HW #3.

a) α IS ANGLE BETWEEN FREESTREAM AND x-AXIS

θ IS ANGLE BETWEEN X-AXIS AND X_E-DIRECTION ↖ GROUND

HERE FREESTREAM IS ALWAYS || X_E-DIRECTION

$\Rightarrow \alpha = \theta$

b) HERE $u = u_0$ = CONST

$\dot{u} = 0$

RECALL w REALLY MEANS $\hat{w} = \frac{w}{u_0} = \alpha$

$\Rightarrow w = \alpha = \Delta\theta$

$\dot{w} = \Delta\dot{\theta}$ ONLY NEED ONE EQUATION FOR ONE OF THEM

$$\Rightarrow \underline{x} = \begin{bmatrix} w \\ q \end{bmatrix} \qquad \underline{\dot{x}} = \begin{bmatrix} \dot{w} \\ \dot{q} \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + m u_0}{m - Z_{\dot{w}}} \\ \frac{1}{I_y}\left[M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}}\right] & \frac{1}{I_y}\left[M_q + \frac{M_{\dot{w}}(Z_q + m u_0)}{m - Z_{\dot{w}}}\right] \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

CHARACT. EQ. $\det\left(\lambda \underline{\underline{I}} - \underline{\underline{A}}\right) = 0$

$$\det \begin{bmatrix} \lambda - A_{11} & A_{12} \\ A_{21} & \lambda - A_{22} \end{bmatrix} = 0$$

$$(\lambda - A_{11})(\lambda - A_{22}) - A_{12}A_{21} = 0$$

$$\lambda^2 - \lambda(A_{11} + A_{22}) - A_{12}A_{21} = 0$$

$$\lambda = \frac{(A_{11}+A_{22}) \pm \sqrt{(A_{11}+A_{22})^2 + 4A_{12}A_{21}}}{2}$$

$$(A_{11}+A_{22}) = \frac{M_q}{I_y} + \frac{1}{m - Z_{\dot{w}}}\left(Z_w + \frac{M_{\dot{w}}(Z_q + m u_0)}{I_y}\right)$$

$$A_{12}A_{21} = \frac{1}{I_y}\left[M_w \frac{Z_q + m u_0}{m - Z_{\dot{w}}} + \frac{M_{\dot{w}} Z_w (Z_q + m u_0)}{(m - Z_{\dot{w}})^2}\right]$$

ASSUME $\Delta < 0$ INSIDE SQUARE ROOT

$$\Rightarrow \lambda_{1,2} = m \pm i\omega$$

w/ $$\omega = \frac{\sqrt{-(A_{11}+A_{22})^2 - 4A_{12}A_{21}}}{2}$$

$$m = \frac{A_{11}+A_{22}}{2}$$

$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{4\pi}{\sqrt{-(A_{11}+A_{22})^2 - 4A_{12}A_{21}}}$$

$$t_{HALF} = \frac{0.693}{|m|} = \frac{1.386}{-(A_{11}+A_{22})}$$

where, from Table 4.1 and Section 6.2,

$$\frac{1}{t^*} = \frac{2u_o}{\bar{c}}$$

$$= \frac{2 \times 774}{27.31}$$

$$= 56.683 \qquad (12)$$

The following table lists the results of applying (11) to Table 6.9 using the data from the above table.

Mode	ψ	$\hat{v}$	$\psi + \hat{v}$	$\frac{y_E}{u_o t^*}$
Spiral	0.997 +0i	-0.00119 +0i	0.9958 +0i	$7.735\times10^3 \angle 180°$
Roll	-0.0562 +0i	-0.0198 +0i	-0.0760 +0i	$7.659 \angle 0°$
Dutch Roll	-0.28162 +0.12716i	0.29110 -0.15543i	0.00948 -0.02827i	$1.78 \angle -163.5°$

6.3 From (6.8,6) for static spiral stability

$$E = (C_{\ell_\beta} C_{n_r} - C_{\ell_r} C_{n_\beta}) \cos\theta_o$$

$$+ (C_{\ell_p} C_{n_\beta} - C_{\ell_\beta} C_{n_p}) \sin\theta_o$$

$$> 0 \qquad (1)$$

Let

$$E = A \cos \theta_o + B \sin \theta_o \tag{2}$$

Also assume

$$L = W = \frac{1}{2} \rho V^2 S C_L$$

Thus

$$V = (2W/\rho S)^{1/2} C_L^{-1/2} \tag{3}$$

at sea level, from App. D

$$\rho = 2.3769 \times 10^{-3} \text{ slug/ft}^3$$

Thus

$$V = \left(\frac{2 \times 2400}{2.3769 \times 10^{-3} \times 160} \right)^{1/2} C_L^{-1/2}$$

$$= 112.35\, C_L^{-1/2} \text{ fps} \tag{4}$$

The results from applying (2) and (4) based on Table 7.2 are given below.

