problem 3 (BO section 3.3, 3.24f)

Problem:

Find series expansions of all the solutions to the following differential equations about $x = 0$. Try

to sum in closed form any infinite series that appear.

$$(\sin x)\, y'' - 2\, (\cos x)\, y' - (\sin x)\, y = 0$$

Answer:

4 Problem 4 (BO section 3.4, 3.27,3.33(b,f)

4.1 Problem 3.27

Derive (3.4.28). Where 2.4.28 is the solution of example 5 which is stated here:

Local behavior of solutions near an irregular singular point of a general nth-order Schrodinger equation. In this example we derive an extremely simple and important formula for the leading behavior of solutions to the nth-order Schrodinger equation

$$\frac{d^n y}{dx^n} = Q(x)\, y$$

near an irregular singular point at x_0.

The exponential substitution $y = e^{S(x)}$ and the asymptotic approximations $\frac{d^k S}{dx^k} \ll (S')^k$ as $x \to x_0$ for $k = 2, 3, \cdots, n$ give the asymptotic differential equation $(S')^n \sim Q(x)$ $(x \to x_0)$. Thus, $S(x) \sim \omega \int^x Q(t)^{\frac{1}{n}}\, dt$ $(x \to x_0)$ $(x \to x_0)$, where ω is an nth root of unity. This result determines the n possible controlling factors of $y(x)$. The leading behavior of $y(x)$ is found in the usual way (see Prob. 3.27) to be

$$y(x) \sim cQ(x)^{\frac{1-n}{2n}} \exp\left(\omega \int^x Q(t)^{\frac{1}{n}}\, dt\right), (x \to x_0)$$

Answer:

4.2 Problem 3.33b

Find the leading behaviors as $x \to 0^+$ of the following equations

$$x^4 y''' - 3x^2 y' + 2y = 0$$

Answer:

4.3 Problem 3.33f

Find the leading behaviors as $x \to 0^+$ of the following equations

$$x^4 y'' - x^2 y' + \frac{1}{4} y = 0$$

Answer:

5 Problem 5 (BO section 3.5, 3.39(b),3.46(a)

5.1 Problem 3.39(b)

Find the leading asymptotic behaviors as $x \to \infty$ of the following equations

$$xy''' = y'$$

Answer:

5.2 Problem 3.46(a)

What is the leading behavior of solutions to $y'' + \frac{y'}{x^{\frac{3}{2}}} - \frac{y}{x^2} = 0$ as $x \to \infty$? Show that it is inconsistent to assume that $S'' \ll S'^2\,(x \to \infty)$. However, show that the approximate equation $S'' + (S')^2 \sim \frac{1}{x^2}\,(x \to \infty)$ can be solved exactly by assuming a solution of the form $S' = \frac{c}{x}$

Answer:

6 Problem 6 (BO section 3.6, 3.49(b))

Find the leading behavior as $x \to \infty$ of the general solution to each of the following equations

$$y'' + x^3 y' + xy = 2x^4 e^{-x^2}$$

Answer: