quiz 2

EMA 545
Mechanical Vibrations

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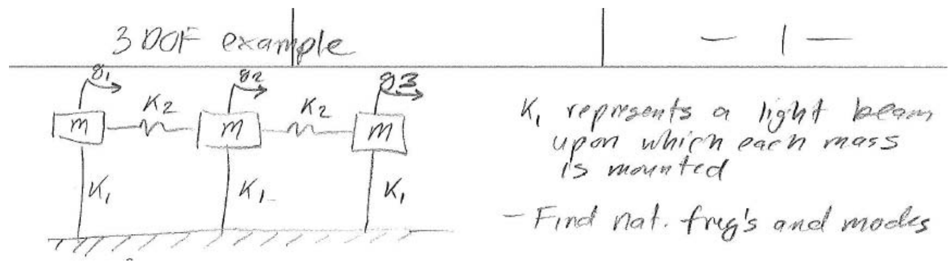
Contents

1 Problem description

Consider this 3 DOF system



Suppose a harmonic force $f(t) = A\cos(\varpi t)$ is applied to the mass in the center. Use modal analysis to do the following:

1. Find the uncoupled modal equations of motion. Consider the steady state solution for each of these equations. Sketch the modal amplitude (X_j in the book on page 275) for each mode versus frequency. A hand sketch is sufficient.

2. Use that result to sketch the frequency response of each of the masses, in other words the complex amplitude Y_n versus ω

2 Answer part (1)

A summary of the steps needed for full modal analysis is first given. In these steps, a column vector is shown as bold letter $\mathbf{Y}$ and a matrix is shown as $[M]$. In this summary, the system is assumed to have n degree of freedom.

The steps are

1. Determine the system of equations of motion and set up $[M]\mathbf{Y}'' + [C]\mathbf{Y}' + [K]\mathbf{Y} = \mathbf{F}$ in matrix form.

2. Solve the eigenvalue problem $\det\left([K] - \omega^2[M]\right) = 0$ in order to determine the n natural frequencies.

3. For each natural frequency ω_j determine the corresponding j^{th} eigenvector $_j$ by solving $\left([K] - \omega_j^2[M]\right)_j = \mathbf{0}$. In this step, the first component of $_j$ is set to 1 and the other components are solved relative to it.

4. Obtain the normalized eigenvectors Φ_j for each $_j$ using $\Phi_j = \frac{_j}{\sqrt{u_j}}$ where $u_j = {}_j^T[M]_j$. Each u_j will be a scalar.

5. Set up the modal transformation matrix $[\Phi] = [\Phi_1\Phi_2 \cdots \Phi_n]$. This will be an $n \times n$ matrix.

6. The transformation from normal solution $y(t)$ to modal $\eta(t)$ will be $\mathbf{Y} = [\Phi]$ and $= [\Phi]^{-1}\mathbf{Y} = [\Phi]^T[M]\mathbf{Y}$

7. Apply the above transformation on the original equations of motions in matrix form to obtain the equations of motion in modal coordinates $[\Phi]^T[M]\,[\Phi]\mathbf{Y}'' + [\Phi]^T[C][\Phi]\mathbf{Y}' + [\Phi]^T[C][\Phi]\mathbf{Y} = [\Phi]^T\mathbf{F}$. This becomes $\mathbf{I}''(t) + \left[\tilde{C}\right]'(t) + \left[\tilde{K}\right](t) = [\Phi]^T\mathbf{F}$ where $\mathbf{I}$ is the identity matrix, $\left[\tilde{C}\right]$ is a diagonal damping matrix obtained using a method such as weak damping approximation and $\left[\tilde{K}\right]$ is diagonal matrix with diagonal that contains the natural frequencies squared ω_j^2 in each of entries.