C_L	C_{ℓ_β}	C_{n_r}	C_{ℓ_r}	C_{n_β}	A
0.15	-.08266	-.04854	.02625	.01364	.003654
0.55	-.1193	-.05520	.1347	.01840	.004107
0.95	-.1560	-.06948	.2431	.02860	.003886
1.35	-.1927	-.09138	.3515	.04424	.002059
1.70	-.2248	-.1168	.4463	.06239	-.001588

C_L	C_{ℓ_p}	C_{n_β}	C_{ℓ_β}	C_{n_p}	B
0.15	-.441	.01364	-.08266	-.01558	-.007303
0.55	-.441	.01840	-.1193	-.05422	-.01458
0.95	-.441	.02860	-.1560	-.09286	-.02710
1.35	-.441	.04424	-.1927	-.1315	-.04485
1.70	-.441	.06239	-.2248	-.1653	-.06467

V (fps)	C_L	E (-10°)	E (0°)	E (10°)
290	0.15	.004867	.003654	.002330
151	0.55	.006576	.004107	.001513
115	0.95	.008533	.003886	-.0008789
96.7	1.35	.009816	.002059	-.005760
86.2	1.70	.009666	-.001588	-.01279

See diagram 6.3 for plots of E vs. V.

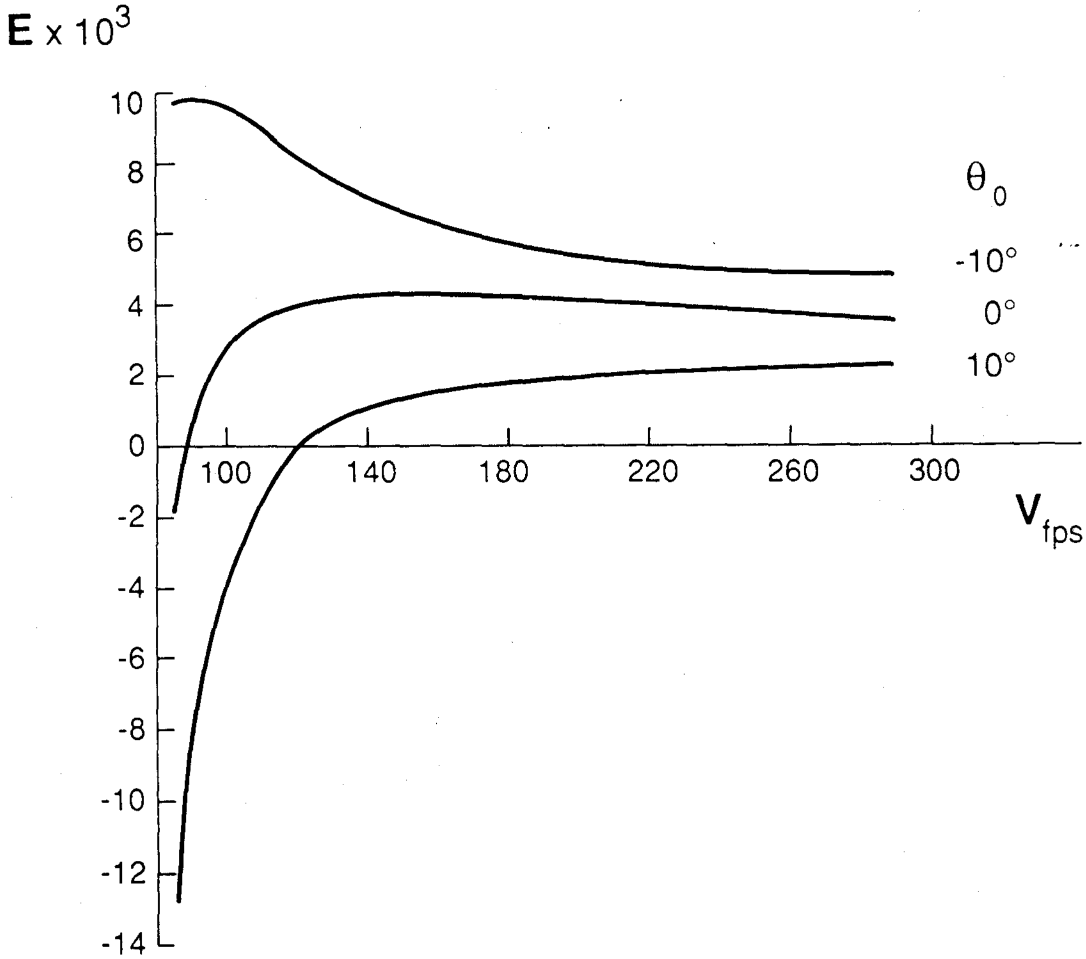

Diagram 6.3

6.4 Write the N^{th} order characteristic equation as

$$(\lambda - \lambda_N)\,(\lambda - \lambda_{N-1})\,...\,(\lambda - \lambda_1) = 0 \qquad (1)$$

For N = 1 this becomes

$$(\lambda - \lambda_1) = 0 \qquad (2)$$

and the coefficient of λ^{N-1} or λ^0 is $-\lambda_1$. For N = 2 (1) becomes

$$(\lambda - \lambda_2)(\lambda - \lambda_1) = 0 \tag{3}$$

or

$$\lambda^2 + \lambda(-\lambda_1 - \lambda_2) + \lambda_1\lambda_2 = 0 \tag{4}$$

and the coefficient of λ^{N-1} or λ^1 is $-(\lambda_1 + \lambda_2)$. For N = n let the coefficient of λ^{n-1} be C_n. Thus for N = n+1 (1) becomes

$$(\lambda - \lambda_{n+1})(\lambda^n + C_n\lambda^{n-1} + \dots) = 0 \tag{5}$$

or

$$\lambda^{n+1} + \lambda^n(C_n - \lambda_{n+1}) + \dots = 0 \tag{6}$$

and the coefficient of λ^{N-1} or λ^n is

$$C_{n+1} = C_n - \lambda_{n+1} \tag{7}$$

From (7) it follows that the coefficient of λ^{N-1} is the negative of the sum of the N roots of (1). Since these roots are either real or complex conjugate pairs it follows that this is also the negative of the sum of the real parts of the N roots of (1).

6.5 From (6.8,6) it is seen that the spiral is stable if $E > 0$. The critical climb angle θ_{o_c} causes $E = 0$. Combine (6.8,6) with the derivative data in Table 6.6 to obtain

$$E = 0.0174 \cos\theta_o - 0.07551 \sin\theta_o \tag{1}$$

Set (1) equal to zero and solve for θ_{o_c}, thus

$$\tan\theta_{o_c} = 0.0174/0.07551$$

$$\theta_{o_c} = 12.98^\circ \quad = 0.2053 \text{ RAD} \tag{2}$$

Assume $\theta_o = 0$ in horizontal flight, thus from (6.8,6)

$$E = C_{\ell_\beta} C_{n_r} - C_{\ell_r} C_{n_\beta} \tag{3}$$

Γ	↑	↓	
C_{ℓ_β}	↓	↓	STAB
C_{ℓ_r}	↑		DESTAB

The effect of Γ on the stability derivatives in (3) can be seen from App. B.9 and B.11. Only C_{ℓ_β} and C_{ℓ_r} are affected. For increasing Γ, C_{ℓ_β} becomes more negative and C_{ℓ_r} more positive. For $C_{n_r} = -0.2737$ and $C_{n_\beta} = 0.1946$ from Table 6.6, it follows that for increasing Γ, the C_{ℓ_β} effect is stabilizing and the C_{ℓ_r} effect is destabilizing (since $E > 0$ for stability). Based on the results of Exercise 5.3, the C_{ℓ_β} effect should normally dominate.

6.6 From (A.4,12) with $\phi = \psi = 0$ and θ small

$$\mathbf{L}_{BE} = \begin{bmatrix} 1 & 0 & -\theta \\ 0 & 1 & 0 \\ \theta & 0 & 1 \end{bmatrix} \tag{1}$$

Now

$$\mathbf{W}_B = \mathbf{L}_{BE}\,\mathbf{W}_E \tag{2}$$

From (1), (2) and (6.9,4)

$$\mathbf{W}_B = \begin{bmatrix} 1 & 0 & -\theta \\ 0 & 1 & 0 \\ \theta & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (W_o + \Gamma z_E)$$

$$= \begin{bmatrix} 1 \\ 0 \\ \theta \end{bmatrix} (W_o + \Gamma z_E)$$

$$\approx \begin{bmatrix} W_o + \Gamma z_E \\ 0 \\ W_o\theta \end{bmatrix} \tag{3}$$

since $\Gamma\theta z_E$ is second order and can be dropped.