8. For steady state solution in modal coordinates, the loading vector $[\Phi]^T\mathbf{F}$ is assumed to be $\mathbf{Q} = [\Phi]^T\mathbf{F} = \text{Re}\left(\hat{\mathbf{Q}}e^{i\varpi t}\right)$ where $\hat{\mathbf{Q}}$ is the complex amplitude of the loading vector in modal coordinates. Therefore, the steady state solution is $_{ss}(t) = \text{Re}\left(\hat{\mathbf{X}}e^{i\varpi t}\right)$ where $\hat{\mathbf{X}}$ is the complex amplitude of each modal response is $\hat{X}_j = \frac{{}_j^T\mathbf{F}}{-\varpi^2 + i2\zeta_j\omega_j\varpi + \omega_j^2}$.

For a system with no damping this simplifies to $\hat{X}_j = \frac{{}_j^T\mathbf{F}}{-\varpi^2+\omega_j^2}$. In here, ${}_j^T$ represents the transpose of the j^{th} column of the modal transformation matrix $[\Phi]$, or the transpose of the j^{th} mass normalized eigenvector, and ω_j is the j^{th} natural frequency.

9. Now the steady state solution in modal coordinate is used to obtain the solution in normal coordinates since $\mathbf{Y} = [\Phi]$. Therefore $\mathbf{Y}_{ss} = \mathrm{Re}\left(\hat{\mathbf{X}}e^{i\varpi t}\right) = \mathrm{Re}\left([\Phi]\hat{\mathbf{X}}e^{i\varpi t}\right) = \mathrm{Re}\left(\hat{\mathbf{Y}}e^{i\varpi t}\right)$. In component form $\mathbf{Y}_{ss} = \mathrm{Re}\left(\left(\sum_{j=1}^{n} {}_j\hat{X}_j\right)e^{i\varpi t}\right)$

The EOM are derived in the hand out given. The force $f(t)$ acting on the second mass is now added, resulting in the following equations of motion for the system

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} q_1'' \\ q_2'' \\ q_3'' \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_1+2k_2 & -k_2 \\ 0 & -k_2 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ A\cos(\varpi t) \\ 0 \end{Bmatrix}$$

The first step is to obtain the natural frequencies of the system. This is done by solving the eigenvalue problem $\det\left([K] - \omega^2[M]\right) = 0$. The solutions are also given in handout. They are $\omega_1^2 = \frac{k_1}{m}, \omega_2^2 = \frac{k_1+k_2}{m}, \omega_3^2 = \frac{k_1+3k_2}{m}$. The non mass normalized eigenvectors associated with these eigenvalues are found as

$$_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, {}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, {}_3 = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$$

The next step is to mass normalize the eigenvectors as follows

$$\mu_1 = {}_1^T[M]{}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 3m$$

$$\mu_2 = {}_2^T[M]{}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = 2m$$

$$\mu_3 = {}_3^T[M]{}_3 = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix} = 6m$$

Hence the mass normalized eigenvectors are

$$_1 = \frac{1}{\sqrt{\mu_1}} = \frac{1}{\sqrt{3m}}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$_2 = \frac{2}{\sqrt{\mu_2}} = \frac{1}{\sqrt{2m}}\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

$$_3 = \frac{3}{\sqrt{\mu_3}} = \frac{1}{\sqrt{6m}}\begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$$

Hence the modal transformation matrix $[\Phi]$ is

$$[\Phi] = \left[{}_{123}\right] = \frac{1}{\sqrt{m}}\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.577 & 0.707 & 0.408 \\ 0.577 & 0 & -0.816 \\ 0.577 & -0.707 & 0.408 \end{bmatrix}$$

The modal EOM's are now found using the modal transformation matrix $[\Phi]$

$$[\Phi]^T[M][\Phi]\left\{\eta''\right\} + [\Phi]^T[K][\Phi]\left\{\eta\right\} = [\Phi]^T\mathbf{Q}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = [\Phi]^T\begin{Bmatrix} 0 \\ A\cos(\varpi t) \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + \frac{1}{m}\begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_1 + k_2 & 0 \\ 0 & 0 & k_1 + 3k_2 \end{bmatrix}\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.577 & 0.707 & 0.408 \\ 0.577 & 0 & -0.816 \\ 0.577 & -0.707 & 0.408 \end{bmatrix}^T\begin{Bmatrix} 0 \\ A\cos(\varpi t) \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + \frac{1}{m}\begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_1 + k_2 & 0 \\ 0 & 0 & k_1 + 3k_2 \end{bmatrix}\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.577\,A\cos(\varpi t) \\ 0 \\ -0.816\,A\cos(\varpi t) \end{Bmatrix}$$

Therefore, the 3 uncoupled modal EOM's are

$$\eta_1''(t) + \frac{k_1}{m}\eta_1(t) = \frac{0.577\,A}{\sqrt{m}}\cos(\varpi t)$$
$$\eta_2''(t) + \frac{k_1 + k_2}{m}\eta_2(t) = 0$$
$$\eta_3''(t) + \frac{k_1 + 3k_2}{m}\eta_3(t) = -\frac{0.816\,A}{\sqrt{m}}\cos(\varpi t)$$

To complete the solution, the above EOM are written as follows by using complex form for the loading vector

$$\eta_1''(t) + \frac{k_1}{m}\eta_1(t) = \mathrm{Re}\left(\frac{0.577\,A}{\sqrt{m}}e^{i\varpi t}\right)$$
$$\eta_2''(t) + \frac{k_1 + k_2}{m}\eta_2(t) = 0$$
$$\eta_3''(t) + \frac{k_1 + 3k_2}{m}\eta_3(t) = \mathrm{Re}\left(\frac{-0.816\,A}{\sqrt{m}}e^{i\varpi t}\right)$$

Assuming the steady state solution is

$$= \mathrm{Re}\left(\hat{\mathbf{X}}e^{i\varpi t}\right)$$

or in expanded form

$$\eta_1(t) = \mathrm{Re}\left(\hat{X}_1 e^{i\varpi t}\right)$$
$$\eta_2(t) = \mathrm{Re}\left(\hat{X}_2 e^{i\varpi t}\right)$$
$$\eta_3(t) = \mathrm{Re}\left(\hat{X}_3 e^{i\varpi t}\right)$$

Where

$$\hat{X}_1 = \frac{\frac{0.577\,A}{\sqrt{m}}}{\omega_1^2 + 2i\zeta_1\omega_1\varpi - \varpi^2}$$
$$\hat{X}_2 = 0$$
$$\hat{X}_3 = \frac{\frac{-0.816\,A}{\sqrt{m}}}{\omega_3^2 + 2i\zeta_3\omega_3\varpi - \varpi^2}$$