6.7 From (4.9,19) with $I_{zx} = \theta_o = 0$ form

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\delta_a \tag{1}$$

where

$$\mathbf{x} = [v \quad p \quad r \quad \phi]^T \tag{2}$$

$$\mathbf{A} = \begin{bmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \frac{Y_r}{m} - u_o & g \\ \frac{L_v}{I_x} & \frac{L_p}{I_x} & \frac{L_r}{I_x} & 0 \\ \frac{N_v}{I_z} & \frac{N_p}{I_z} & \frac{N_r}{I_z} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{3}$$

$$\mathbf{B} = \begin{bmatrix} \frac{Y_{\delta_a}}{m} \\ \frac{L_{\delta_a}}{I_x} \\ \frac{N_{\delta_a}}{I_z} \\ 0 \end{bmatrix} \quad (4)$$

From Table 4.4

$$Y_v = \frac{1}{2}\rho u_0 S C_{y\beta}$$

$$Y_p = \frac{1}{4}\rho u_0 b S C_{y_p}$$

$$Y_r = \frac{1}{4}\rho u_0 b S C_{y_r}$$

$$L_v = \frac{1}{2}\rho u_0 b S C_{\ell\beta}$$

$$L_p = \frac{1}{4}\rho u_0 b^2 S C_{\ell_p}$$

$$L_r = \frac{1}{4}\rho u_0 b^2 S C_{\ell_r} \quad (5)$$

$$N_v = \frac{1}{2}\rho u_0 b S C_{n\beta}$$

$$N_p = \frac{1}{4}\rho u_0 b^2 S C_{n_p}$$

$$N_r = \frac{1}{4}\rho u_0 b^2 S C_{n_r}$$

$$Y_{\delta_a} = \frac{1}{2}\rho u_0^2 S C_{y\delta_a}$$

$$L_{\delta_a} = \frac{1}{2}\rho u_0^2 b S C_{\ell\delta_a}$$

$$N_{\delta_a} = \frac{1}{2}\rho u_0^2 b S C_{n\delta_a}$$

From App. D $\rho_{sea\ level} = 2.3769\times10^{-3}$ slug/ft^3. Since L = W, thus

$$C_L = \frac{W}{\frac{1}{2}\rho u_0^2 S}$$

$$= \frac{2400}{\frac{1}{2}\times 2.3769\times 10^{-3}\times\left(\frac{77.3}{.3048}\right)^2\times 160} \quad (6)$$

$$= 0.196$$

From Table 7.2 and (6)

$$C_{y\beta} = -0.14 \quad (7)$$

$$C_{y_p} = -0.039$$

$$C_{y_r} = 0.165$$

$$C_{\ell\beta} = -0.0689 - 0.0917\, C_L$$

$$= -0.0869$$

$$C_{\ell_p} = -0.441$$

$$C_{\ell_r} = -0.0144 + 0.271\ C_L$$
$$= 0.0387$$

$$C_{n_\beta} = 0.01326 + 0.017\ C_L{}^2$$
$$= 0.0139$$

$$C_{n_p} = -0.00109 - 0.0966\ C_L$$
$$= -0.0200$$

$$C_{n_r} = -0.048 - 0.0238\ C_L{}^2$$
$$= -0.0489$$

[redacted]

[redacted]

[redacted]

$$\frac{1}{2}\rho u_o S = \frac{1}{2} \times 2.3769 \times 10^{-3} \times \left(\frac{77.3}{.3048}\right) \times 160 \qquad (8)$$
$$= 48.22$$

$$\frac{1}{4}\rho u_o b S = \frac{30}{2} \times 48.22$$
$$= 723.3$$

$$\frac{1}{2}\rho u_0 bS = 1{,}446.6$$

$$\frac{1}{4}\rho u_0 b^2 S = 21{,}701$$

$$\frac{1}{2}\rho u_0^2 bS = 366{,}903$$

From (5), (7) and (8)

$$Y_v = 48.22 \times (-0.14) = -6.751 \quad (9)$$

$$Y_p = 723.3 \times (-0.039) = -28.209$$

$$Y_r = 723.3 \times (0.165) = 119.345$$

$$L_v = 1{,}446.6 \times (-0.0869) = -125.710$$

$$L_p = 21{,}701 \times (-0.441) = -9{,}570.14$$

$$L_r = 21{,}701 \times (0.0387) = 839.829$$

$$N_v = 1{,}446.6 \times (0.0139) = 20.108$$

$$N_p = 21{,}701 \times (-0.0200) = -434.020$$

$$N_r = 21{,}701 \times (-0.0489) = -1{,}061.18$$

[redacted]

[redacted]

[redacted]

$$m = 2400/32.2 = 74.534 \text{ slugs}$$

$$I_x = 170 \text{ slug ft}^2 \qquad g = 32.2 \text{ ft/s}^2$$

$$I_z = 1{,}312 \text{ slug ft}^2 \qquad u_o = \frac{77.3}{.3048} = 253.6 \text{ fps}$$