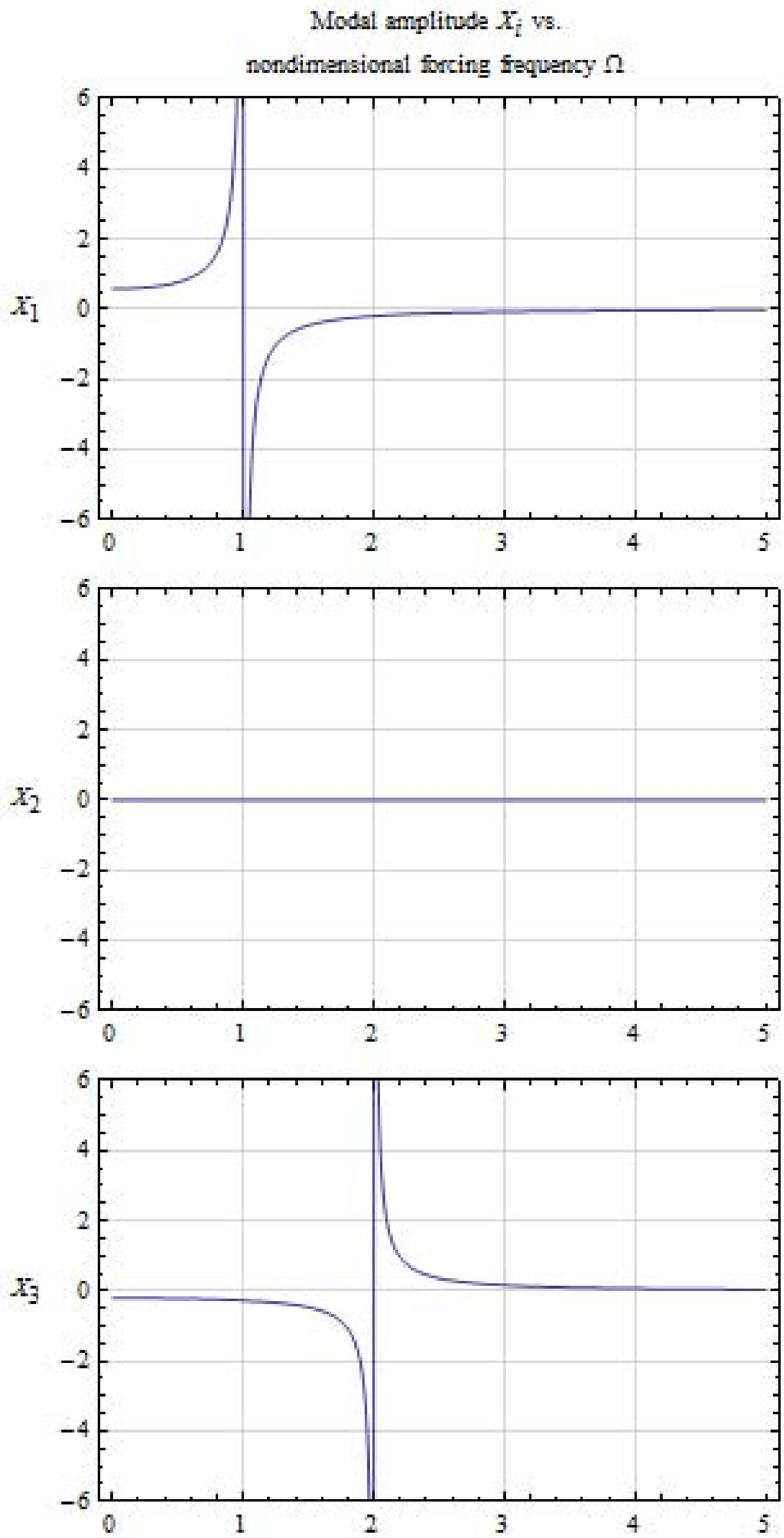
Dividing the numerator and the denominator by ω_i^2 where $i = 1,2,3$ and using $r_i = \frac{\varpi}{\omega_i}$ and letting $\zeta = 0$ since no damping exists, results in

$$\hat{X}_1 = \frac{A\sqrt{m}}{k_1}\left(\frac{0.577}{1 - m\frac{\varpi^2}{k_1}}\right)$$
$$\hat{X}_2 = 0$$
$$\hat{X}_3 = \frac{A\sqrt{m}}{k_1 + 3k_2}\left(\frac{-0.816}{1 - m\frac{\varpi^2}{k_1+3k_2}}\right)$$

To sketch these amplitudes, the equations are normalized. This is in effect the same as setting $m = 1, k_1 = k_2 = 1, A = 1$ resulting in

$$\hat{\mathbf{X}} = \begin{Bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3 \end{Bmatrix} = \begin{Bmatrix} \frac{0.577}{1-\varpi^2} \\ 0 \\ \frac{1}{4}\left(\frac{-0.816}{1-\frac{\varpi^2}{4}}\right) \end{Bmatrix}$$

Here is a plot of each X_i vs ϖ. The x-axis is the nondimensional forcing frequency Ω



Since there is no damping, resonance will occur at $\Omega = 1$ in first mode and at $\Omega = 2$ for mode 3.

3 Answer part (2)

The transformation from modal coordinates to normal coordinates is

$$\mathbf{q} = [\Phi]$$

In expanded form

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} \Phi_1^T\{\eta\} \\ \Phi_2^T\{\eta\} \\ \Phi_3^T\{\eta\} \end{Bmatrix}$$

But $[\Phi] = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.577 & 0.707 & 0.408 \\ 0.577 & 0 & -0.816 \\ 0.577 & -0.707 & 0.408 \end{bmatrix}$ and $= \mathrm{Re}\left(\hat{\mathbf{X}}e^{i\varpi t}\right)$ hence the above becomes

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} \begin{Bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{Bmatrix}^T \begin{Bmatrix} \mathrm{Re}\left(\hat{X}_1 e^{i\varpi t}\right) \\ \mathrm{Re}\left(\hat{X}_2 e^{i\varpi t}\right) \\ \mathrm{Re}\left(\hat{X}_3 e^{i\varpi t}\right) \end{Bmatrix} \\ \begin{Bmatrix} 0.707 \\ 0 \\ -0.707 \end{Bmatrix}^T \begin{Bmatrix} \mathrm{Re}\left(\hat{X}_1 e^{i\varpi t}\right) \\ \mathrm{Re}\left(\hat{X}_2 e^{i\varpi t}\right) \\ \mathrm{Re}\left(\hat{X}_3 e^{i\varpi t}\right) \end{Bmatrix} \\ \begin{Bmatrix} 0.408 \\ -0.816 \\ 0.408 \end{Bmatrix}^T \begin{Bmatrix} \mathrm{Re}\left(\hat{X}_1 e^{i\varpi t}\right) \\ \mathrm{Re}\left(\hat{X}_2 e^{i\varpi t}\right) \\ \mathrm{Re}\left(\hat{X}_3 e^{i\varpi t}\right) \end{Bmatrix} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.577\ \mathrm{Re}\left(X_1 e^{it\varpi}\right) + 0.577\ \mathrm{Re}\left(X_2 e^{it\varpi}\right) + 0.577\ \mathrm{Re}\left(X_3 e^{it\varpi}\right) \\ 0.707\ \mathrm{Re}\left(X_1 e^{it\varpi}\right) - 0.707\ \mathrm{Re}\left(X_3 e^{it\varpi}\right) \\ 0.408\ \mathrm{Re}\left(X_1 e^{it\varpi}\right) - 0.816\ \mathrm{Re}\left(X_2 e^{it\varpi}\right) + 0.408\ \mathrm{Re}\left(X_3 e^{it\varpi}\right) \end{Bmatrix}$$