From (3) and (9)

$$\mathbf{A} = \begin{bmatrix} -0.0906 & -0.378 & -252 & 32.2 \\ -0.739 & -56.3 & 4.94 & 0 \\ 0.0153 & -0.331 & -0.809 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad (10)$$

From (4) and (9)

$$\mathbf{B} = \begin{bmatrix} 0 \\ -115 \\ 1.40 \\ 0 \end{bmatrix} \qquad (11)$$

NOW CAN FIND $\underline{\underline{A}}$'S EIGENVALUES AND EIGENVECTORS USING A MODIFIED VERSION OF THE UNCONTROLLED_LONGITUDINAL.m MATLAB SCRIPT

NOTE THAT THE ELEMENTS OF $\underline{\underline{A}}$ WERE OBTAINED USING ENGLISH UNITS. THUS WE'LL HAVE TO DO THE SAME FOR THE NORMALIZATIONS.

HERE

$u_0 = 77.3 \text{ m/s} = 253.61 \text{ ft/s}$ (CRUISE SPEED)

$b = 30 \text{ ft}$ (WINGSPAN)

AND THE NORMALIZATIONS ARE

$$\hat{v} = \frac{v}{u_0} \qquad \hat{p} = \frac{p}{\frac{2u_0}{b}} \qquad \hat{r} = \frac{r}{\frac{2u_0}{b}} \qquad \hat{\phi} = \phi$$

```
hw4_prob3_w_gravity

A =

  -0.0906   -0.3780 -252.0000   32.2000
  -0.7390  -56.3000    4.9400         0
   0.0153   -0.3310   -0.8090         0
        0    1.0000         0         0


eigenvalues =

 -56.3028                 0                  0                 0
        0        -0.4207 + 2.2839i           0                 0
        0                 0          -0.4207 - 2.2839i         0
        0                 0                  0           -0.0554


eigenvectors =

  -0.0435     -0.9999             -0.9999             0.9807
  -0.9989      0.0131 + 0.0002i   0.0131 - 0.0002i   -0.0107
  -0.0059     -0.0014 + 0.0084i  -0.0014 - 0.0084i    0.0246
   0.0177     -0.0009 - 0.0056i  -0.0009 + 0.0056i    0.1936


      Eigenvalue          Damping     Frequency

 -5.63e+01                  1.00e+00     5.63e+01
 -4.21e-01 + 2.28e+00i     1.81e-01     2.32e+00
 -4.21e-01 - 2.28e+00i     1.81e-01     2.32e+00
 -5.54e-02                  1.00e+00     5.54e-02

(Frequencies expressed in rad/TimeUnit)


period =

     Inf      Inf      Inf      Inf
     Inf   2.7510      Inf      Inf
     Inf      Inf  -2.7510      Inf
     Inf      Inf      Inf      Inf


thalf =

   0.0123     -Inf     -Inf     -Inf
     -Inf   1.6472     -Inf     -Inf
     -Inf     -Inf   1.6472     -Inf
     -Inf     -Inf     -Inf  12.5134
```

nhalf =

0	NaN	NaN	NaN
NaN	-0.5988	NaN	NaN
NaN	NaN	0.5988	NaN
NaN	NaN	NaN	0

Summary

r arbitrarily chosen with unit amplitude and zero phase

	Dimensional Ratios	Dimensionless Ratios	Phases
Roll			
v	7.3219	0.4881	0
p	167.9934	167.9934	0
r	1.0000	1.0000	0
phi	2.9837	50.4472	-180.0000
Dutch Roll			
v	117.9715	7.8648	80.1557
p	1.5462	1.5462	-98.9582
r	1.0000	1.0000	0
phi	0.6658	11.2568	-199.3952
Spiral			
v	39.8336	2.6556	0
p	0.4355	0.4355	180.0000
r	1.0000	1.0000	0
phi	7.8645	132.9684	0

$$N_v(\lambda_3) = -309.33(-.33692 + 2.284i)(106.58 + 2.284i) \qquad (20)$$
$$= 76{,}132\angle 279.62^\circ$$

$$N_p(\lambda_3) = -115(-.4207 + 2.284i)[(-.00097 + 2.284i)^2 + 1.2166^2]$$
$$= -115(-.4207 + 2.284i)(-3.7365 - .0044310i)$$
$$= 997.94\angle 100.50^\circ$$

$$N_r(\lambda_3) = 1.4(-.82245 + 2.284i)(.0764 + 2.284i)(83.064 + 2.284i)$$
$$= 645.38\angle 199.46^\circ$$

$$N_\phi(\lambda_3) = N_p(\lambda_3)/(-.4207 + 2.284i)$$
$$= 429.70\angle 0.06^\circ$$

From (20)

$$\frac{v}{r} = 117.96\angle 80.16^\circ \qquad \frac{\hat{v}}{\hat{r}} = 7.86\angle 80.16^\circ$$

$$\frac{p}{r} = 1.546\angle -98.96^\circ \qquad \frac{\hat{p}}{\hat{r}} = 1.546\angle -98.96^\circ \qquad (21)$$

$$\frac{\phi}{r} = 0.666\angle -199.4^\circ \qquad \frac{\phi}{\hat{r}} = 11.26\angle -199.4^\circ$$

When gravity is absent g = 0 in (3). The variable ϕ can then be eliminated from the system equations by forming

$$\dot{\mathbf{x}} = \mathbf{A}'\mathbf{x} + \mathbf{B}'\delta_a \qquad (22)$$

where

$$\mathbf{x} = [v \quad p \quad r]^T \qquad (23)$$

and A' is 3×3. Bank angle can be found from the auxiliary equation

$$\phi = \int p\,dt + \phi_0 \tag{24}$$

Also, since gravity is now absent it follows that

$$L = W = mg = 0 \tag{25}$$

and

$$C_L = 0 \tag{26}$$

Thus from Table 7.2 the following derivatives which depend on C_L are altered

$$C_{\ell_\beta} = -0.0689 \tag{27}$$

$$C_{\ell_r} = -0.0144$$

$$C_{n_\beta} = 0.01326$$

$$C_{n_p} = -0.00109$$

$$C_{n_r} = -0.048$$

From (5), (8) and (27)

$$L_v = 1{,}446.6. \times (-0.0689) = -99.671 \tag{28}$$

$$L_r = 21{,}701 \times (-0.0144) = -312.494$$

$$N_v = 1{,}446.6 \times (0.01326) = 19.182$$

$$N_p = 21{,}701 \times (-0.00109) = -23.654$$

$$N_r = 21{,}701 \times (-0.048) = -1{,}041.65$$

The modes of the system are determined from (22) and there are now only 3 distinct roots to the characteristic equation. In (22) (from (3), (9) and (28))

$$\mathbf{A}' = \begin{bmatrix} -0.0906 & -0.378 & -252 \\ -0.586 & -56.3 & -1.84 \\ 0.0146 & -0.0180 & -0.794 \end{bmatrix} \quad (29)$$

$$\mathbf{B}' = \begin{bmatrix} 0 \\ 115 \\ 1.40 \end{bmatrix} \quad (30)$$

The resulting eigenvalues are

$$\lambda_1 = -56.31 \qquad \text{(Roll)} \quad (31)$$

$$\lambda_{2,3} = -0.43961 \pm 1.8977i \qquad \text{(Dutch Roll)} \quad (32)$$

For $g = 0$, λ_1 and $\lambda_{2,3}$ look like λ_2 and $\lambda_{3,4}$ for $g = 32.2$ ft/s^2, and thus they were named accordingly. The numerators of the transfer functions $\bar{x}_i/\bar{\delta}_a$ are

```
hw4_prob3_no_gravity

A =

   -0.0906   -0.3780 -252.0000
   -0.5860  -56.3000   -1.8400
    0.0146   -0.0180   -0.7940


eigenvalues =

  -0.4396 + 1.8977i        0                  0
        0           -0.4396 - 1.8977i        0
        0                  0           -56.3054


eigenvectors =

   0.9999             0.9999             0.0082
  -0.0105 + 0.0006i  -0.0105 - 0.0006i   1.0000
   0.0014 - 0.0075i   0.0014 + 0.0075i   0.0003


       Eigenvalue          Damping     Frequency

 -4.40e-01 + 1.90e+00i     2.26e-01     1.95e+00
 -4.40e-01 - 1.90e+00i     2.26e-01     1.95e+00
 -5.63e+01                 1.00e+00     5.63e+01

(Frequencies expressed in rad/TimeUnit)


period =

    3.3109       Inf       Inf
       Inf   -3.3109       Inf
       Inf       Inf       Inf


thalf =

    1.5764      -Inf      -Inf
      -Inf    1.5764      -Inf
      -Inf      -Inf    0.0123


nhalf =

   -0.4761       NaN       NaN
       NaN    0.4761       NaN
       NaN       NaN         0
```