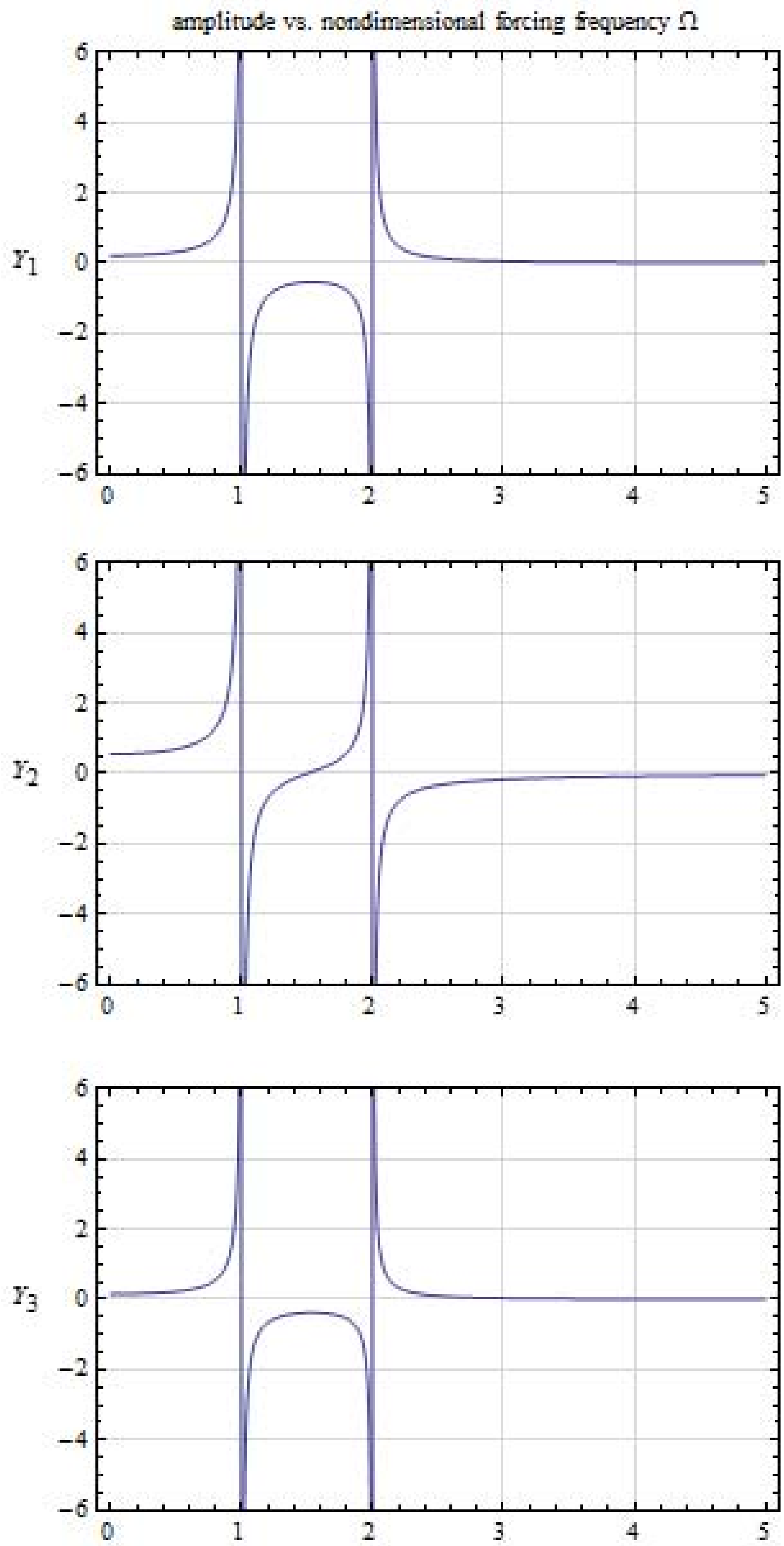
$$= \mathrm{Re}\left(\begin{Bmatrix} 0.577\,X_1 + 0.577\,X_2 + 0.577\,X_3 \\ 0.707\,X_1 - 0.707\,X_3 \\ 0.408\,X_1 - 0.816\,X_2 + 0.408\,X_3 \end{Bmatrix} e^{i\varpi t}\right)$$

Comparing the above to $\mathbf{q}_{ss} = \mathrm{Re}\left(\mathbf{Y}e^{i\varpi t}\right)$ shows that

$$\mathbf{Y} = \begin{Bmatrix} 0.577\,X_1 + 0.577\,X_2 + 0.577\,X_3 \\ 0.707\,X_1 - 0.707\,X_3 \\ 0.408\,X_1 - 0.816\,X_2 + 0.408\,X_3 \end{Bmatrix}$$

To plot each Y_i, let $m = 1, k_1 = 1, k_2 = 1, A = 1$, and letting $X_2 = 0$ as found earlier, results in

$$\mathbf{Y} = \left\{\begin{pmatrix} 0.577\,\frac{0.577}{1-\omega^2} + \frac{0.577}{4}\left(\frac{-0.816}{1-\frac{\omega^2}{4}}\right) \\ 0.707\frac{0.577}{1-\omega^2} - \frac{0.707}{4}\left(\frac{-0.816}{1-\frac{\omega^2}{4}}\right) \\ 0.408\,\frac{0.577}{1-\omega^2} + \frac{0.408}{4}\left(\frac{-0.816}{1-\frac{\omega^2}{4}}\right) \end{pmatrix}\right\}$$



The above shows that when the nondimensional frequency Ω is not close to a one of the nondimensional natural frequencies, then the Y values have comparable magnitudes. For nondimensional frequency Ω larger than 3 all amplitude are zero, which means the whole system does not oscillate any more in steady state.