Summary

r arbitrarily chosen with unit amplitude and zero phase

Dimensional Ratios	Dimensionless Ratios	Phases
Roll		
1.0e+03 *		
v 0.0254	0.0017	0
p 3.1045	3.1045	0
r 0.0010	0.0010	0
Dutch Roll		
1.0e+03 *		
v 0.1305	0.0087	0.0795
p 0.0014	0.0014	0.2562
r 0.0010	0.0010	0

$\lambda_2 = -0.43961 + 1.8977i$ Dutch Roll Mode

$$N_v(\lambda_2) = -309.33(65.343 + 1.8977i) \qquad (36)$$

$$= 20{,}221\angle 181.66^\circ$$

$$N_p(\lambda_2) = 115[(.01389 + 1.8977i)^2 + 1.3228^2]$$

$$= -115(-1.8513 + .05272i)$$

$$= 212.99\angle -1.63^\circ$$

$$N_r(\lambda_2) = 1.4(-.34500 + 1.8977i)(57.335 + 1.8977i)$$

$$= 154.91\angle 102.20^\circ$$

From (36)

$$\frac{v}{r} = 130.5\angle 79.46^\circ \qquad \frac{\hat{v}}{\hat{r}} = 8.70\angle 79.46^\circ$$

$$\frac{p}{r} = 1.375\angle -103.83^\circ \qquad \frac{\hat{p}}{\hat{r}} = 1.375\angle -103.83^\circ \qquad (37)$$

The biggest effect of setting g = 0 was the disappearance of the spiral mode. Comparing (19) with (35) and (21) with (37) it can be seen that deleting gravity had no major impact on the Dutch roll mode but it did alter the roll mode (there is now relatively less yaw response).

6.8 From Exercise 4.10 the linearized equations for the hovercraft are

$$\Delta\dot{u} = -g\theta \qquad (1)$$

$$\dot{v} = -g\phi - u_o r \tag{2}$$

$$\dot{w} = \frac{Z_z}{m}\Delta z_E + u_o q \tag{3}$$

$$\Delta\dot{z}_E = -u_o\theta + w \tag{4}$$

$$\dot{\phi} = p \tag{5}$$

$$\dot{\theta} = q \tag{6}$$

$$\dot{\psi} = r \tag{7}$$

$$\dot{p} = \frac{L_\phi}{I_x}\phi - \frac{H}{I_x}q \tag{8}$$

$$\dot{q} = \frac{M_\theta}{I_y}\theta + \frac{H}{I_y}p \tag{9}$$

$$\dot{r} = 0 \tag{10}$$

From the above it can be seen that (5), (6), (8) and (9) represent a self-contained subset of equations containing all the parameters required to solve this exercise. Thus consider these 4 equations and represent them by

$$\mathbf{x} = [\theta \quad q \quad \phi \quad p]^T \tag{11}$$

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \tag{12}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{M_\theta}{I_y} & 0 & 0 & \frac{H}{I_y} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{H}{I_x} & \frac{L_\phi}{I_x} & 0 \end{bmatrix} \tag{13}$$

Thus the characteristic equation is

$$|\mathbf{I}\,s - \mathbf{A}| = 0 \tag{14}$$

From (13) and (14) it follows that

$$\begin{vmatrix} s & -1 & 0 & 0 \\ -\frac{M_\theta}{I_y} & s & 0 & -\frac{H}{I_y} \\ 0 & 0 & s & -1 \\ 0 & \frac{H}{I_x} & -\frac{L_\phi}{I_x} & s \end{vmatrix} = 0 \tag{15}$$

Expand (15) on the first row

$$s\begin{vmatrix} s & 0 & -\frac{H}{I_y} \\ 0 & s & -1 \\ \frac{H}{I_x} & -\frac{L_\phi}{I_x} & s \end{vmatrix} + \begin{vmatrix} -\frac{M_\theta}{I_y} & 0 & -\frac{H}{I_y} \\ 0 & s & -1 \\ 0 & -\frac{L_\phi}{I_x} & s \end{vmatrix} = 0 \qquad (16)$$

Expand the determinants on their second rows in (16)

$$s^2\begin{vmatrix} s & -\frac{H}{I_y} \\ \frac{H}{I_x} & s \end{vmatrix} + s\begin{vmatrix} s & 0 \\ \frac{H}{I_x} & -\frac{L_\phi}{I_x} \end{vmatrix} + s\begin{vmatrix} -\frac{M_\theta}{I_y} & -\frac{H}{I_y} \\ 0 & s \end{vmatrix} + \begin{vmatrix} -\frac{M_\theta}{I_y} & 0 \\ 0 & -\frac{L_\phi}{I_x} \end{vmatrix} = 0 \qquad (17)$$

and (17) becomes on expansion

$$s^2\left(s^2 + \frac{H^2}{I_xI_y}\right) - s^2\frac{L_\phi}{I_x} - s^2\frac{M_\theta}{I_y} + \frac{M_\theta L_\phi}{I_xI_y} = 0 \qquad (18)$$

or

$$s^4 + s^2\left(\frac{H^2}{I_xI_y} - \frac{L_\phi}{I_x} - \frac{M_\theta}{I_y}\right) + \frac{M_\theta L_\phi}{I_xI_y} = 0 \qquad (19)$$

Let (19) be represented by

$$s^4 + s^2b + c = 0 \qquad (20)$$

Thus

$$s^2 = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c} \qquad (21)$$

Consider the case where $\left(\frac{b}{2}\right)^2 < c$: thus

$$s^2 = -\frac{b}{2} \pm id \qquad (22)$$

where $d = \sqrt{c - \left(\frac{b}{2}\right)^2}$, and (22) is a complex number. Let (22) be represented by

$$s^2 = ae^{i\beta^+} \quad \text{and} \quad s^2 = ae^{i\beta^-} \qquad (23)$$

where "a" is real and positive.

Thus the solutions to (23) are

$$s_{1,2} = \pm\sqrt{a}\, e^{i(\beta^+/2)}$$
$$s_{3,4} = \pm\sqrt{a}\, e^{i(\beta^-/2)} \qquad (24)$$

and

$$\mathrm{Re}\{s_{1,2}\} = \pm\sqrt{a}\cos\frac{\beta^+}{2}$$
$$\mathrm{Re}\{s_{3,4}\} = \pm\sqrt{a}\cos\frac{\beta^-}{2} \qquad (25)$$

Since there are always positive real parts in (25) the system is unstable when $\left(\frac{b}{2}\right)^2 < c$ or

$$(H^2 - L_\phi I_y - M_\theta I_x)^2 < 4M_\theta L_\phi I_x I_y \qquad (26)$$

Consider the case where $\left(\frac{b}{2}\right)^2 > c$: thus s^2 is real with

$$s^2 = -\frac{b}{2} \pm e \tag{27}$$

where

$$e = \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

Since c is given as positive (i.e., M_θ and L_ϕ positive) thus

and

$$0 < e < \left| \frac{b}{2} \right| \tag{28}$$

It follows from (27) and (28) that the polarity of s^2 is the same as that of (-b). Thus, if b is negative (b < 0). Then for f real

$$s^2 = f$$
$$f > 0 \tag{29}$$
$$s = \pm\sqrt{f}$$

and some solutions s_i to (27) will have positive real parts and the system is unstable.

If b is positive (b > 0) then

$$s^2 < 0 \tag{30}$$

and all solutions s_i to (27) will be pure imaginary, i.e., zero real parts, and the system will be neutrally stable and hence bounded.

From the above it is seen that to prevent instability requires

$$\left(\frac{b}{2}\right)^2 > c \tag{31}$$

and

$$b > 0 \tag{32}$$

For M_θ and $L_\phi > 0$ this can be achieved by increasing H to a suitable level. This can be seen from:

(31) becomes
$$(H^2 - L_\phi I_y - M_\theta I_x)^2 > 4M_\theta L_\phi I_x I_y \tag{33}$$

(32) becomes
$$H^2 - L_\phi I_y - M_\theta I_x > 0 \tag{34}$$

6.9 In the instant following the point in time when the headwind suddenly vanishes, the aircraft's inertial velocity $\mathbf{V}^E$ is unchanged and the airspeed is suddenly reduced in magnitude by an amount equal to the headwind speed. The governing equations following the removal of the headwind are those presented in Chapter 4 (the no-wind case). Thus for the given control settings the initial airspeed is too low and the aircraft will respond as if it had a negative Δu for its initial conditions. In general, the release of a dynamic system of linear differential equations from non-zero initial conditions will result in a response which is a linear combination of its modal responses. From the longitudinal modes described in Section 6.2 it can be seen that (see Fig. 6.3) Δu is almost absent from the short-period mode while it figures prominently in the phugoid mode. Thus any excitation of the short-period mode should be minimal, and since it is highly damped, it would soon disappear. On the other hand, the lightly damped phugoid mode should be strongly excited.

The steady state flight path angle without the headwind will be less steep than that in the presence of the headwind. If it is assumed that the headwind is larger than a small perturbation in u, then the initial response will be governed by the nonlinear flight equations. The initial drop in airspeed will cause the aircraft to lose lift and fall below the original flight path. This will be followed by an oscillation in flight path angle that will soon become dominated by the phugoid response as the response becomes that of the linearized equations. Note that flight path angle